Streaming Algorithms for Connectivity Augmentation Problems

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Problem Statement

Connectivity Augmentation Problem (k-CAP)

Input:

- $(k-1)$-edge-connected graph $G = (V, E)$, and
- set of weighted links L, where weights are in $\{0,1, \dots, W\}$

Output: min-weight $L' \subset L$ s.t. $G' = (V, E \cup L')$ is k-edge connected

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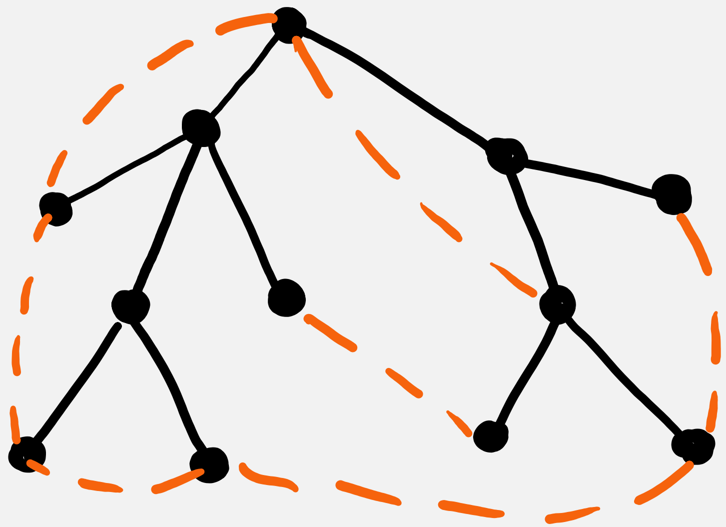

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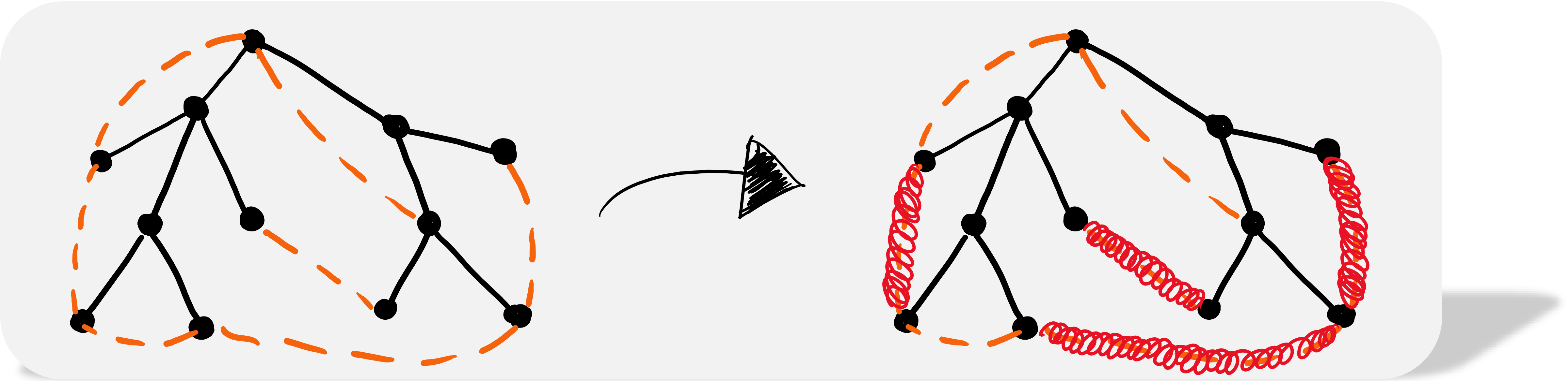

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➢ **Minimum Spanning Tree** and **Tree Augmentation Problem (TAP)** are special cases.

$k = 1$

$k = 2$

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General Network Design Problem (aka SNDP)

k-ECSS:
$r(uv) = k$ for all $u, v \in V$

Input:

- graph $G=(V,E)$ with a weight function $w: E \to \{0,\ldots,W\}$, and
- connectivity requirement $r: V \times V \to \mathbb{Z}_{\geq 0}$

Output: min-weight $H \subset G$ s.t. $\forall s,t \in V$, H contains $r(st)$ edge-disjoint st-paths

Motivations

- Wireless/Telecommunication Networks
- Transportation Networks

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- Its study led to fundamental theoretical advances in combinatorial optimization, algorithms and mathematical programming:
  - Primal-Dual
  - Iterative Rounding
- Recent Advancements on TAP and CAP [Byrka, Grandoni, Jabal Ameli'20] [Cecchetto, Traub, Zenklusen'21] [Traub, Zenklusen'22a,b] [Traub, Zenklusen'23] [Garg, Grandoni, Jabal Ameli'23]

What is Known (Offline)?

