The long path to $\sqrt{d}$ monotonicity testers

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AWS

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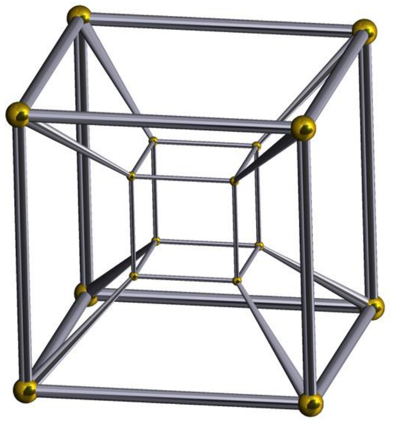
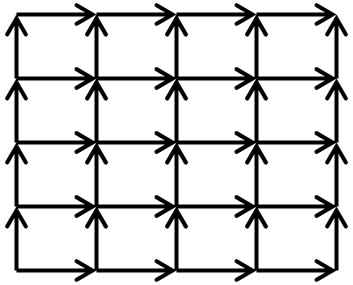
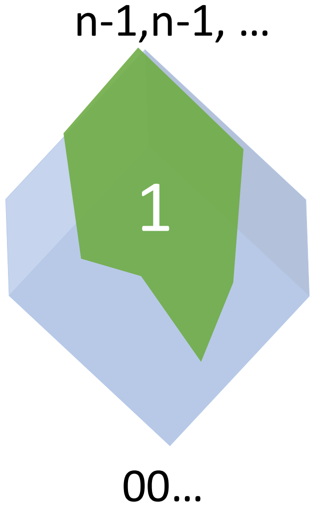
Monotone hypergrid functions

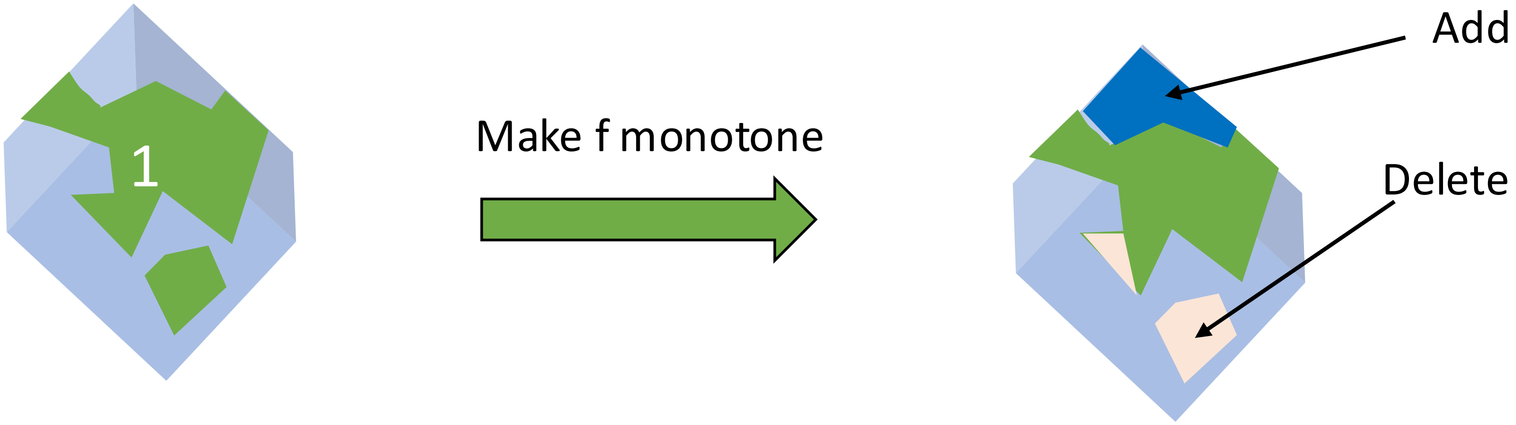
- $f: D \to \{0,1\}$
- **Hypergrid:** $[n]^d$
  (includes line, tesseract, etc.)
- **n = 2, Boolean Hypercube**: $\{0,1\}^d$
- $x \prec y$ iff $x_i \leq y_i$ for all i
- f is monotone: if x ≺ y, f(x) ≤ f(y)
- Equivalently, think of f as indicator for set







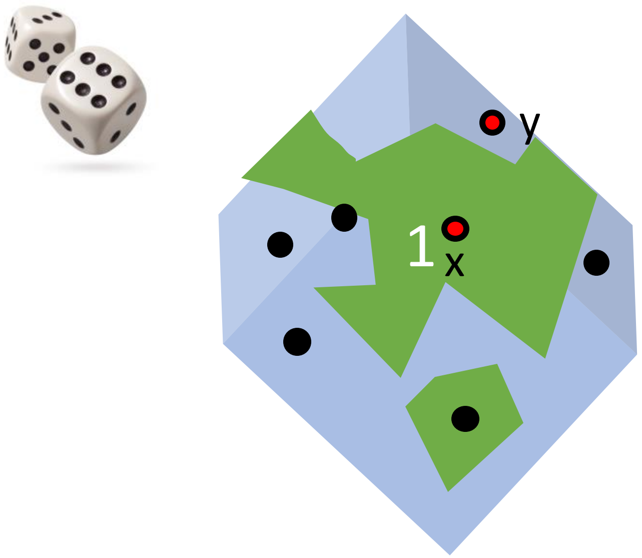
The distance to monotonicity



- Distance to monotonicity = (min changes to make set monotone)/n^d
- ε_f in [0,1)
  - Amen

- $\varepsilon_f = \min_{g \text{ monotone}} |f - g|_0 / n^d$

Monotonicity testing



$x \prec y$

$f(x) = 1$

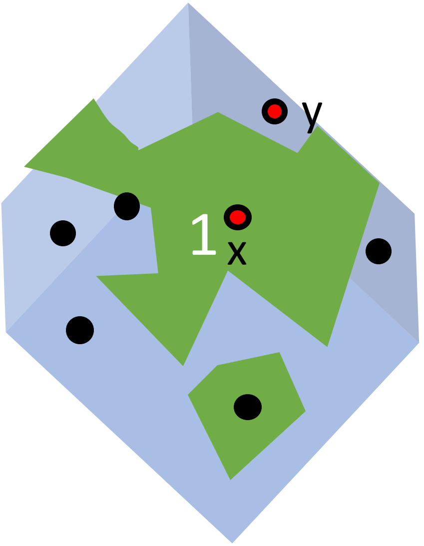
$f(y) = 0$

- [Bshouty-Tamon96, Lange-Vasilyan23] Learning monotone functions needs $> \exp(\sqrt{d})$
- Given ε and query access to f:

  Distinguish monotone ($\varepsilon_f = 0$) vs far from monotone ($\varepsilon_f > \varepsilon$) whp

- One-sided tester: given f such that $\varepsilon_f > \varepsilon$, discover a “violation” whp
- Non-adaptive: all queries are made in advance

Big question



$x \prec y$

$f(x) = 1$

$f(y) = 0$

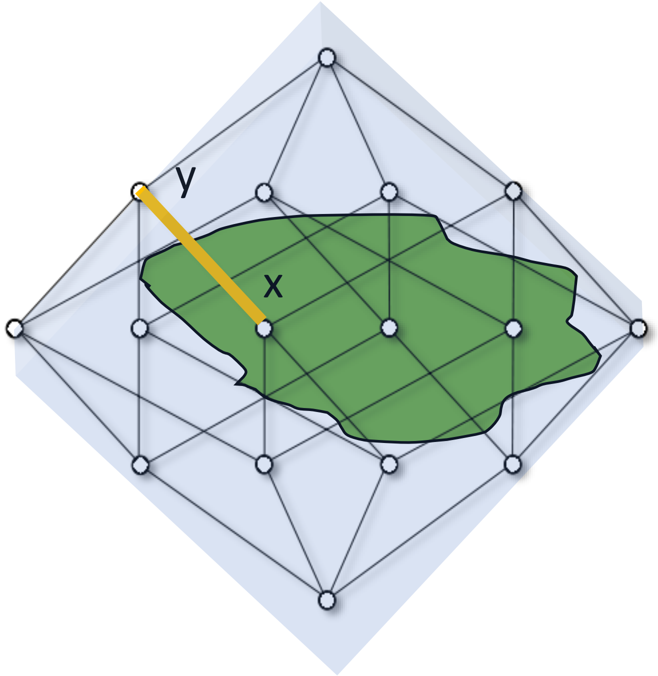
- [Goldreich-Goldwasser-Lehman-Ron-Samorodnitsky 00, Raskhonikova 99, Dodis-Goldreich-Lehman-Raskhodnikova-Ron-Samorodnitsky 99]

What is the (non-adaptive) complexity of monotonicity testing?

- Hypergrid domain, Boolean range

- > 20 papers and two decades of history

The Edge Tester



- [GGLRS00, DGGLRS00]
- Sample an edge of hypercube (x, y) u.a.r
- Query $f(x), f(y)$
- Reject if violation

Theorem: the probability of finding violation is > ε_f /d

So if $\varepsilon_f > \varepsilon$, $O(d/\varepsilon)$ samples suffice to detect violation whp

- [DGGLRS00] $O(d \log^2 d /\varepsilon)$ monotonicity tester for hypergrids

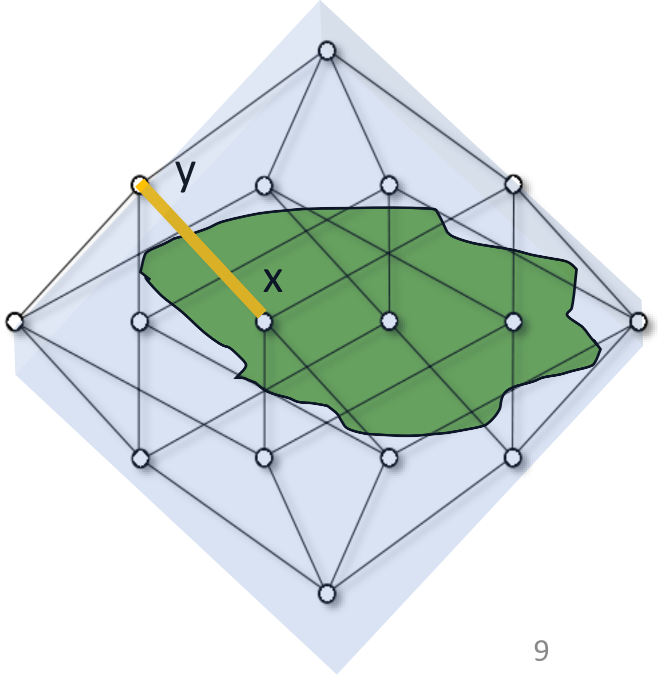
Can we beat d?

- [Blais-Brody-Matulef 12] $\Omega(d)$ lower bound when range is real
- [Chakrabarty-S 13a, Chakrabarty-S 13b]

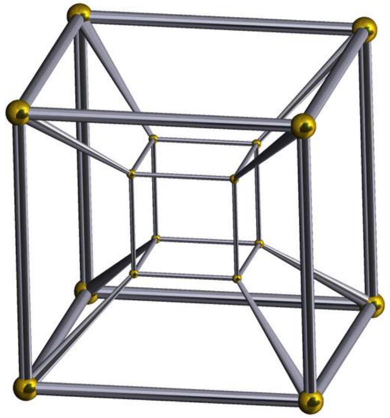
  Hypergrid domain, arbitrary range, the complexity is $\Theta(d \log n / \varepsilon)$
- Non-adaptive, one-sided upper bound. Adaptive, two-sided lower bound
  - Finis. End of story
  - Optimal tester for hypercube is edge tester

- Does Boolean range make complexity lower?
  - [Fischer-Lehman-Newman-Raskhodnikova-Rubinfeld 02]

  $\Omega(\sqrt{d})$ one-sided lower bound
  - Can one get o(d) complexities?



The mystery of root d, for hypercubes



- [Chakrabarty-S 13] $d^{7/8}$ query tester
- [Chen-Servedio-Tan 14] $d^{5/6}$ query tester

- [Khot-Minzer-Safra 15] $\sqrt{d}$ query tester
  - Essentially, same tester as before. All in the analysis

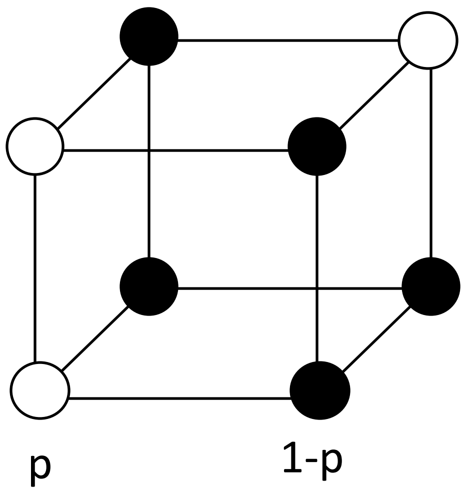
- [Chen-De-Servedio-Tan 15, Chen-Waingarten-Xie 17] $\Omega(\sqrt{d})$ two-sided non-adaptive lower bound

- The theory of Directed Isoperimetric Theorems
  - Seemingly hypercube specific; can they generalize?

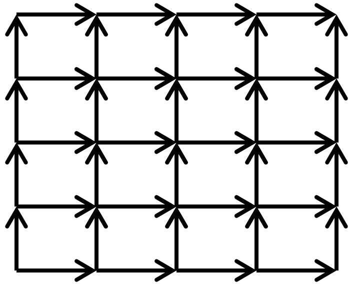
Getting to root d, for hypergrids

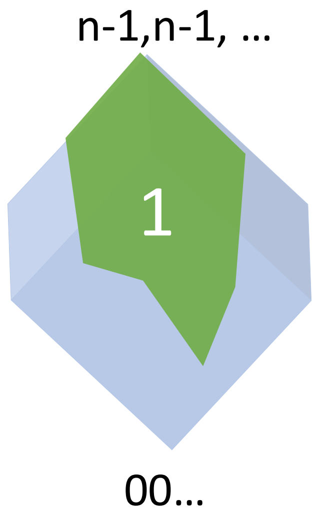
- [DGGLRS00] O(d $\log^2$d /ε) monotonicity tester for hypergrid
- [Berman-Raskhodnikova-Yaroslavtsev 17] O(d log d)
- [Black-Chakrabarty-S 18, BCS 20] $d^{5/6}$ query tester
- [Braverman-Khot-Kindler-Minzer 23, BCS 23] $poly(n)\sqrt{d}$ query tester
- [BCS 24] $d^{1/2+o(1)}$ query tester
  - All results have different testers!

The generality of hypergrids



- **Hypergrid:** $[n]^d$
  Results for uniform distribution automatically translate to any product distribution
  - Otherwise uniform distribution on hypercube looks like special case





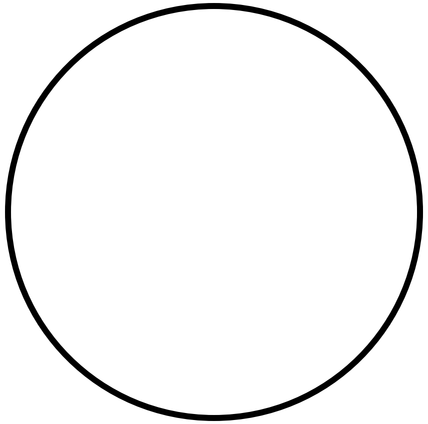
- Can even set n = ∞, so domain is R^d
  - Monotonicity testing for measurable functions over product distributions

Why did the baby grid not get any sugar?

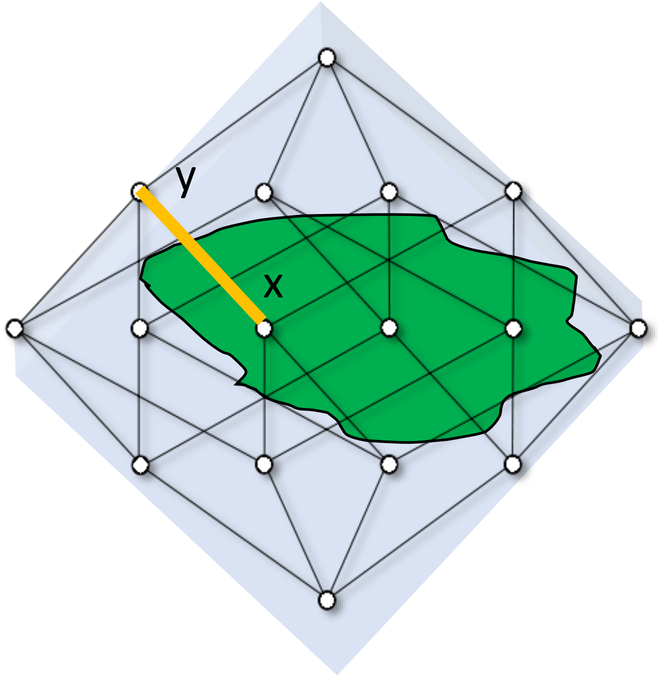
Because it would become a hypergrid.