- **General Network Design Problem:**
  - ❑ 2-approximation [Jain'98]
- **Connectivity Augmentation Problem**
  - ❑ **Unweighted:** 1.393-approximation [Cecchetto, Traub, Zenklusen'21]
  - ❑ **Weighted:** $(1.5 + \epsilon)$-approximation [Traub, Zenklusen'23]
  - ❑ **Hardness:** APX-hard

Preliminaries: Streaming Model

Graph Streaming Model

- graph edges arrive in a stream, one by one (in an arbitrary order)
- using sublinear space, $O(n \operatorname{polylog}(n))$ space (known as **semi-streaming**)

Network Design in Streaming: increasing reliability of large-scale networks

- unlike graph problems such as MST, Matching, Cut, Sparsifiers, not much is known
- testing connectivity

Streaming Models: **Possible Computation Models**

Connectivity Augmentation Problem

- Link Arrival Streams
  - ❑ G is given to the algorithm (its space is not counted in the space complexity)
  - ❑ links L arrive in a stream, one by one

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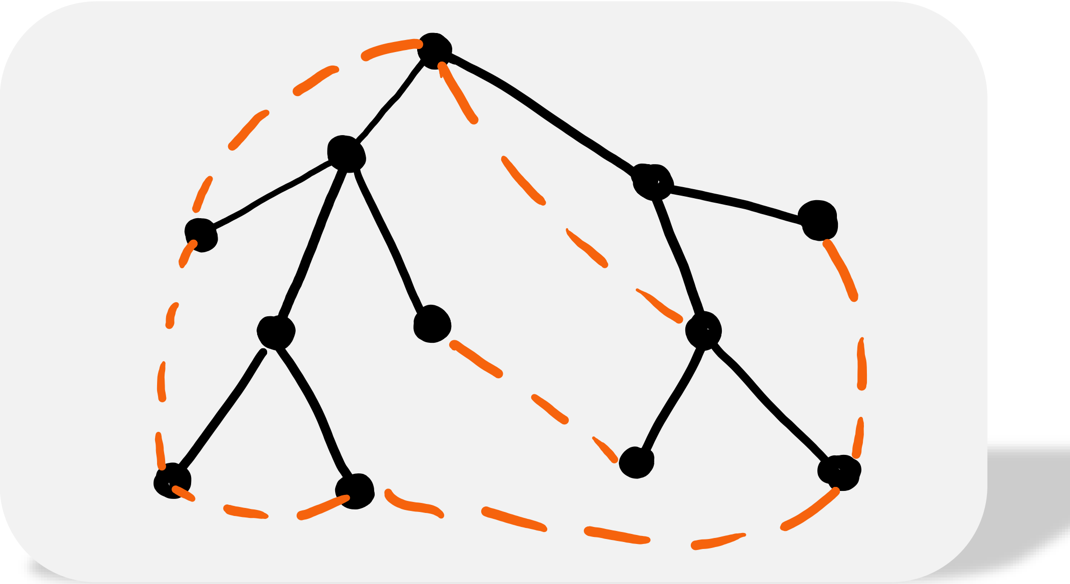

Streaming Models: Possible Computation Models

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  - ❑ links L arrive in a stream, one by one
  - ❑ compact representation exists:
    - o k-edge-connectivity certificate in $O(nk)$ space
    - o cactus representation of min-cuts in $O(n)$ space

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  - ❑ both G and L arrive in a stream, in an arbitrary order

Streaming Models: **Possible Computation Models**

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For General Network Design Problem, the only relevant model is **edge arrival**

- (Fully) Edge Arrival Streams
  - ❑ both G and L arrive in a stream, in an arbitrary order

Results

k-CAP, General Network Design, and Spanners

Our Results I: Weighted Connectivity Augmentation

	Approximation	Space	Model
k-CAP in one pass	$2+\epsilon$ $2-\epsilon$	$O(\frac{n}{\epsilon}\log n)$ $\Omega(n^2)$ bits	Link Arrival

- To get any approximation, $\Omega(n)$ space is needed.