Directed Isoperimetry

Surfaces, volumes, and why o(d) is possible



The Edge Tester



- [GGLRS00, DGLRRS99]
- Sample an edge of hypercube (x, y) u.a.r
- Query $f(x), f(y)$
- Reject if violation

Theorem: the probability of finding violation is $> \varepsilon_f / d$

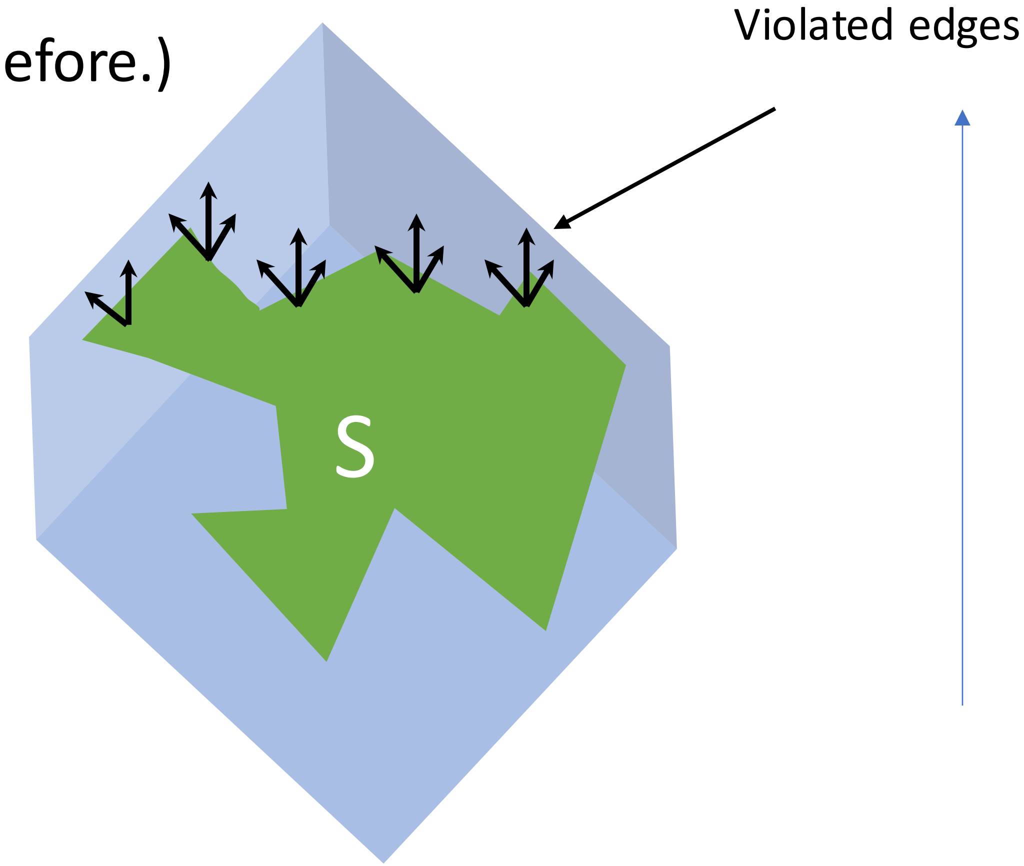
A Geometric Interpretation

Abusing notation, S is the set of 1s. (We still use ε_f as before.)

Theorem: The number of **violated** hypercube edges is = $\Omega(\varepsilon_f \cdot 2^d)$

$$\mathrm{Inf}_f^+ = \frac{\left|\vec{E}(S, S^c)\right|}{2^d}$$

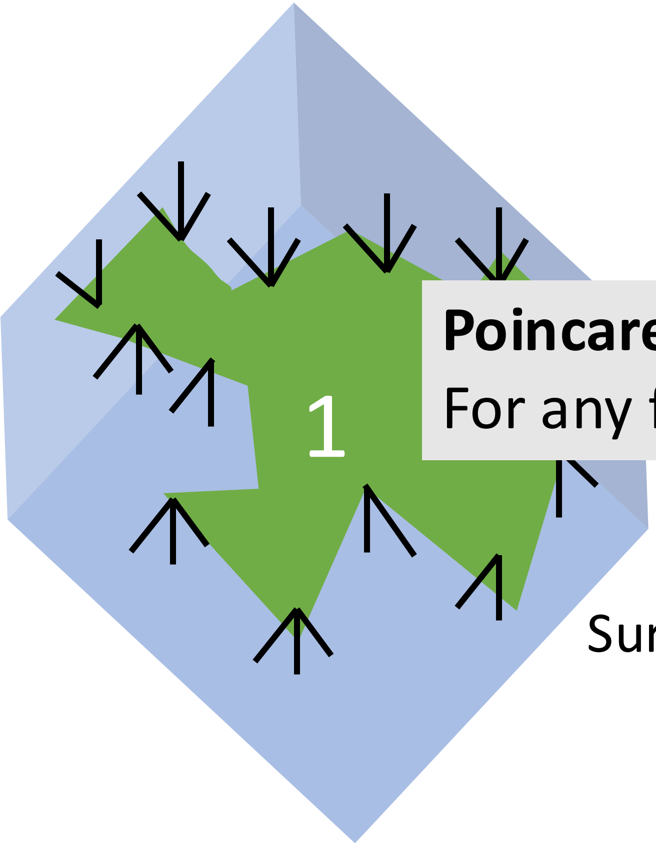
Theorem: $\mathrm{Inf}_f^+ = \Omega(\varepsilon_f)$

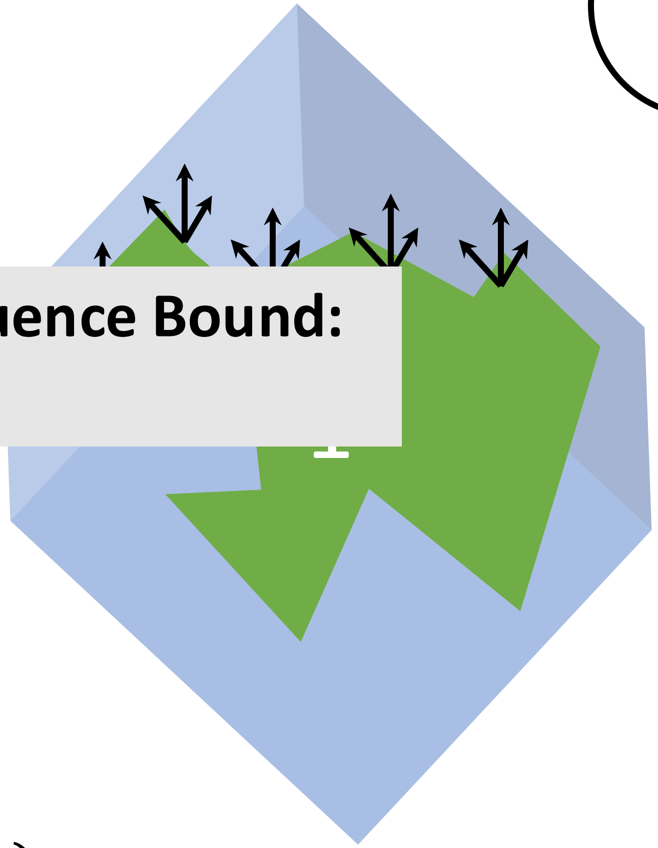


A far from monotone function

Undirected Isoperimetry: Poincare

$$\mathrm{Inf}_f = \frac{|E(S,S^c)|}{2^d}$$





Poincare Inequality/Edge Expansion/Influence Bound:

For any f, $\mathrm{Inf}_f = \Omega(var_f)$

Surface area > (Function of) Volume

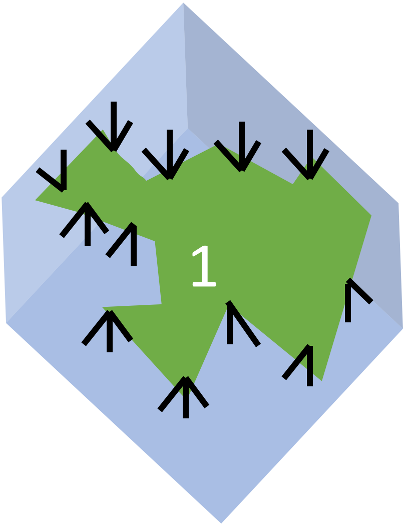
$$var_f = \frac{\#1's}{2^d} \cdot \frac{\#0's}{2^d} = \frac{|S|}{2^d} \cdot (1 - \frac{|S|}{2^d})$$

Influential Edges

GGLRS = Directed Poincare

Undirected Hypercube

- **Poincare:** $\mathrm{Inf}_f = \Omega(var_f)$

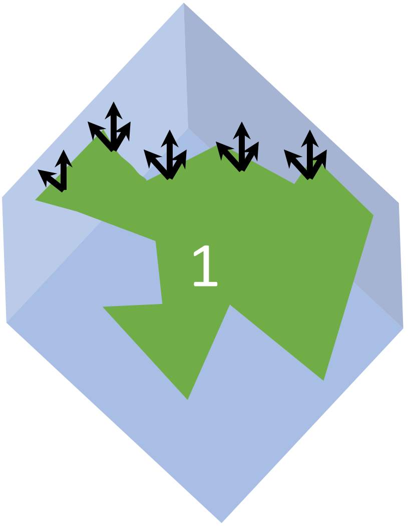


Directed Hypercube

- **GGLRS:** $\mathrm{Inf}_f^+ = \Omega(\varepsilon_f)$

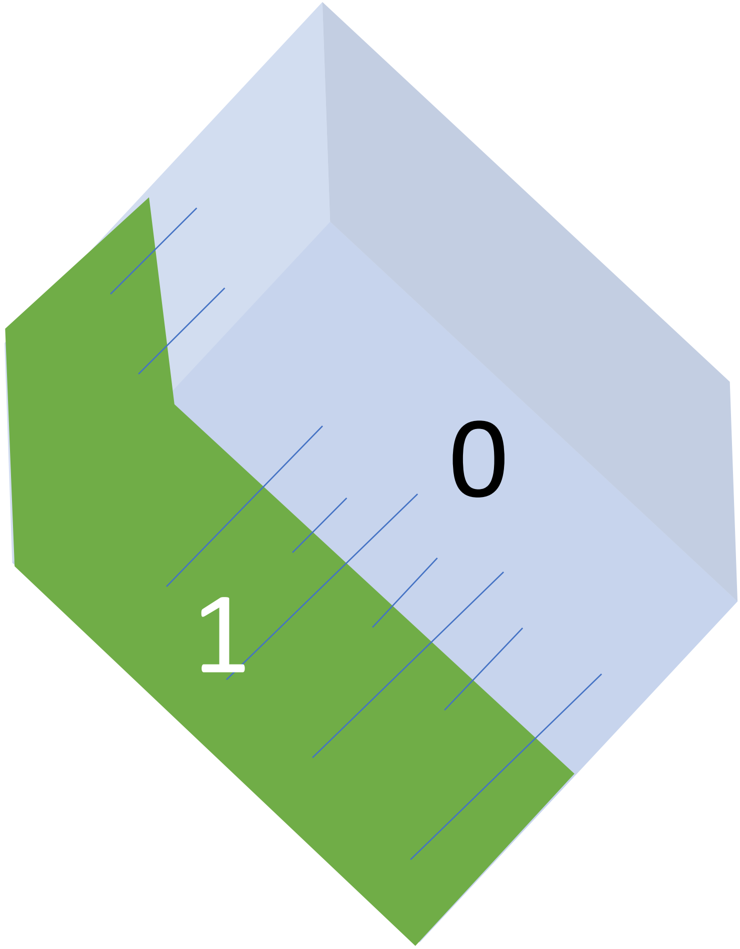
Directed expansion/
surface area

Directed volume…?



(Directed) Poincare is tight

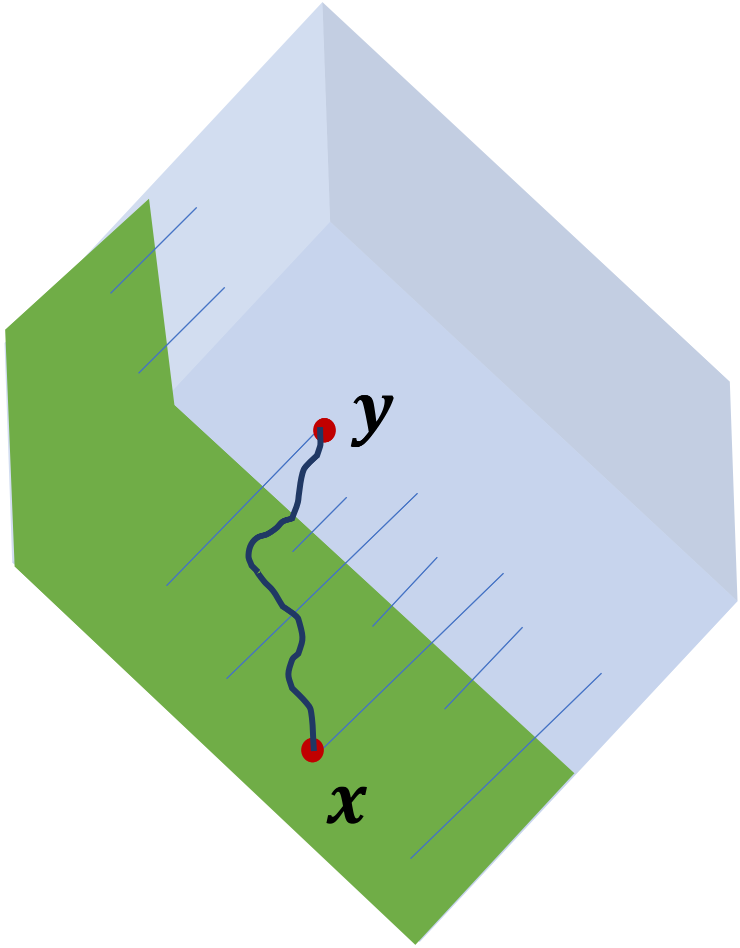
- Distance = $\frac{1}{2}$
- Number of viol edges = 2^{d-1}
    - Total edges = $d \cdot 2^{d-1}$
- Edge tester needs $\Omega(d)$ queries to catch violation



Bypassing the (anti)-Dictator

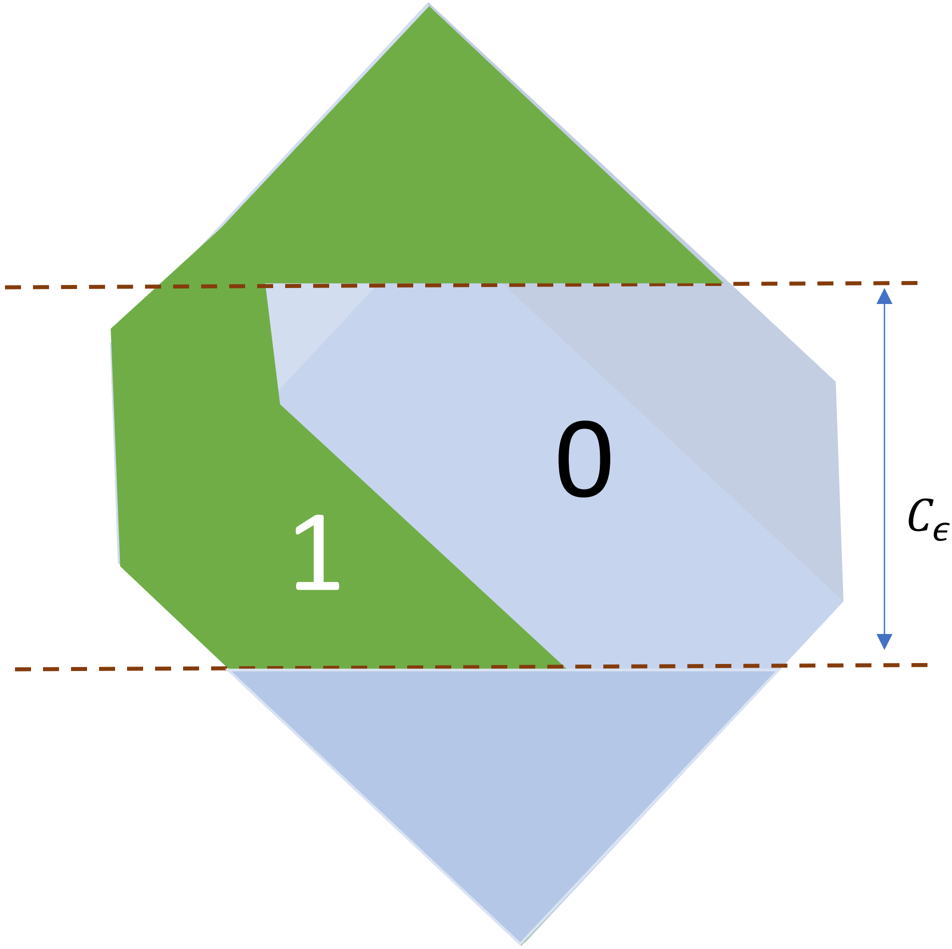
- Sample x
- Walk “up” to get y
- Query $f(x), f(y)$ and Test.