Unlike the offline setting, the 2-approximation is a barrier in link arrival streams

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k-CAP in one pass	$2+\epsilon$	$O(\frac{n}{\epsilon}\log n)$	Link Arrival
	$2-\epsilon$	$\Omega(n^2)$ bits	
	$O(t)$	$\tilde{O}(nk + n^{1+\frac{1}{t}})$	Edge Arrival
		$\Omega(nk + n^{1+\frac{1}{t}})$ bits	

- **An Interesting Special Case:** All links arrive before existing (zero cost) edges

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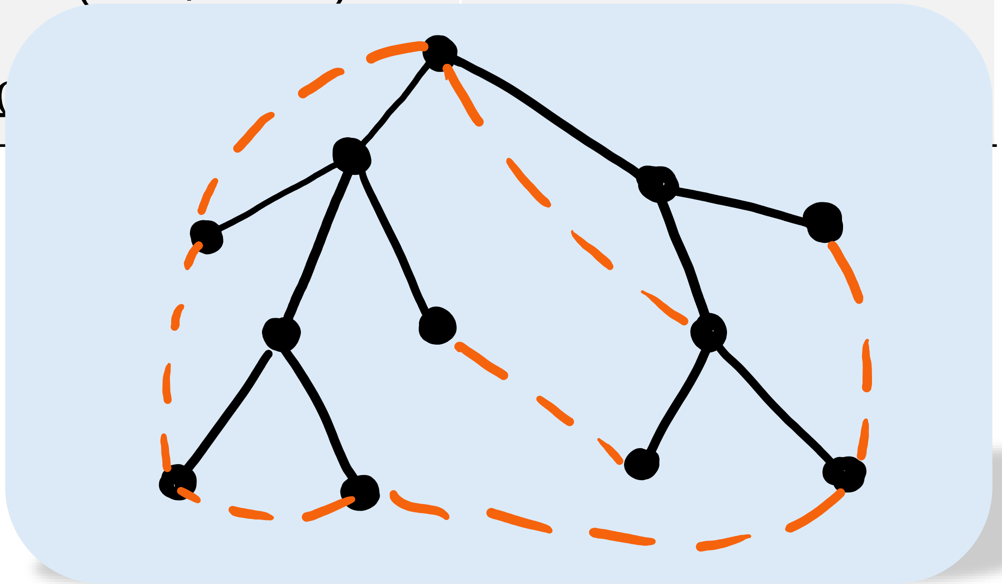

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- **An Interesting Special Case:** All links arrive before existing (zero cost) edges
- nk denotes the amount of space required to construct the k-connectivity certificate

Spanner is **the sparsifier** for connectivity augmentation problem.

Our Results II: (Weighted) Spanners

- Tight bound in unweighted case: $O(t)$-spanners with $O(n^{1+\frac{1}{t}})$ edges
- **Weighted Case:** Existing methods give $O(t)$-spanners with $O(n^{1+\frac{1}{t}} \cdot \log W)$ edges

	Distortion	Space	Model
Weighted Spanners in one pass	$O(t)$	$\tilde{O}(n^{1+\frac{1}{t}})$ $\Omega(n^{1+\frac{1}{t}})$ bits	Edge Arrival

Results III: Applications to General Network Design

	Passes	Approximation	Space	Model
SNDP	1	$O(t \log k)$ $O(t)$	$\tilde{O}(kn^{1+\frac{1}{t}})$ $\Omega(n^{1+\frac{1}{t}})$ bits	Edge Arrival
k-ECSS	k	$O(\log k)$	$O(nk \log n)$	Edge Arrival

Algorithms Overview

Weighted Spanners / k-CAP in Streams

Weighted Spanner Problem

Input:

- edge-weighted graph $G = (V, E)$ where weights belong to $\{1, \dots, W\}$,
- distortion parameter t

Output: subgraph $H \subset G$ s.t. for every $u, v \in V$, $d_H(u, v) \leq t \cdot d_G(u, v)$

Weighted Spanner Problem

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Algorithm for Unweighted Graphs:

1. Initialize $H = (V, E')$ with $E' = \emptyset$
2. For every edge $e \in E$, add e to H if it does not form a cycle of length $\leq O(t)$ with existing edges in H

Simple streaming-friendly algorithm with space complexity $n^{1+1/t}$

Weighted Spanner Problem

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- edge-weighted graph $G = (V, E)$ where weights belong to $[1, W]$,
- distortion parameter t

Output: subgraph $H \subset G$ s.t. for every $u, v \in V$, $d_H(u, v) \leq t \cdot d_G(u, v)$

Simple Extension of Algorithm for Weighted Graphs:

- Partition edges into $\log W$ classes s.t. edges in each class differ by $O(1)$-factor
- Maintain $O(t)$-spanner in each weight class

Simple streaming-friendly algorithm
with space complexity $O(n^{1+\frac{1}{t}} \cdot \log W)$

Shaving the $\log W$ Factor: Even-Odd Bucketing

Standard Partitioning (into $\log W$ partitions)

Even-Odd Bucketing

$$\cdots \quad \underbrace{[n^{i-2}, n^{i-1})}_{B_{i-2}} \quad \underbrace{[n^{i-1}, n^{i})}_{B_{i-1}} \quad \underbrace{[n^{i}, n^{i+1})}_{B_{i}} \quad \underbrace{[n^{i+1}, n^{i+2})}_{B_{i+1}} \quad \underbrace{[n^{i+2}, n^{i+3})}_{B_{i+2}} \quad \cdots$$

Key Property: It is cheaper to pick n edges from B_{i-2} than to pick a single edge from B_i

- ❑ Separately maintain spanners on **even buckets** and **odd buckets**

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B_{i-2} B_{i-1} B_i B_{i+1} B_{i+2}

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❑ Separately maintain spanners on **even buckets** and **odd buckets**

Outline of Weighted Spanner Algorithm

Step 1. Summarize edges in **even** and **odd** buckets separately **(maintaining a $O(t)$-spanner)**

Step 2. Combine t-spanners of **even** and **odd** buckets

Algorithm for Spanners on Even Buckets

… $[n^{i-2}, n^{i-1})$ $[n^{i-1}, n^{i})$ $[n^{i}, n^{i+1})$ $[n^{i+1}, n^{i+2})$ $[n^{i+2}, n^{i+3})$ …

B_{i-2} B_{i-1} B_i B_{i+1} B_{i+2}

- For any edge $uv \in B_i$:
  - If there exists a uv-path in (even buckets of) $B_{\leq i-2}$, don't keep uv in spanner

Key Property: For every ***uv*-edge** in B_i, a ***uv*-path** (of any length) in $B_{\leq i-2}$ (if exists) has a lower weight

Algorithm for Spanners on Even Buckets

… $[n^{i-2}, n^{i-1})$ $[n^{i-1}, n^{i})$ $[n^{i}, n^{i+1})$ $[n^{i+1}, n^{i+2})$ $[n^{i+2}, n^{i+3})$ …

$\boldsymbol{B_{i-2}}$ $\boldsymbol{B_{i-1}}$ $\boldsymbol{B_{i}}$ $\boldsymbol{B_{i+1}}$ $\boldsymbol{B_{i+2}}$

- For any edge $uv \in \boldsymbol{B_i}$:
  - If there exists a uv-path in (even buckets of) $\boldsymbol{B_{\leq i-2}}$, don't keep uv in spanner
- Need to deal with edges of $\boldsymbol{B_i}$ between different CCs of $G[\boldsymbol{B_{\leq i-2}}]$

Algorithm for Spanners on Even Buckets

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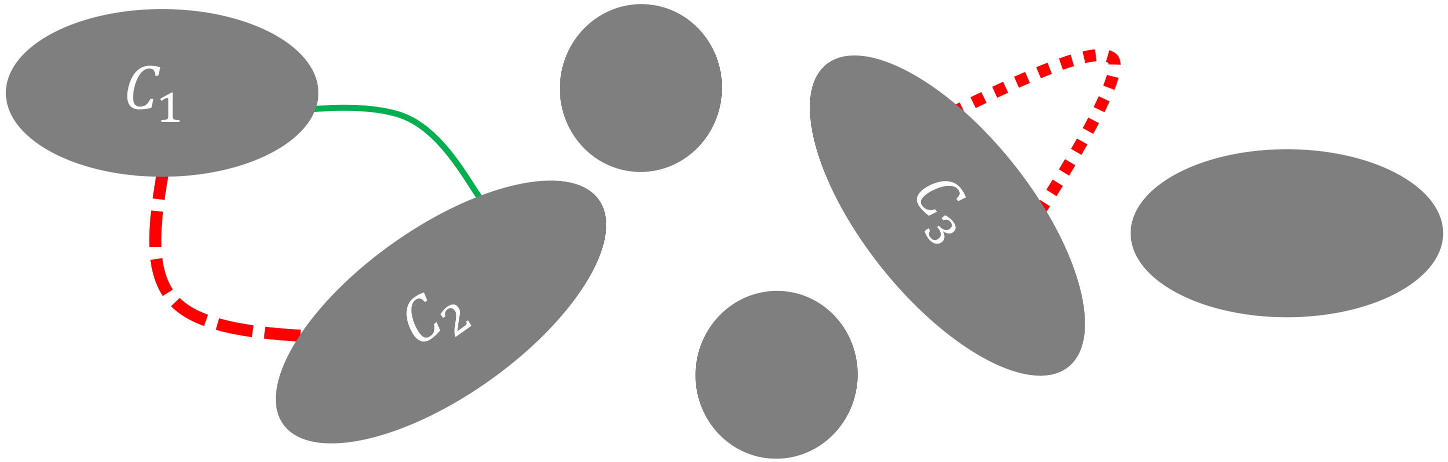
I. Remove edges in bucket i with both endpoints in a same CC