How long should we walk?



Square Root of Dimension

- Sample x
- Walk "up" to get y
- Query $f(x), f(y)$ and Test.



Can't walk more than $\approx \sqrt{d}$

Analysis for the (capped) Anti-Di

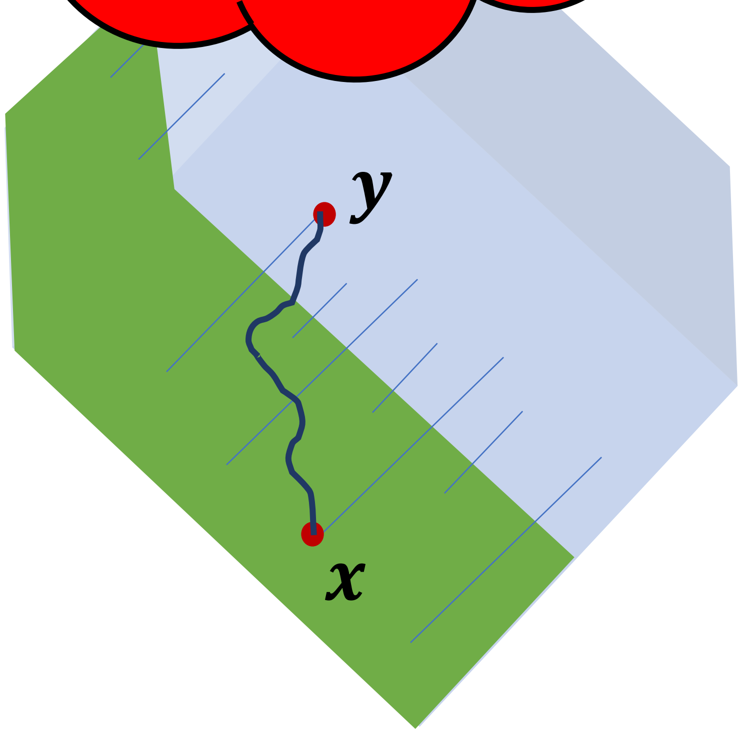
The root to root d is taking a longer root.

- Sample x
- Walk “up” to get y
- Query $f(x), f(y)$ and Test.

- Prob[x is green] $\approx 1/2$
- Prob[crossing] $\approx \sqrt{d} \cdot \frac{1}{d} \approx \frac{1}{\sqrt{d}}$

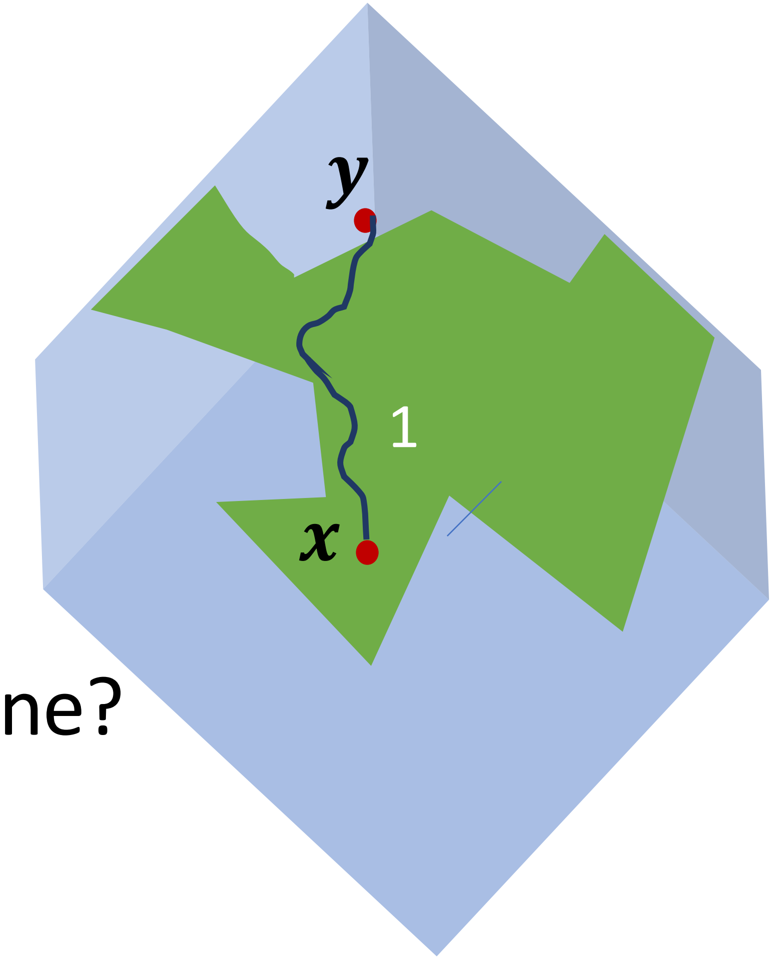
Number of steps

Chance of crossing in each step



How to analyze the "path tester"

Need structural insight
"far from monotone" functions/sets



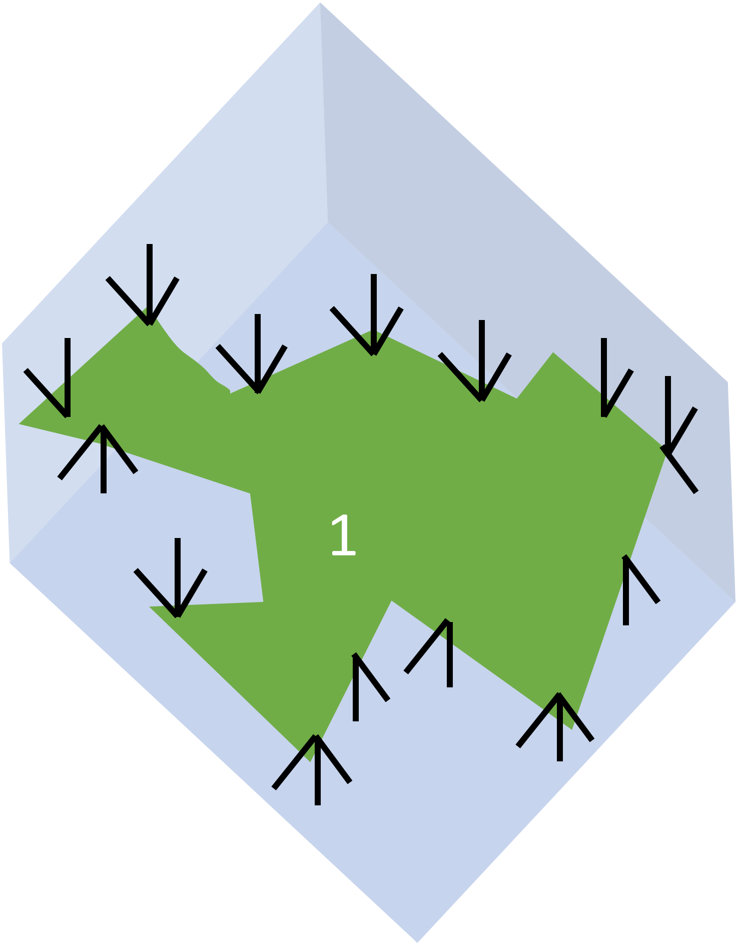
How to "escape" S that is far from monotone?

Directed Isoperimetric Theorems

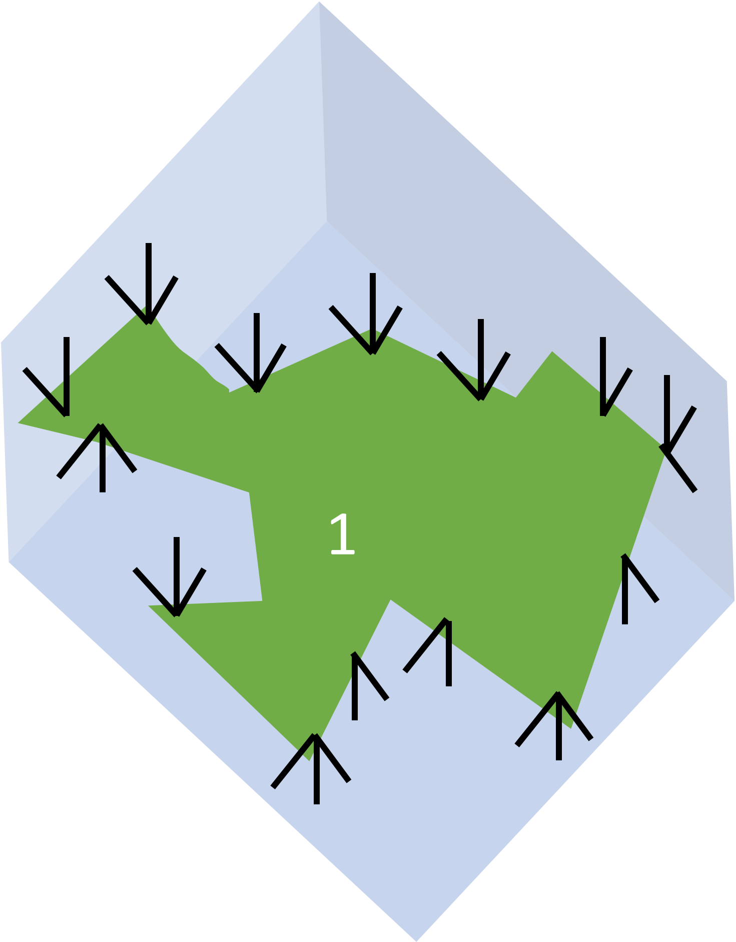
Undirected Hypercube	Directed Hypercube	Query
• Poincare: $\mathrm{Inf}_f = \Omega(var_f)$	• [DGLRRS00, GGLRS00]: $\mathrm{Inf}_f^+ = \Omega(\varepsilon_f)$	d
• [Margulis74]: $\mathrm{Inf}_f \cdot \Gamma_f = \Omega(var_f^2)$	• [Chakrabarty-S 13]: $\mathrm{Inf}_f^+ \cdot \Gamma_f^+ = \Omega(\varepsilon_f^2)$	$d^{5/6}$
• [Talagrand 92]: $\mathrm{E}[\sqrt{\mathrm{Inf}(x)}] = \Omega(var_f)$	• [Khot-Minzer-Safra 15]: $\mathrm{E}[\sqrt{\mathrm{Inf}^+(x)}] = \Omega(\varepsilon_f)$	$\sqrt{d}$

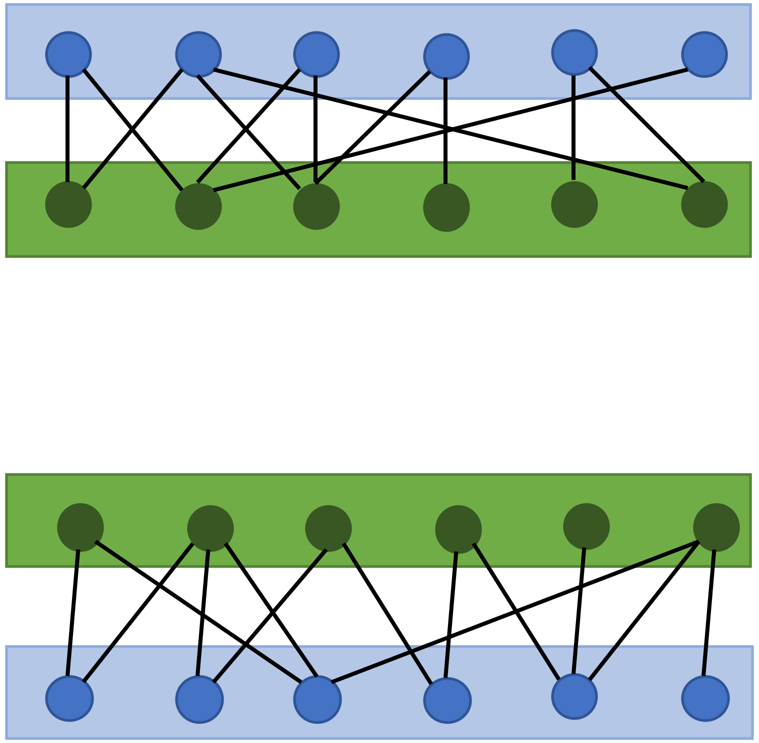
How to measure the boundary?

Boundary as bipartite graph

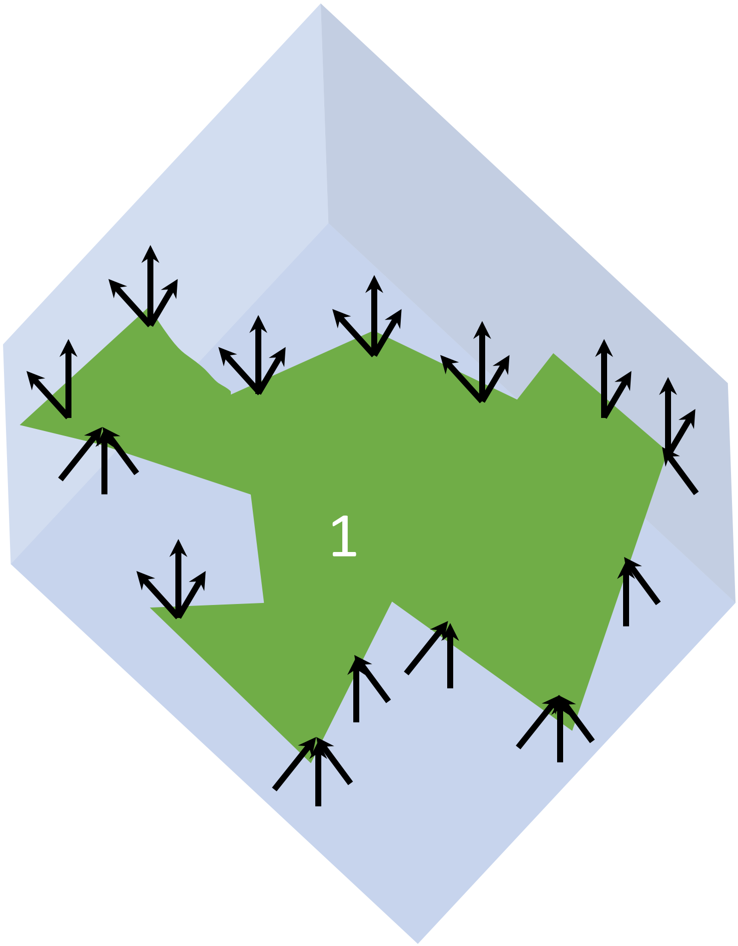


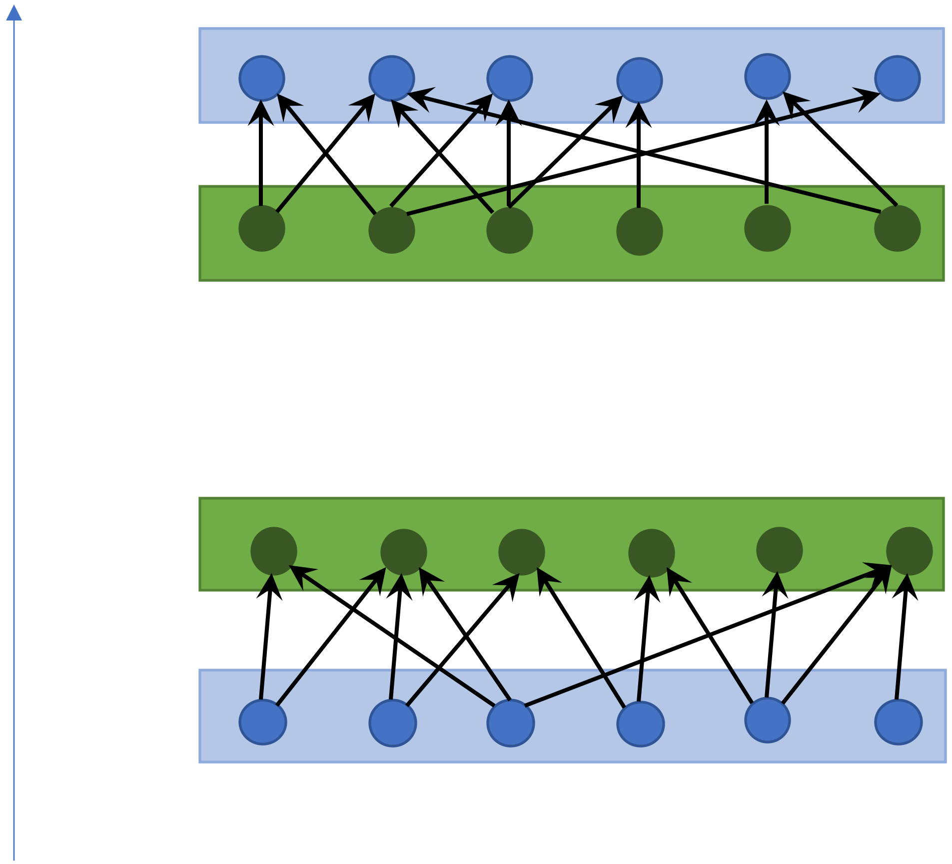
Boundary as bipartite graph



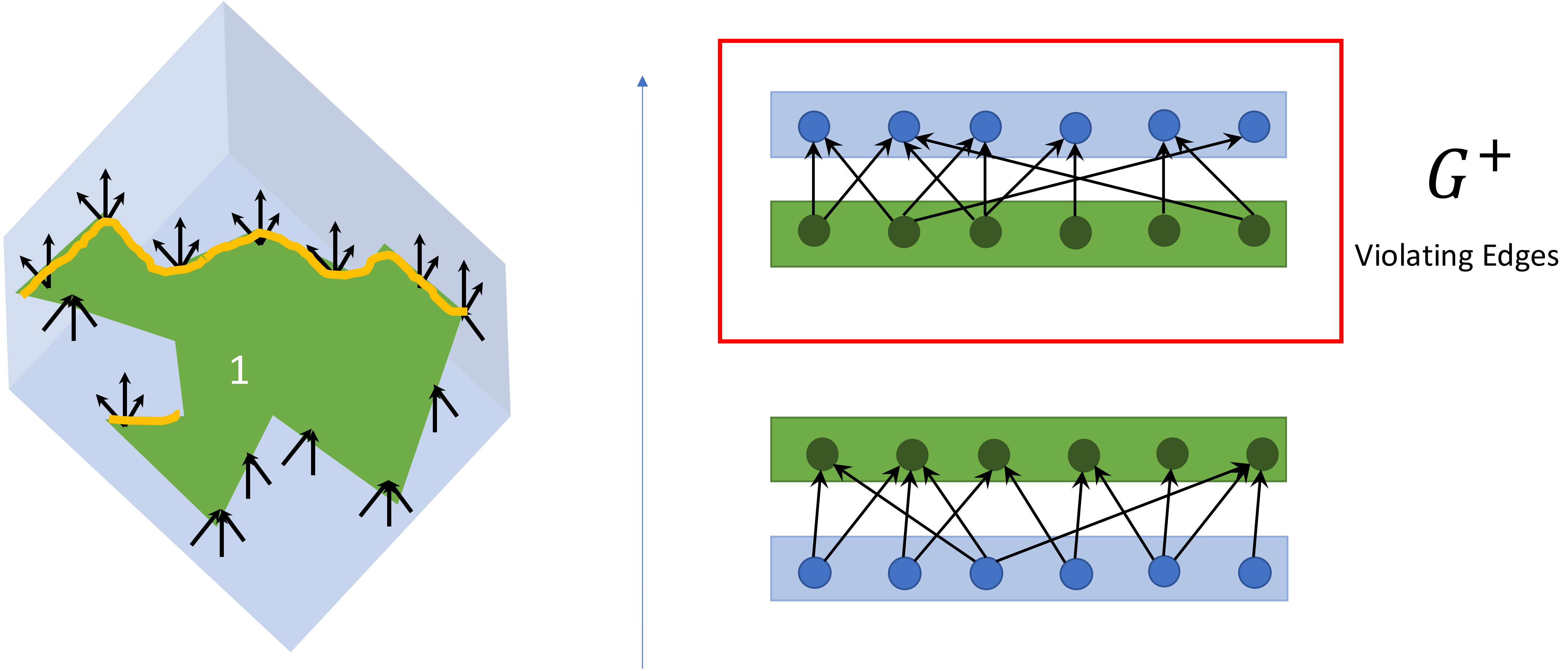


Directed boundary

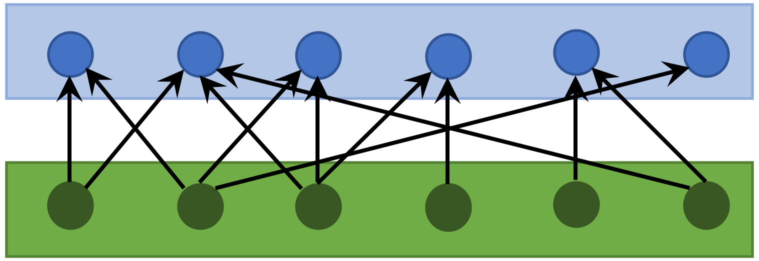




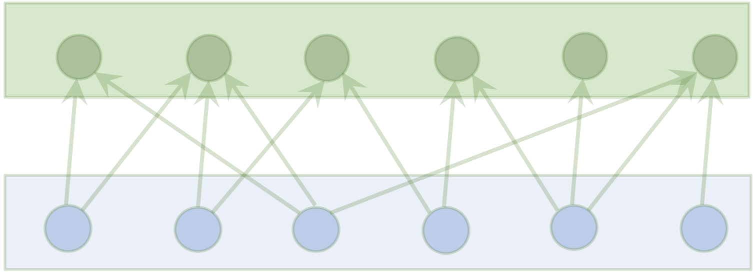
Directed boundary



Directed Influence



$\mathrm{Inf}^+(x) =$ out-degree of x in <u>directed</u> G^+

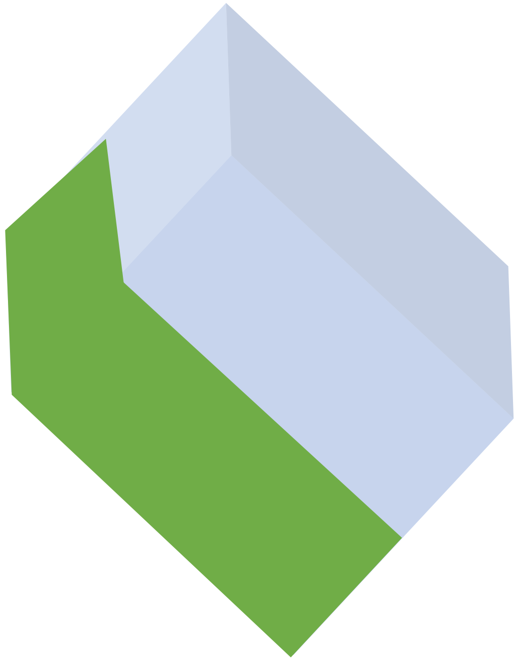


$$\mathrm{Exp}_{x:\mathbf{1}}[\mathrm{Inf}^+(x)] = \frac{E^+(S, S^c)}{2^{\mathrm{d}}} = I_f{}^+$$

$$\mathrm{Exp}_{x:\mathbf{1}}[\mathrm{Inf}(x)] = I_f$$

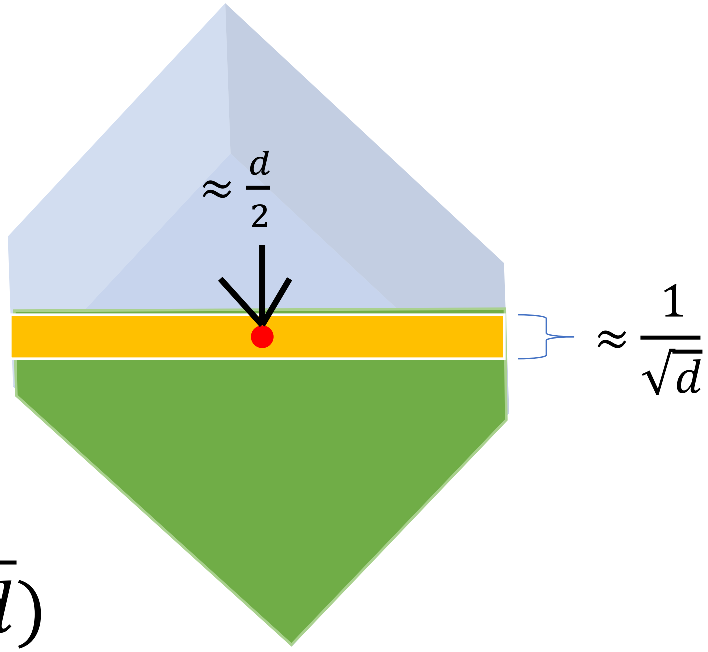
A careful look into influence

Poincare: $I_f := \mathrm{Exp}_{x:\mathbf{1}}[\mathrm{Inf}(x)] = \Omega(var_f)$



$I_f = \Theta(1)$

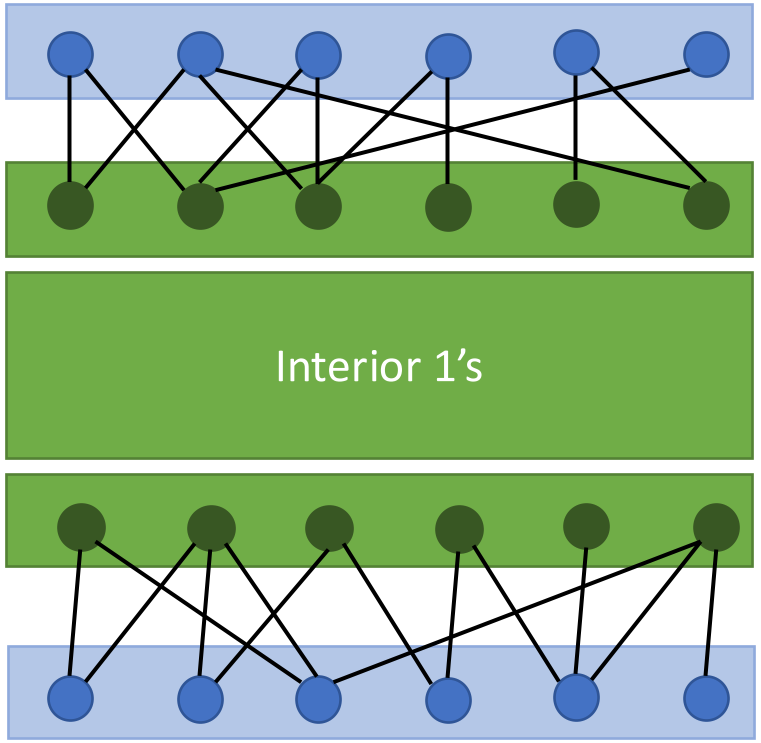
anti-dictator



$I_f = \Theta(\sqrt{d})$

anti-majority

Consider the Vertex Boundary

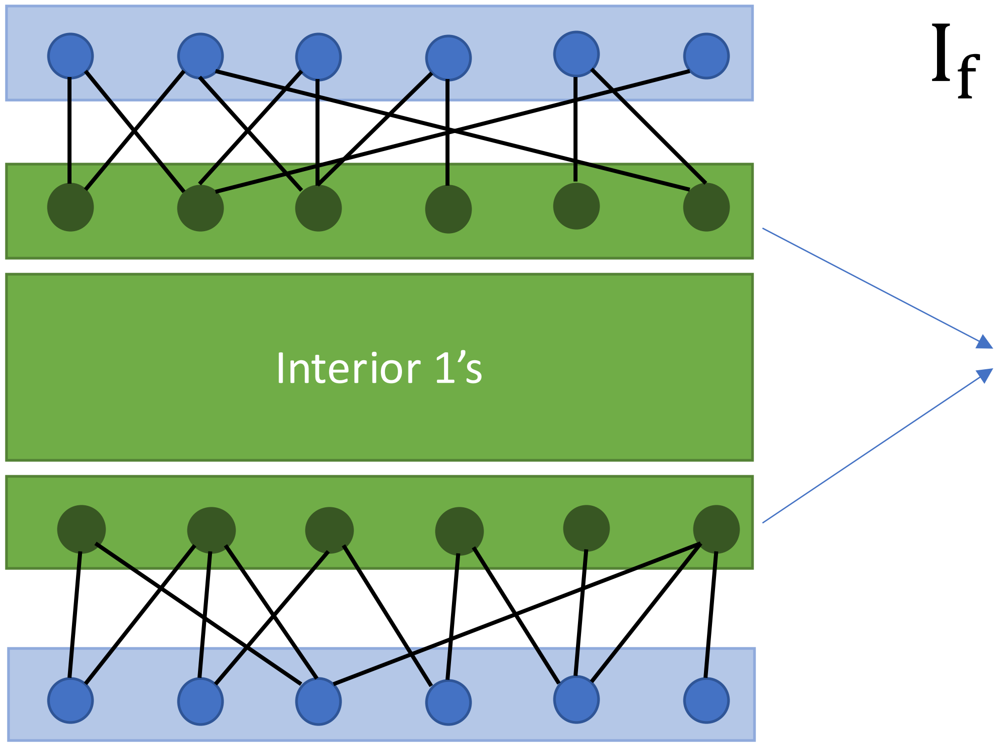


$$\mathrm{I_f} \;\coloneqq\; \mathrm{Exp}_{x:\mathbf{1}}[\mathrm{Inf}(x)]$$

“Edge boundary”

“Vertex Boundary”

Consider the Vertex Boundary



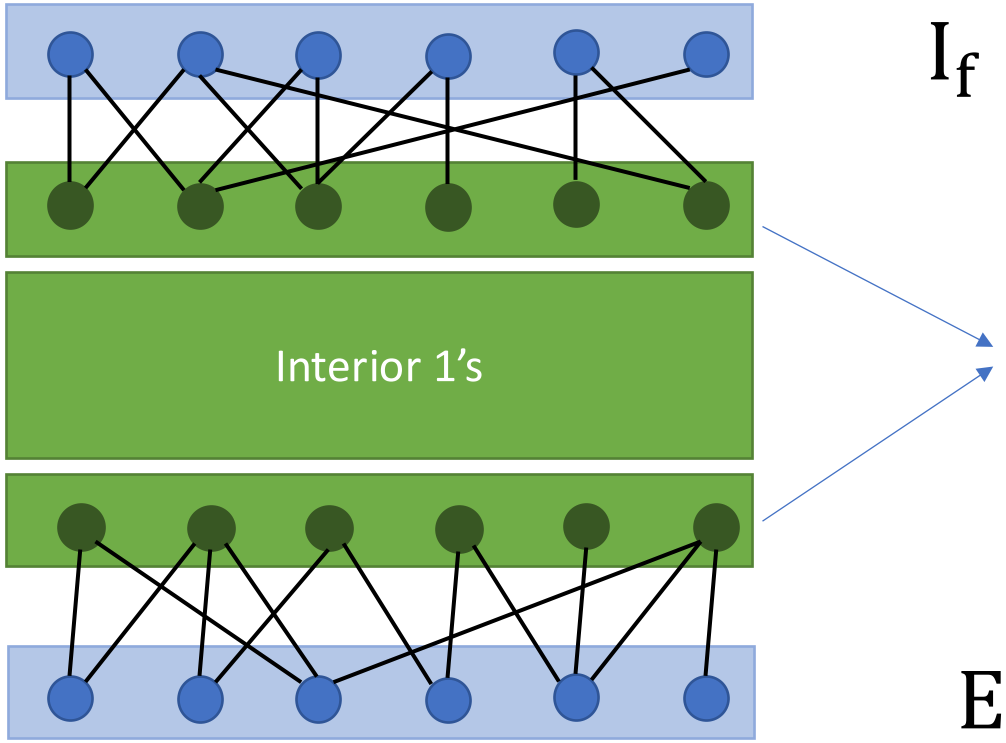
$$\mathrm{I_f} \; := \; \mathrm{Exp}_{x:\mathbf{1}}[\mathrm{Inf}(x)]$$

"Edge boundary"

"Vertex Boundary"

$\frac{1}{2^d} \cdot$ $\{x$: $\mathbf{1}$ such that it has at least one influential edge incident on it$\}$

Consider the Vertex Boundary



$$\mathrm{I_f} \;\coloneqq\; \mathrm{Exp}_{x:\mathbf{1}}[\mathrm{Inf}(x)]$$

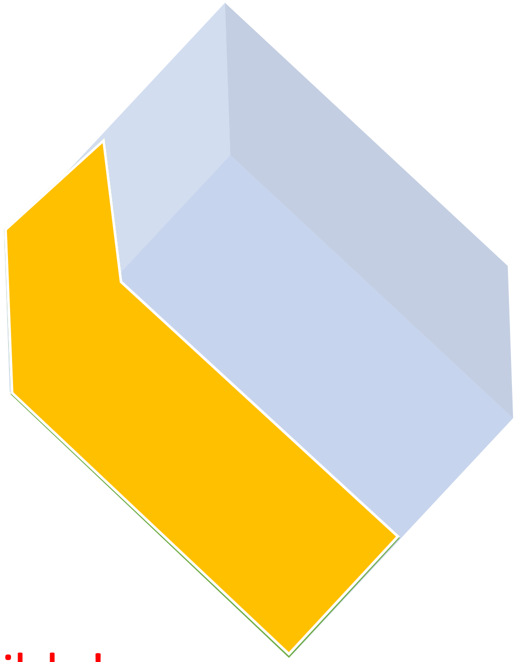
"Edge boundary"

"Vertex Boundary"

$\frac{1}{2^d} \cdot$ $\{x$: **1** such that it has at least one influential edge incident on it$\}$

$$\mathrm{Exp}_{x:\mathbf{1}}[\mathbb{I}\{\mathrm{Inf}(x) > 0\}]$$

Vertex Boundaries



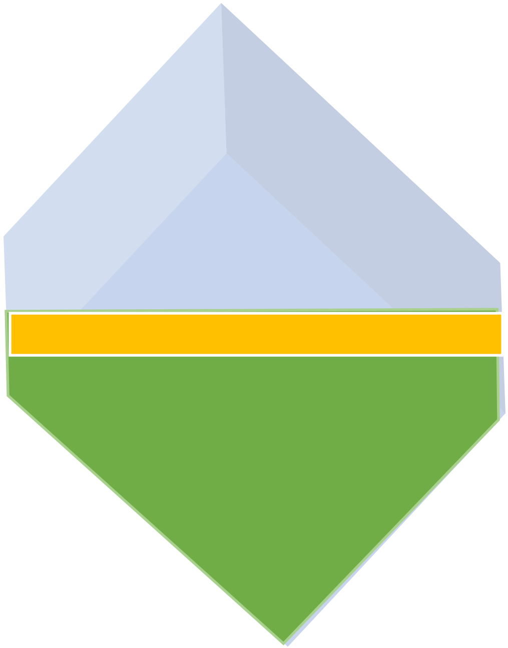
Lowest possible!
(By Poincare)

anti-dictator

Small Edge boundary $= \Theta(1)$

Every vertex in the boundary

Large Vertex boundary $= \ \Theta(1)$



anti-majority

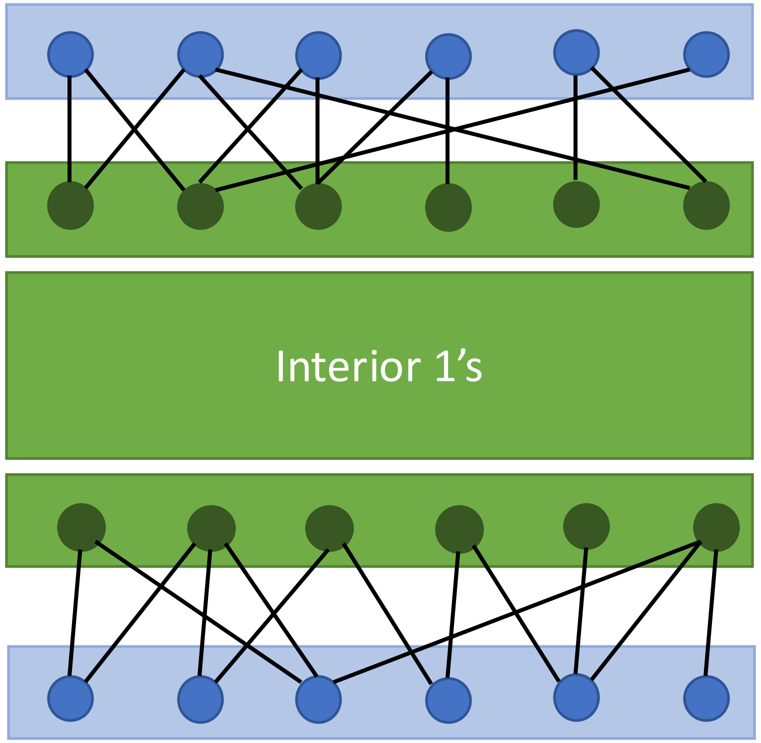
Large Edge boundary $= \ \Theta(\sqrt{d})$

Only $1/\sqrt{d}$ fraction in boundary

Lowest possible!
(By Harper)

Small Vertex boundary $= \Theta(1/\sqrt{d})$

Margulis



$$\mathrm{I}_f \;\coloneqq\; \mathrm{Exp}_{x:1}[\mathrm{Inf}(x)]$$ *“Edge boundary”*

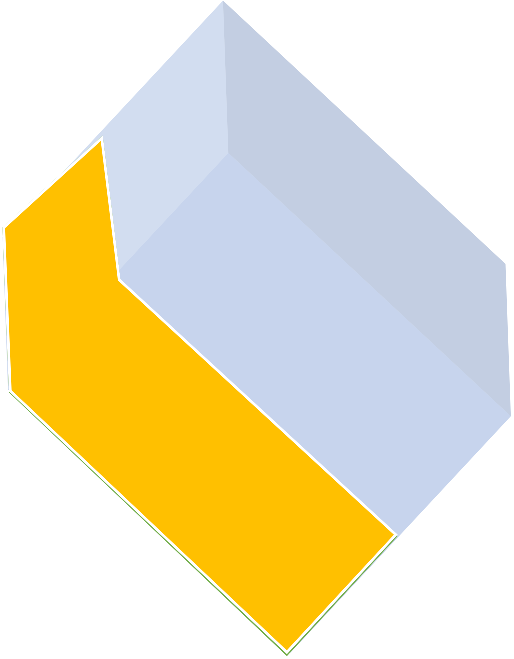
$$\Gamma_f \;\coloneqq\; \mathrm{Exp}_{x:1}[\mathbb{I}\{\mathrm{Inf}(x) > 0\}]$$ *“Vertex Boundary”*

Margulis 1974:

$$I_f \cdot \; \Gamma_f \; = \Omega(var_f^2)$$

Both cannot be simultaneously small!

Vertex Boundaries

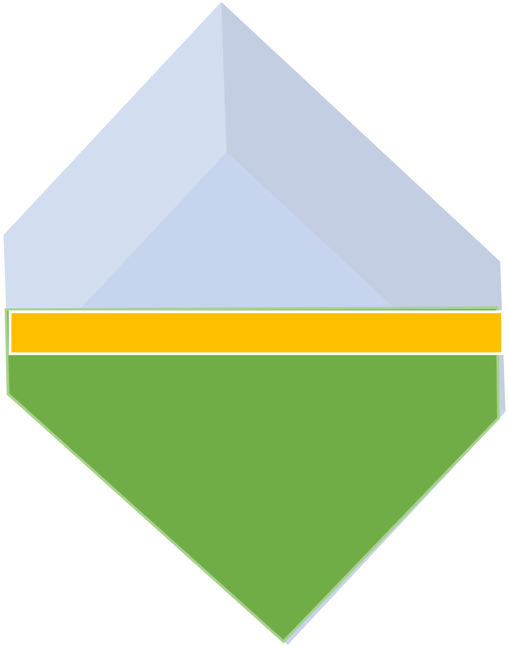


anti-dictator

Small $\mathrm{Exp}_{x:\mathbf{1}}[\mathrm{Inf}(x)] = 1$

Every vertex in the boundary

Large $\mathrm{Exp}_{x:\mathbf{1}}[\mathbb{I}\{\mathrm{Inf}(x) > 0\}] = 1$



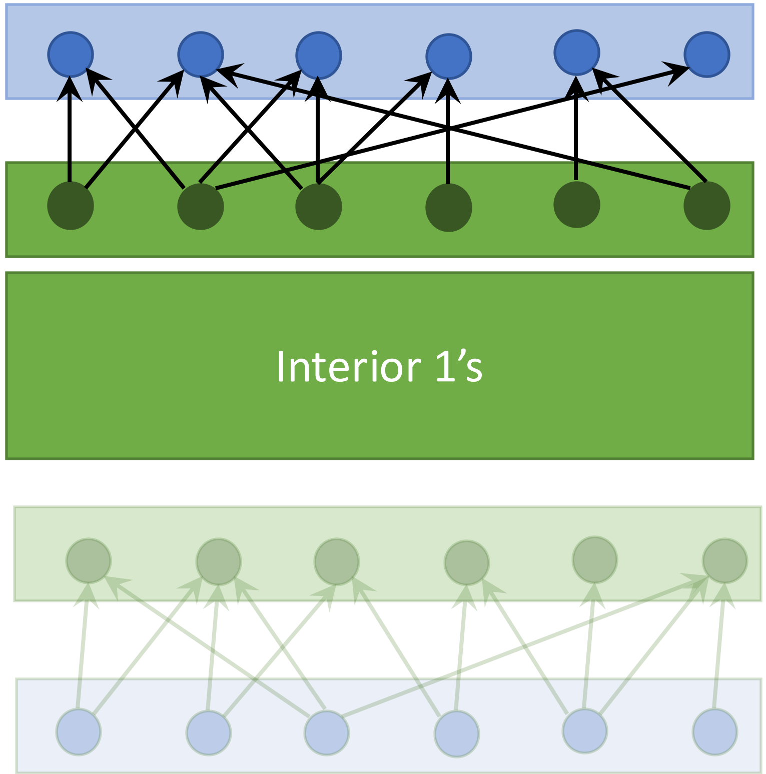
anti-majority

Large $\mathrm{Exp}_{x:\mathbf{1}}[\mathrm{Inf}(x)] = \Theta(\sqrt{d})$

Only $1/\sqrt{d}$ fraction in boundary

Small $\mathrm{Exp}_{x:\mathbf{1}}[\mathbb{I}\{\mathrm{Inf}(x) > 0\}] = \Theta(1/\sqrt{d})$

Directed Margulis



$$\mathrm{I}_{\mathrm{f}}^{+} \;:=\; \mathrm{Exp}_{x:\mathbf{1}}[\mathrm{Inf}^{+}(x)]$$

"Directed Edge boundary"

$$\Gamma_{f}^{+} \;:=\; \mathrm{Exp}_{x:\mathbf{1}}[\mathbb{I}\{\mathrm{Inf}^{+}(x) > 0\}]$$

"Dir. Vrtx. boundary"

[Chakrabarty-S 13]:

$$I_{f}^{+} \cdot \; \Gamma_{f}^{+} \; = \Omega(\varepsilon_{f}^{2})$$

Analysis f the Anti-Dictator

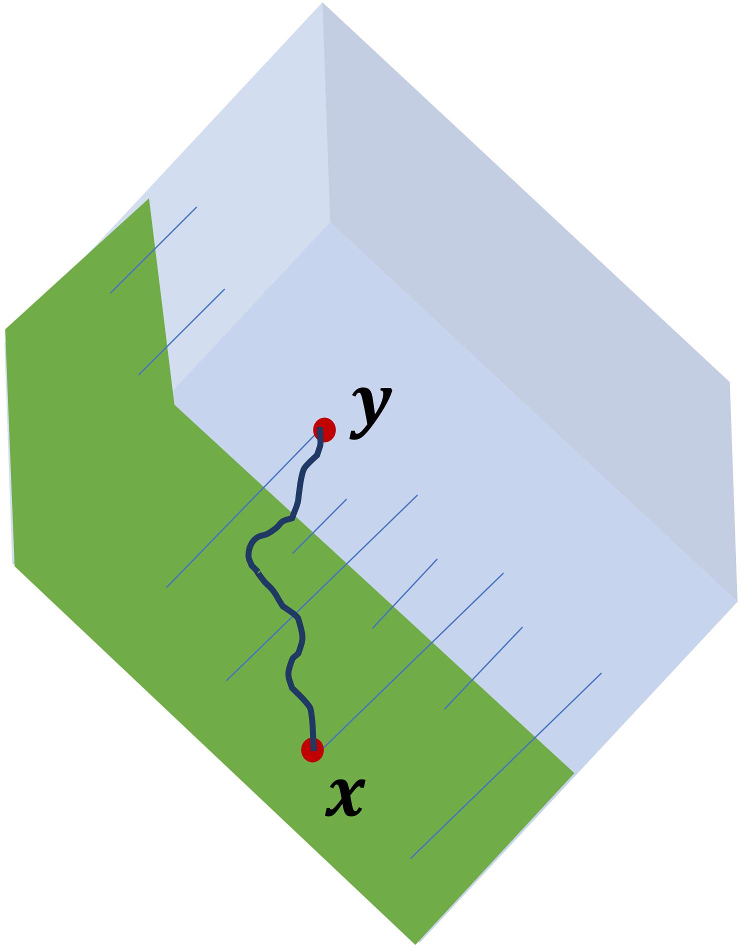
By directed Margulis, if $I_f^+ = O(1)$, constant fraction of vertices on directed boundary.

Analysis like this should work?

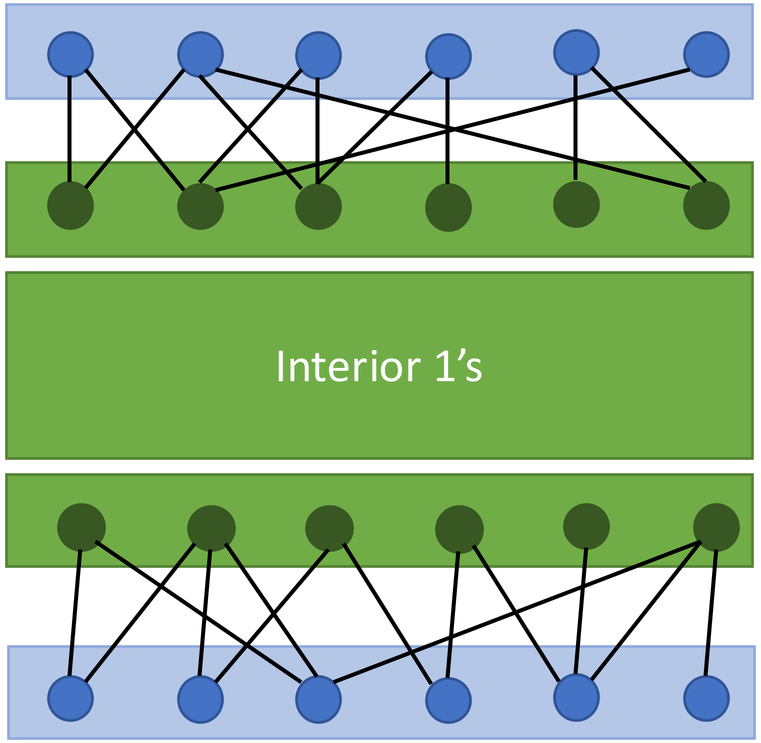
- Prob[x is green] ≈ 1/2
- Prob[crossing] $\approx \sqrt{d} \cdot \frac{1}{d} \approx \frac{1}{\sqrt{d}}$

Number of steps

Chance of crossing in each step



The Talagrand theorem



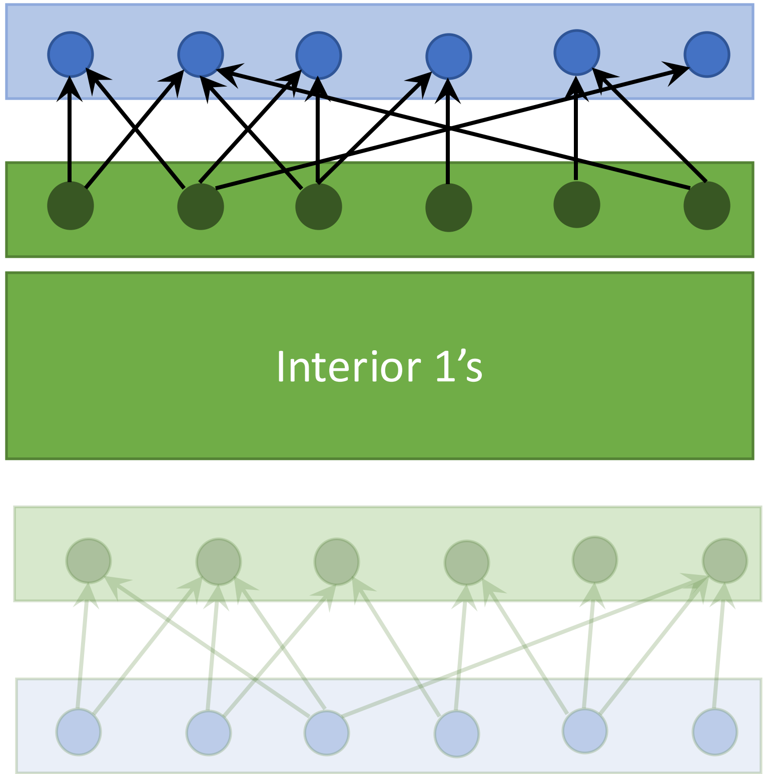
Talagrand 1993: *"Notion of Surface Area"*

$$\mathrm{Tal}_f \;\coloneqq\; \mathrm{Exp}_{x:\mathbf{1}}[\sqrt{\mathrm{Inf}(x)}]$$

$$\mathrm{Tal}_f = \Omega(var_f)$$

Implies Margulis (by Cauchy-Schwartz), which also implies Poincare

Khot-Minzer-Safra



$$\mathrm{Tal}_f^+ \;:=\; \mathrm{Exp}_{x:1}[\sqrt{\mathrm{Inf}^+(x)}]$$

$$= \Omega(\varepsilon_f)$$

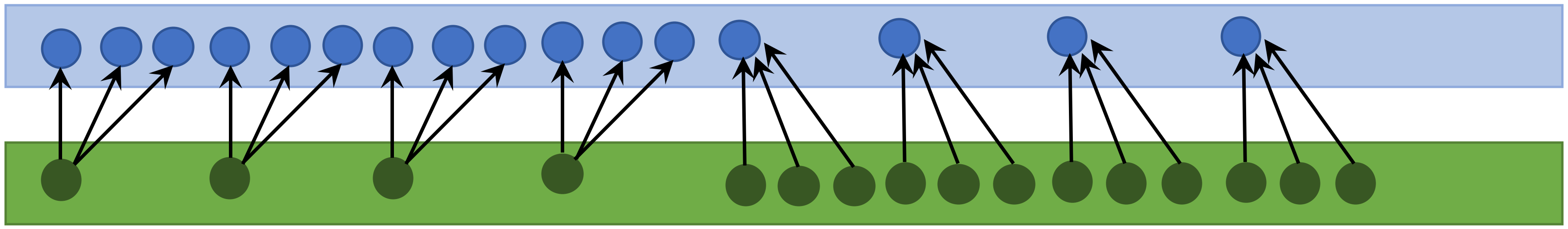
Actually, for any edge bicoloring ψ,

$$\mathrm{Tal}_f^+ \;:= \min_\psi \mathrm{E}_x[\sqrt{\mathrm{Inf}_\psi^+(x)}]$$

KMS lost a log factor, which [Pallavoor-Raskhodnikova-Waingarten 22] removed

Robustness

$$\mathrm{Tal}_f^+ := \mathrm{Exp}_{x:1}\left[\sqrt{\mathrm{Inf}^+(x)}\right] = \Omega(1) \qquad \mathrm{Tal}_f^+ := \mathrm{Exp}_{x:0}\left[\sqrt{\mathrm{Inf}^+(x)}\right] = \Omega(1)$$



2^d vertices

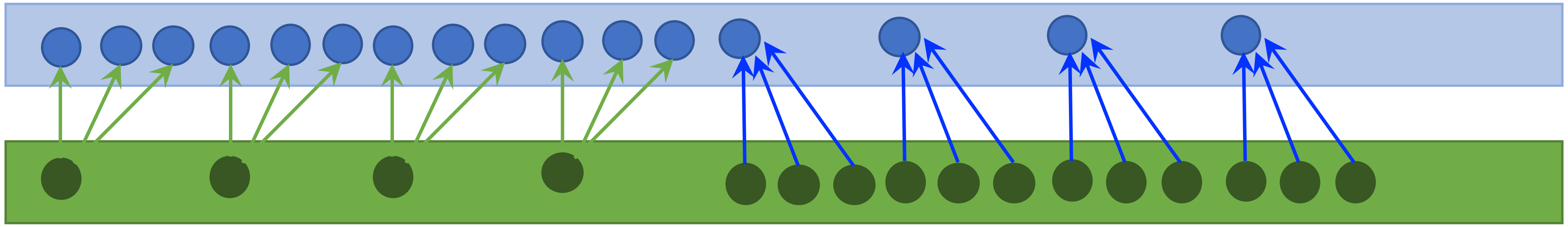
2^d edges

Adversary ψ assigns each edge to 0 or 1, to minimize Talagrand “surface area”

$$\mathrm{Tal}_f^+ := \min_\psi \mathrm{E}_x[\sqrt{\mathrm{Inf}_\psi^+(x)}]$$

Robustness

$\mathrm{Tal}_f^+ \ := \ \mathrm{Exp}_{x:1}\left[\sqrt{\mathrm{Inf}^+(x)}\right] = \ \Omega(1)$ $\quad$ $\mathrm{Tal}_f^+ \ := \ \mathrm{Exp}_{x:0}\left[\sqrt{\mathrm{Inf}^+(x)}\right] = \ \Omega(1)$



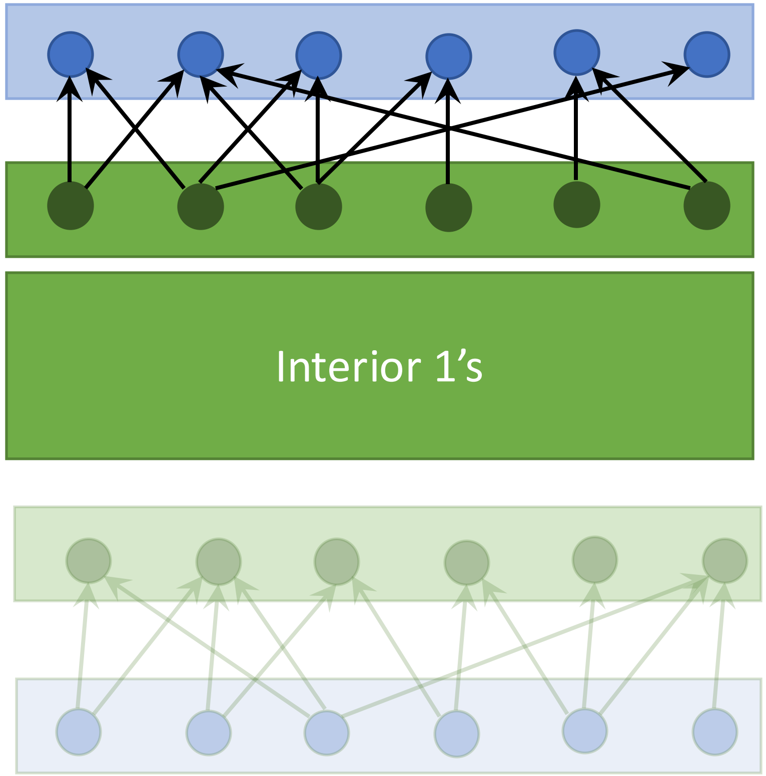
2^d vertices

2^d edges

Adversary ψ assigns each edge to 0 or 1, to minimize Talagrand "surface area"

$$\mathrm{Tal}_f^+ \ := \min_\psi \mathrm{E}_x\left[\sqrt{\mathrm{Inf}_\psi^+(x)}\right] = \frac{2^d}{d}\cdot\sqrt{d} = O(1/\sqrt{d})$$

Khot-Minzer-Safra



Actually, for any edge bicoloring ψ,

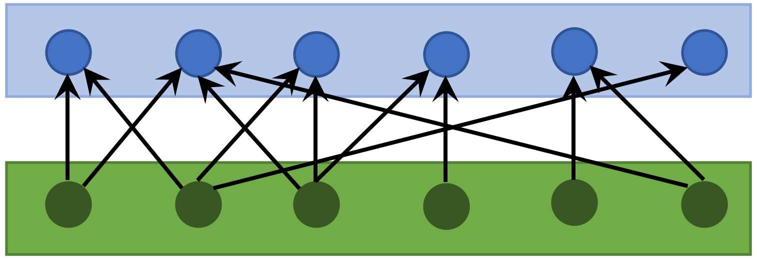
$$\mathrm{Tal}_f^+ := \min_\psi \mathrm{E}_x\left[\sqrt{\mathrm{Inf}_\psi^+(x)}\right] = \Omega(\varepsilon_f)$$

KMS lost a log factor, which [Pallavoor-Raskhodnikova-Waingarten 22] removed

But what does it mean?

$$\mathrm{Tal}_f^+ := \min_\psi \mathrm{E}_x[\sqrt{\mathrm{Inf}_\psi^+(x)]} = \Omega(\varepsilon_f)$$

Assume $\varepsilon_f = \Omega(1)$



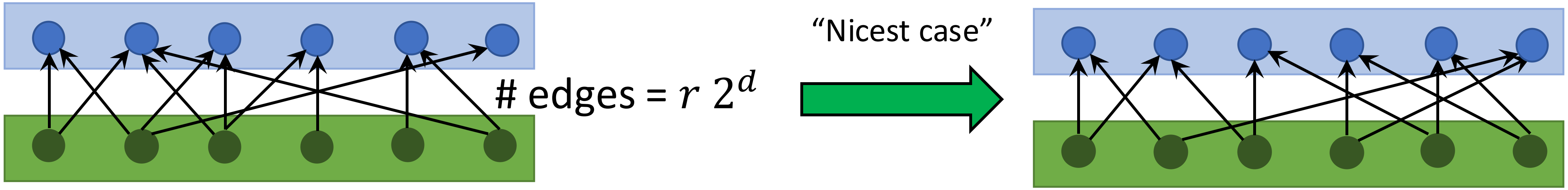
vertices = $r\ 2^d$ $\quad I_f^+ \cdot \Gamma_f^+ = \Omega(1)$ Dir. Margulis

$I_f^+ = \Omega(1)$ Dir. Poincare

vertices = $2^d/r$

For some r > 1, $I_f^+ = r$ and $\Gamma_f^+ = 1/r$

The most regular boundary



edges = $r\ 2^d$

“Nicest case”

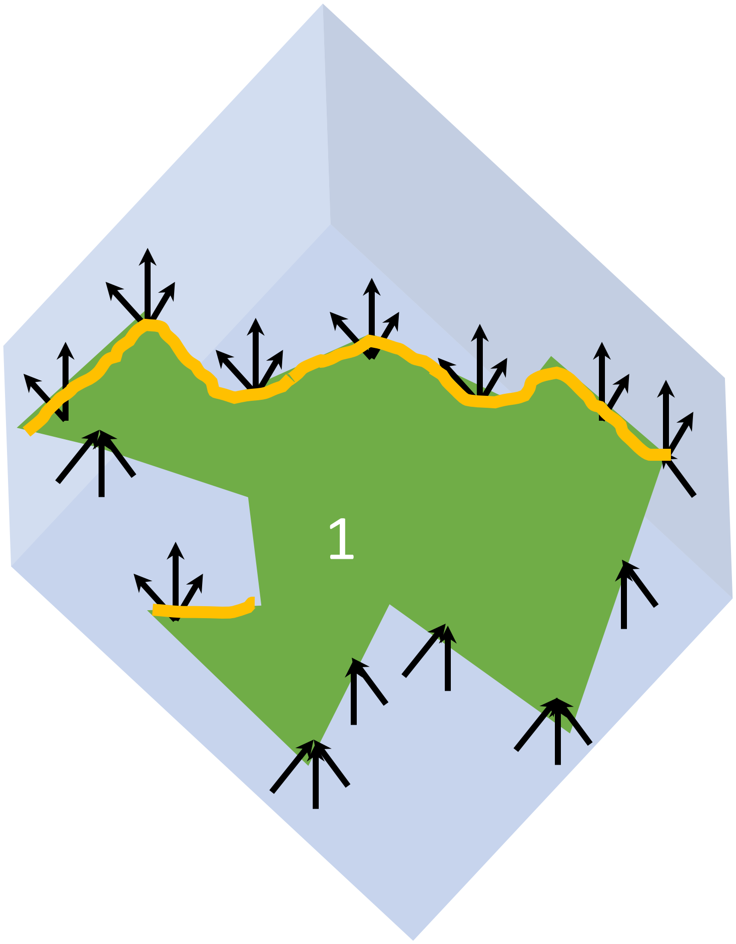
vertices = $2^d/r$

vertices = $2^d/r$

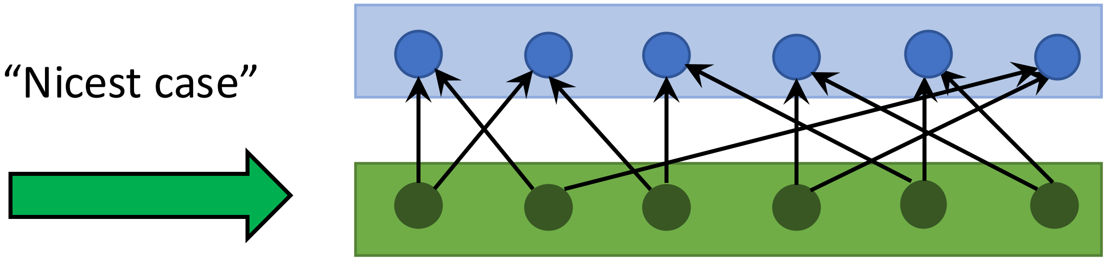
All degrees = r^2

- Robust Talagrand theorem of KMS implies that’s exactly what happens!

Boundaries are always regular!



Assume $\varepsilon_f = \Omega(1)$



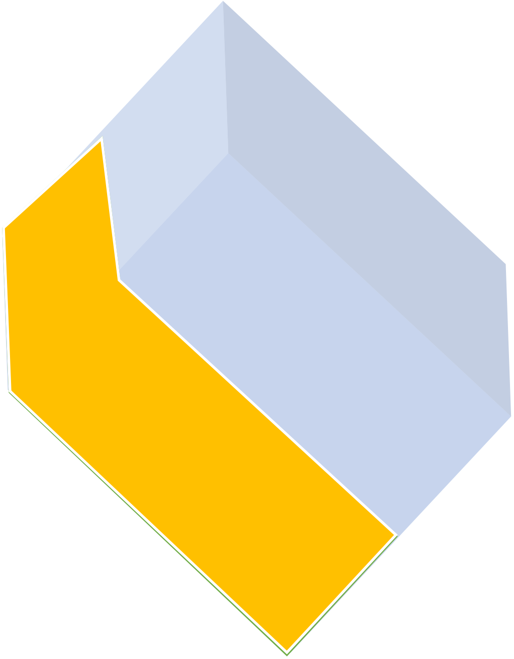
vertices = $2^d/r$

All degrees = r^2

There exists r > 1 such that boundary contains regular bipartite graph with these parameters.

- (Up to $d^{o(1)}$ factors)

Our usual examples

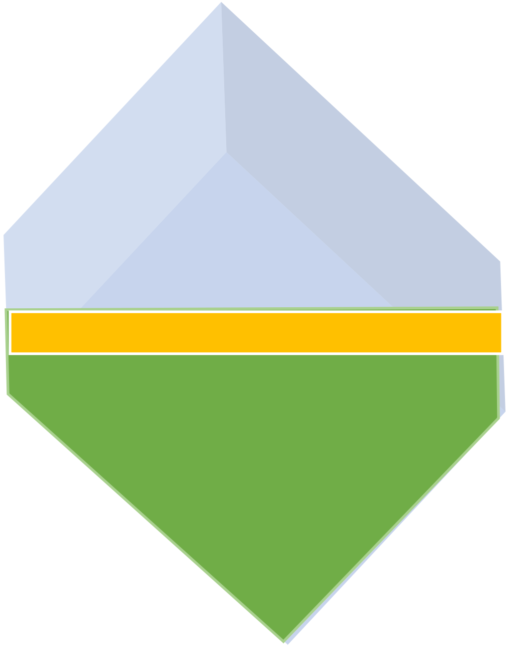


anti-dictator

2^d vertices on boundary,
each with degree 1

Boundary size = $2^d/r$

All degrees = r^2



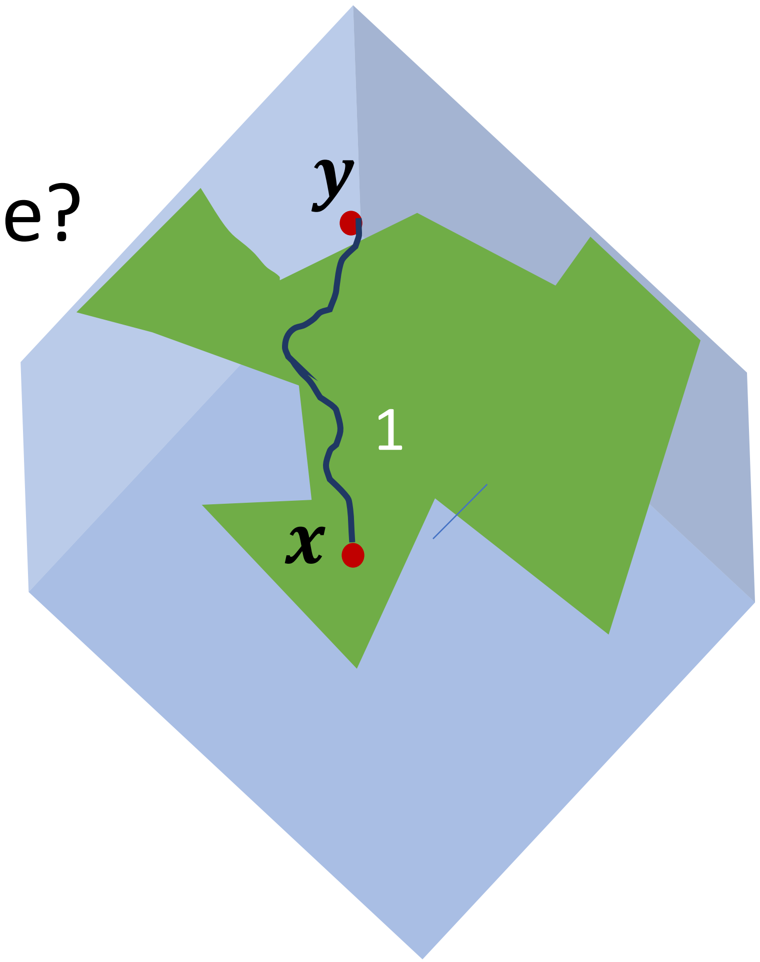
anti-majority

$2^d/\sqrt{d}$ vertices on
boundary,
each with degree d

But what about monotonicity testing?

How to analyze the “path tester”

How to “escape” S that is far from monotone?



Be persistent!

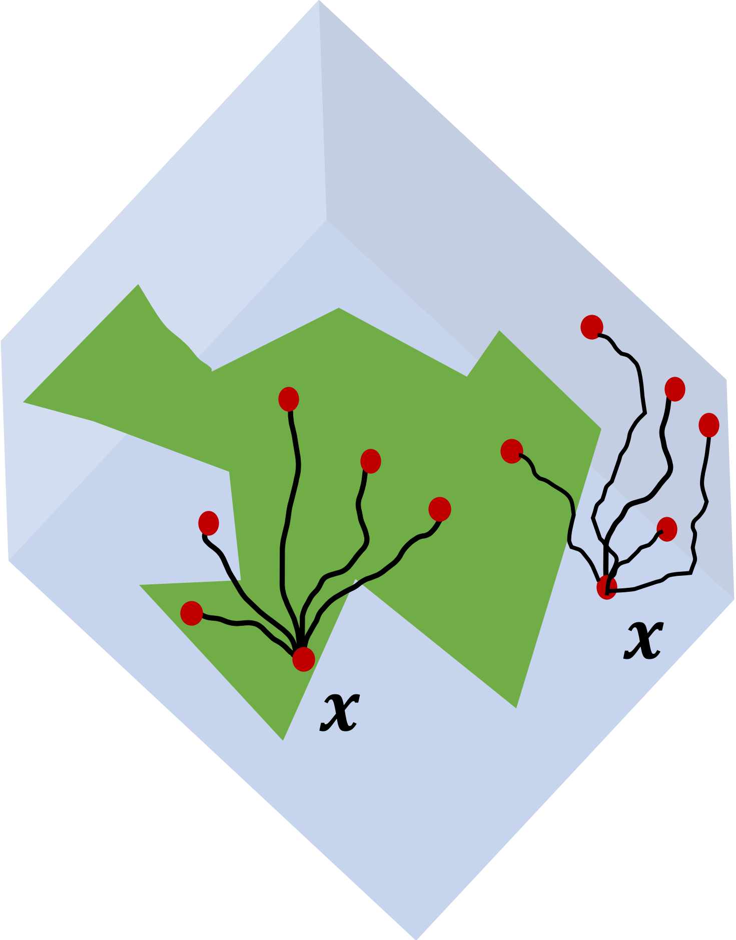
- x is ℓ-persistent, if ℓ-length (directed walk) stays within f(x)-region whp

Pr[single step changes value] = $\frac{I_f}{d}$

Pr[one of ℓ-steps changes value] $\leq \ell \cdot \frac{I_f}{d}$

Fraction of NON ℓ-persistent vertices $= O\left(\ell \cdot \frac{I_f}{d}\right)$

$$\xrightarrow{I_f = O(\sqrt{d})} O\left(\frac{\ell}{\sqrt{d}}\right)$$



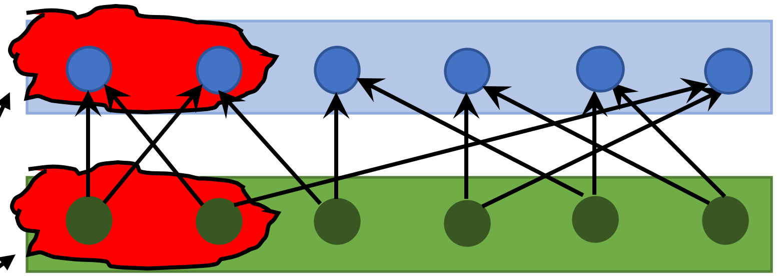
If $I_f \gg \sqrt{d}$ then $I_f^+ \gg \sqrt{d}$

Edge tester itself good

The analysis, in one slide

By (robust) Directed Talagrand, boundary is

Non- ℓ-persistent

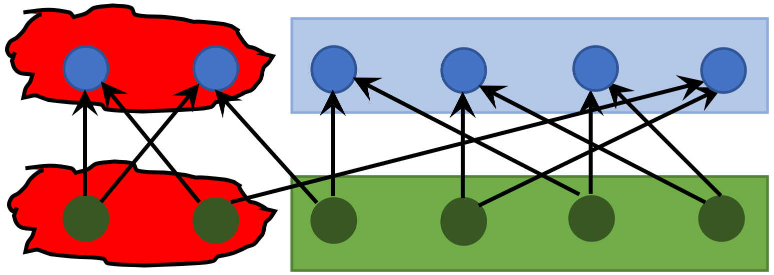


Frac. non-persistent = $O\left(\frac{\ell}{\sqrt{d}}\right)$

Set $\ell \ll \sqrt{d}/r$
so frac. $\ll 1/r$

vertices = $2^d/r$

All degrees = r^2



By regularity of boundary

vertices ≈ $2^d/r$

All degrees ≈ r^2

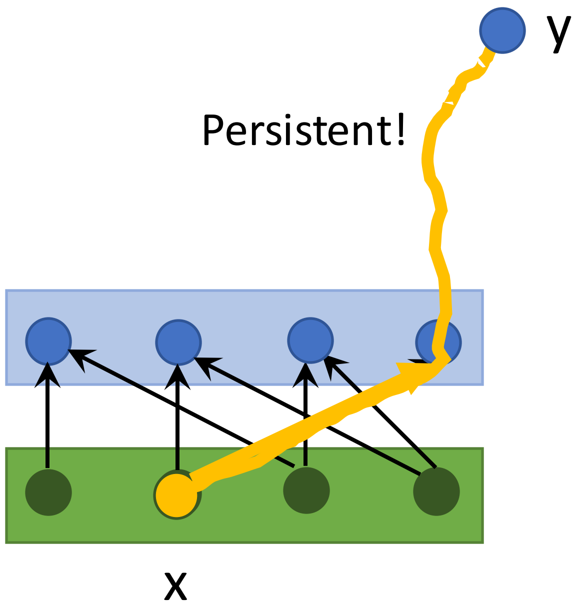
All ℓ-persistent

The analysis, in ~~one~~ two slides

With prob ≈ $1/r$, start with f(x) = 1 in this part

With prob ≈ $r^2/\ell = r/\sqrt{d}$, relevant bit is flipped

When both happen, f(y) = 0



By regularity of boundary

\# vertices ≈ $2^d/r$

All degrees ≈ r^2

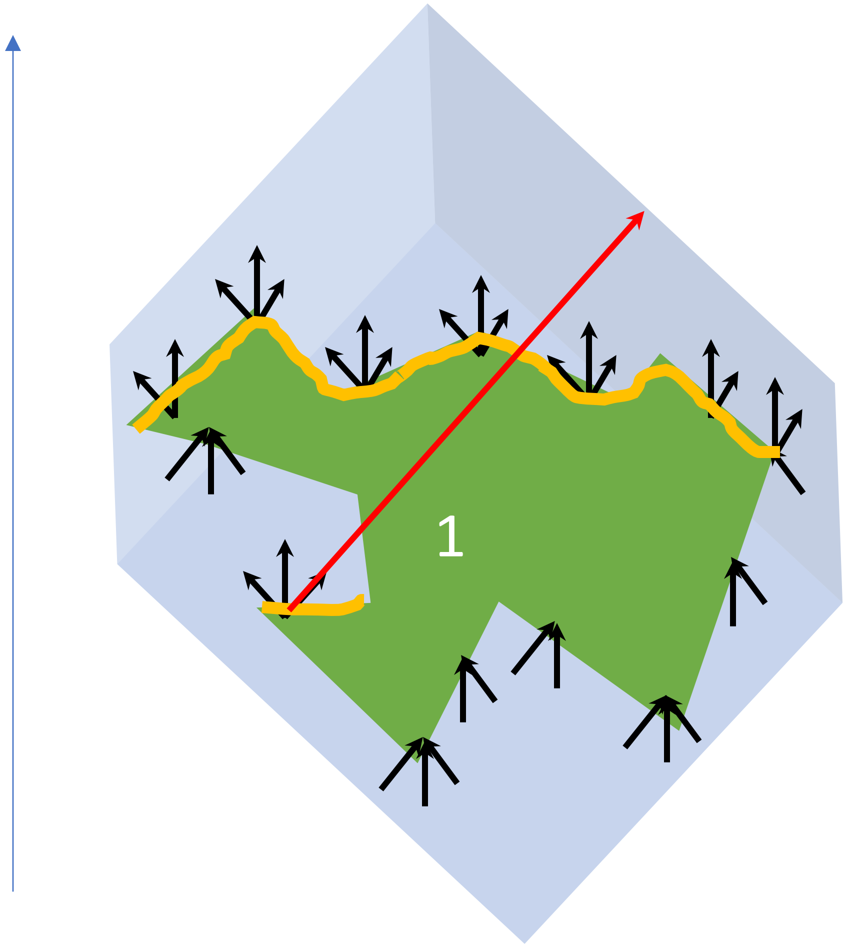
All ℓ-persistent

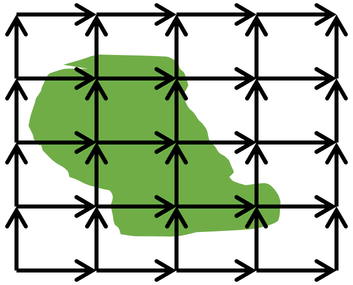
$$\ell = o(\sqrt{d}/r)$$

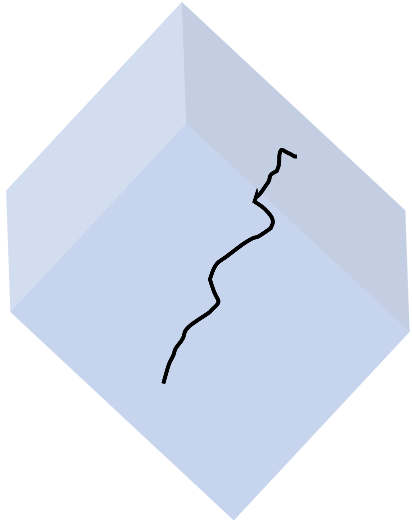
Total prob ≈ $\frac{1}{r} \cdot \frac{r}{\sqrt{d}} = \frac{1}{\sqrt{d}}$

The challenge of hypergrids

What is a boundary?





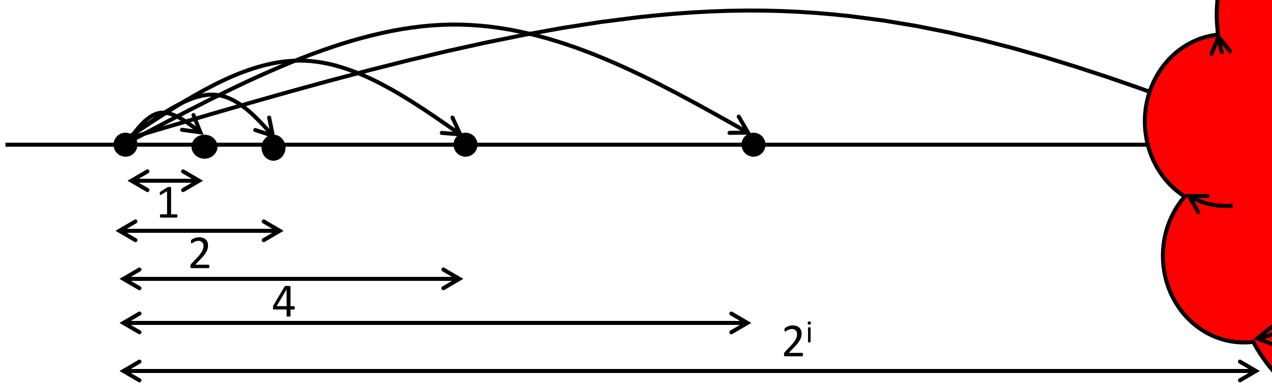


Move in a direction. But for “how much”?

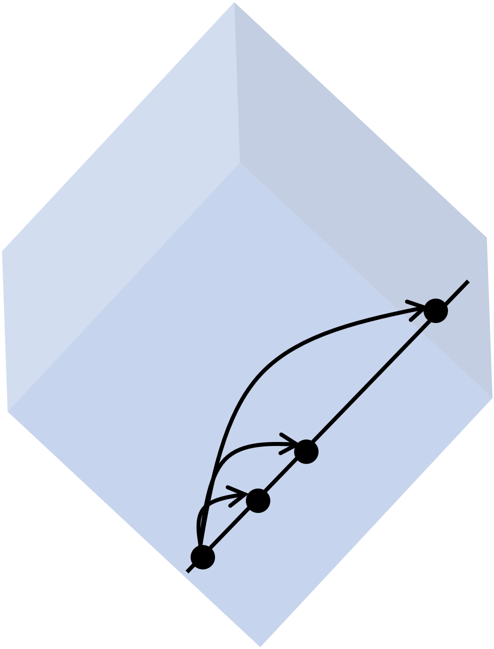
What is influence?

How to define a path tester? What are upward random walks?

The augmentation view



Does directed Talagrand hold?

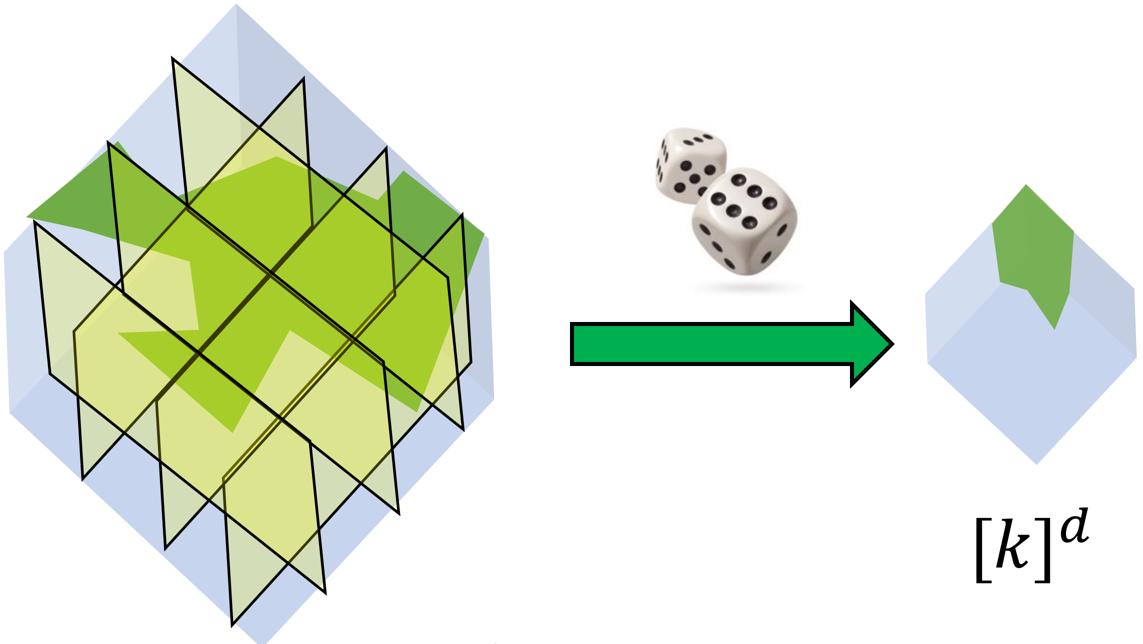


- Degree is now $d \log n$
  - Can we treat it a hypercube/product structure?
- [BCS18] Directed Margulis holds
  - Define boundaries in the graph theoretic way
  - $d^{5/6} \log n$ query tester, using path tester

Directed Margulis proofs more amenable to alternate domains.

[KMS15] very intricate proof; doesn't seem to generalize

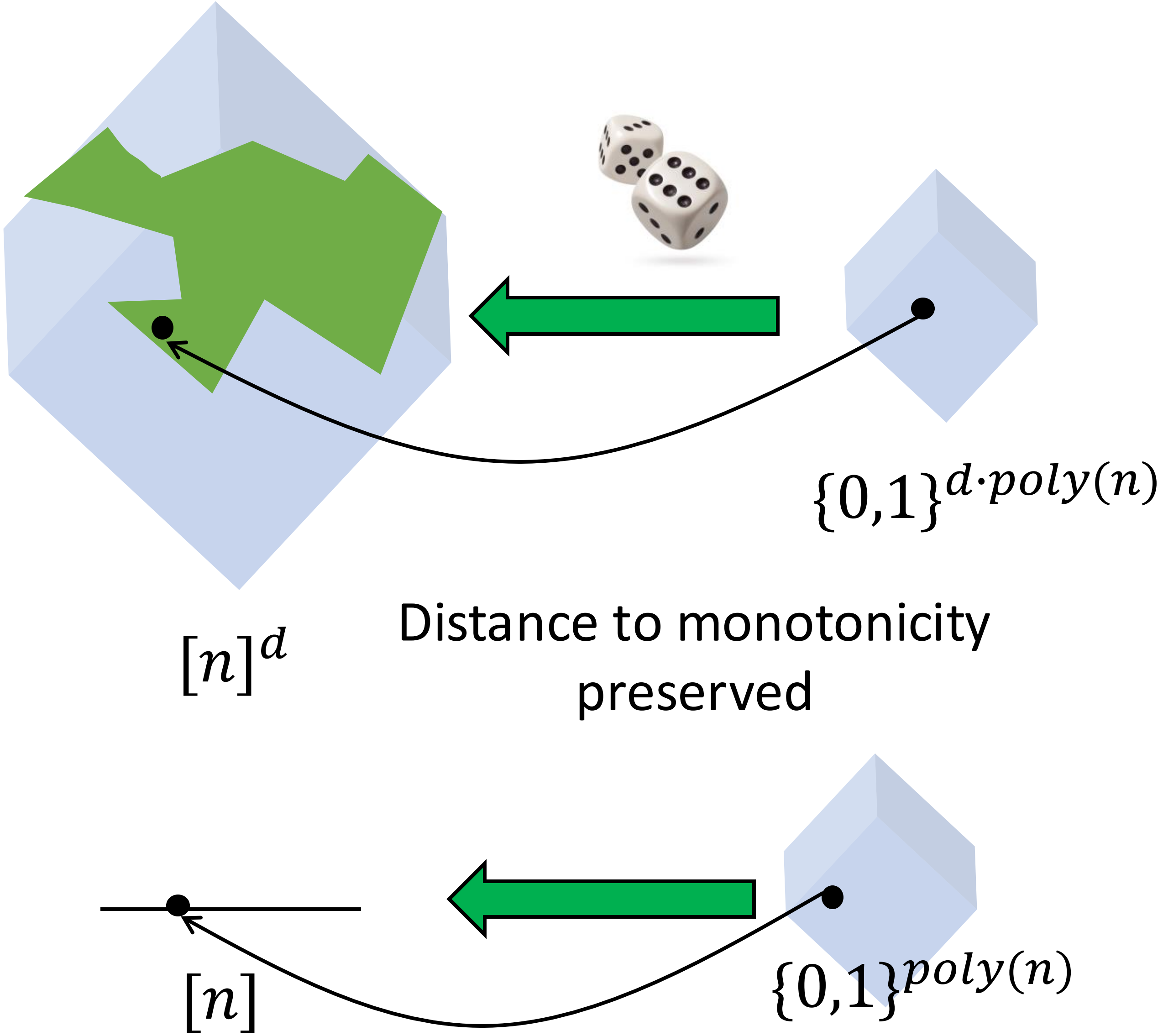
Domain reduction



Is distance to monotonicity preserved?

- Set k=2, and reduce to hypercubes…?
- [BCS20] If k << d, sampled function can be close to monotone
- [BCS20] If k = poly(d), distance is preserved
  - So we can assume n = poly(d)
  - Even reduces from continuous
- [Harms-Yoshida 22] Downsampling

The embedding method

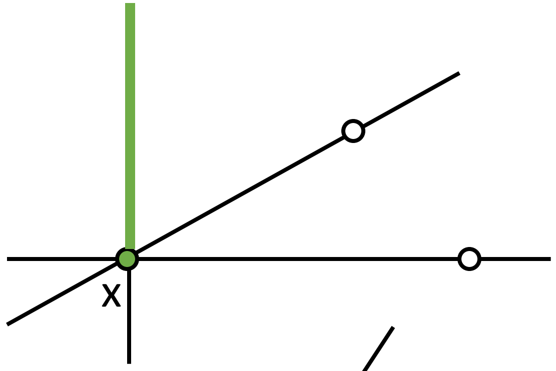


Distance to monotonicity preserved

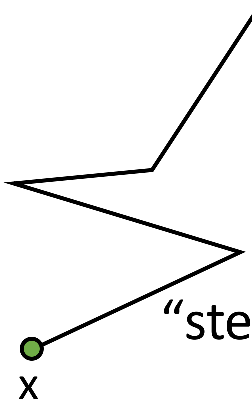
- Embedding [n] into hypercube
- [Braverman-Kindler-Khot-Minzer 23]

- First $\sqrt{d}$ tester for any n > 2
  - poly(n) dependence necessary

Doing isoperimetry



Dimension contains violation



“step” uar in [x$_i$, n]

$\mathrm{Inf}^+(x) =$ # dimensions where x is in violation

[BCS23–1] Tal_f^+

$:= \; \mathrm{Exp}_{x:1}\left[\sqrt{\mathrm{Inf}_f^+(x)}\right]$

$= \Omega(\varepsilon_f / \log n)$

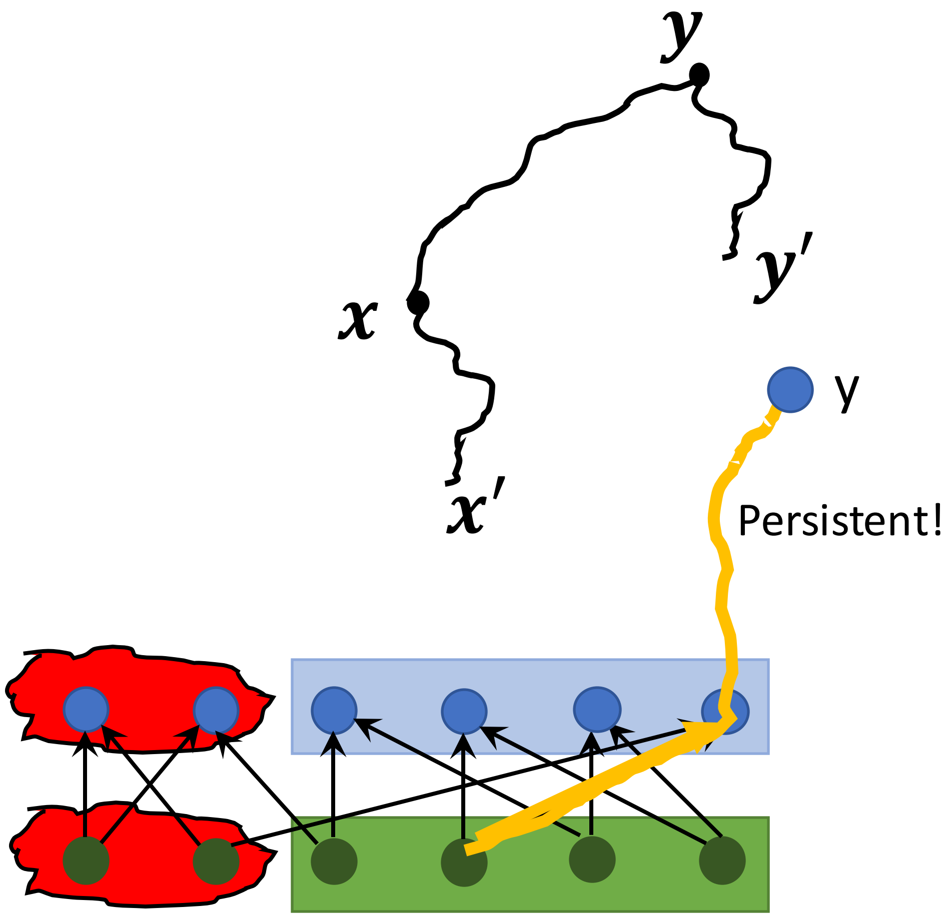
- Also prove the robust version
- Direct tester analysis leads to $n\sqrt{d}$ query tester
  - Path picks uniform random step in each direction

The final step

[BCS 23-2] Getting $d^{\frac{1}{2}+o(1)}$ query tester

- Tester does correlated walks
  - Heavily used in analysis
  - Standard tester probably works

- How to perform this analysis and not lose n factor?
  - New combinatorial tools to analyze the walk using the [BCS23-1] Talagrand theorem

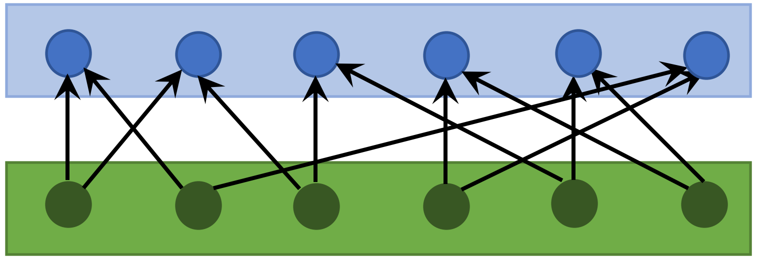


(Mostly) end of story

vertices = $2^d/r$

All degrees = r^2

$$\min_{\psi} \mathrm{E}_x\left[\sqrt{\mathrm{Inf}_{\psi}^{+}(x)}\right] = \Omega(\varepsilon_f / \log n)$$



- $d^{o(1)}$ loss annoying, comes from one specific calculation
- $d^{\frac{1}{2}+o(1)}$ non-adaptive one-sided monotonicity tester
  - Ends a 24 year odyssey
- Nearly matching (two-sided) lower bound

Some odds and ends

- Get rid of the $d^{o(1)}$ factor

- Are hypergrids strictly harder than hypercubes?
  - Only (log d) factors apart

- [Black-Kalemaj-Raskhodnikova 23] Tester for larger ranges, for hypercube
  - We can probably apply the methods for hypergrids

The big question

What is the adaptive complexity of monotonicity testing for hypercubes/grids?

- [Blais-Belovs 16] $\Omega\left(d^{1/4}\right)$ lower bound
- [Chen-Waingarten-Xie 18] $\Omega\left(d^{1/3}\right)$ lower bound
  - Every example has matching adaptive upper bound

- [Chakrabarty-S 16] $O\left(I_f\right)$ query adaptive tester
  - Proves adaptivity gap, for natural families

More on directed isoperimetry

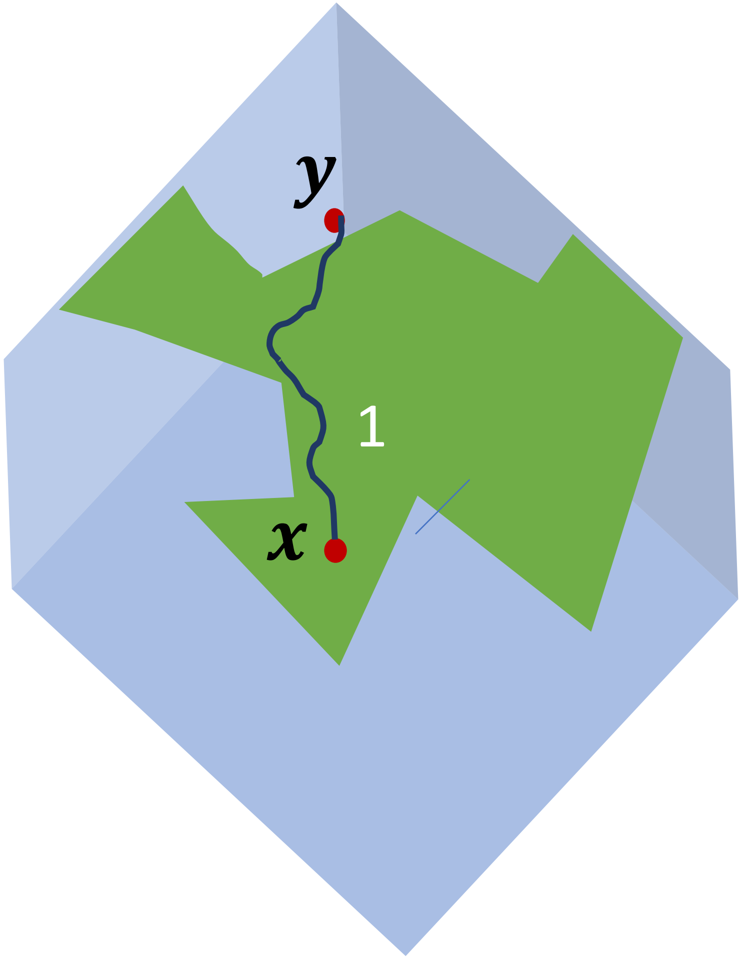
- [Black-Kalemaj-Raskhodnikova 23] Directed Talagrand for real-range

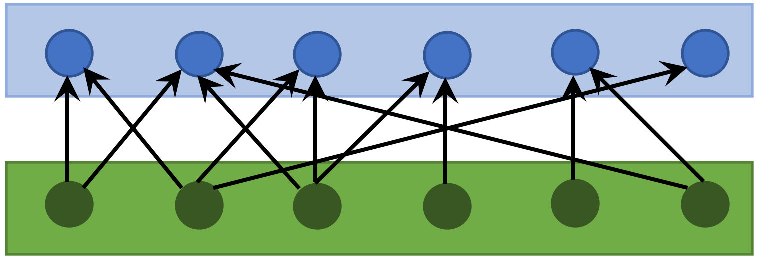
- [Pinto 23, Pinto 24] Differential functions $f\colon [0,1]^d \to [0,1]$, with access to derivatives
  - Draws connections to optimal transport theory, based on undirected Poincare/Talagrand inequalities
- [Canonne-Chen-Levi-Kamath-Waingarten 21] Applications of robustness concept to distribution testing
- [Yoshida 24] Directed isoperimetry for general posets

Generalizing hypercube results

- [Pallavoor-Raskhodnikova-Waingarten 20] Estimating distance to monotonicity, $\sqrt{d}$-approximations in poly(d) time for hypercube
  - Can we get results for hypergrids? Also implies results for product distributions…?

- [Berman-Raskhodnikova-Yaroslavtsev 14] L_p testing





Thank you!

Here, no bad pun