II. Between any pair of CC, only keep the lightest edge from bucket i



Algorithm for Spanners on Even Buckets

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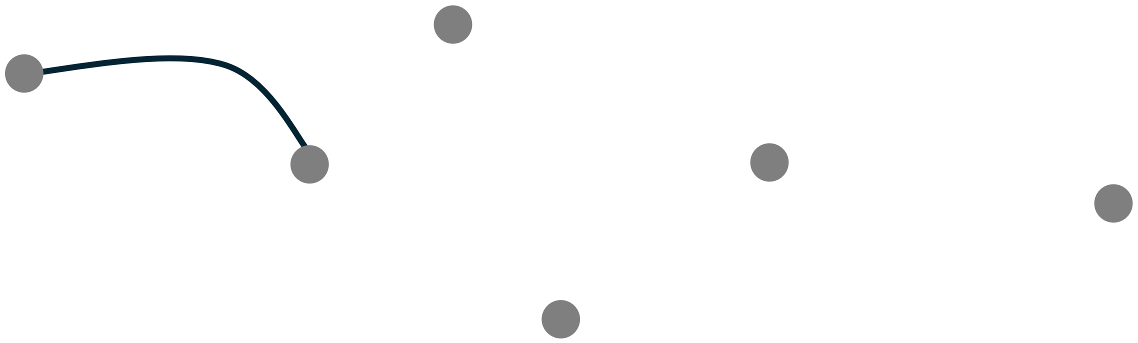
	$\boldsymbol{B}_{i-2}$	$\boldsymbol{B}_{i-1}$	$\boldsymbol{B}_{i}$	$\boldsymbol{B}_{i+1}$	$\boldsymbol{B}_{i+2}$	

- For any edge $uv \in \boldsymbol{B}_i$:
  - If there exists a uv-path in (even buckets of) $\boldsymbol{B}_{\leq i-2}$, don't keep uv in spanner
- Need to deal with edges of $\boldsymbol{B}_i$ between different CCs of $G[\boldsymbol{B}_{\leq i-2}]$
- Contract CCs of $G[\boldsymbol{B}_{\leq i-2}]$ into super nodes



Analysis of the Algorithm on Even Buckets

… $[n^{i-2}, n^{i-1})$ $[n^{i-1}, n^{i})$ $[n^{i}, n^{i+1})$ $[n^{i+1}, n^{i+2})$ $[n^{i+2}, n^{i+3})$ …

$\boldsymbol{B_{i-2}}$ $\boldsymbol{B_{i-1}}$ $\boldsymbol{B_{i}}$ $\boldsymbol{B_{i+1}}$ $\boldsymbol{B_{i+2}}$

- If there are x super nodes in $\boldsymbol{B_i}$, need to store at most $\tilde{O}(x^{1+1/t})$ edges from $\boldsymbol{B_i}$
  - Using the simple weighted spanner algorithm within each $\boldsymbol{B_i}$ (increase the space complexity by $\log n$)
  - If the number of non-zero-degree super nodes w.r.t. $\boldsymbol{B_i}$ is y, the spanner size becomes $\tilde{O}(y^{1+1/t})$
- Using the nested structure of CCs w.r.t. buckets ($\boldsymbol{B_{\leq i}}, \boldsymbol{B_{\leq i-2}}, \ldots$), we can bound the total number of edges by $\tilde{O}(n^{1+1/t})$

Implementation in Streaming

$\cdots \quad [n^{i-2}, n^{i-1}) \quad [n^{i-1}, n^{i}) \quad [n^{i}, n^{i+1}) \quad [n^{i+1}, n^{i+2}) \quad [n^{i+2}, n^{i+3}) \quad \cdots$

$\quad\quad \boldsymbol{B_{i-2}} \quad\quad \boldsymbol{B_{i-1}} \quad\quad \boldsymbol{B_{i}} \quad\quad \boldsymbol{B_{i+1}} \quad\quad \boldsymbol{B_{i+2}}$

- Edges may come in any order and CCs may change for several buckets as we add an edge

- Maintain super nodes in the stream (given the set of so far stored edges H, we can compute them when needed)
- Sparsify the spanners of all heavier buckets once an edge is added to H

Algorithms Overview

k-CAP in Streams

Weighted k-CAP in Streams

- Edge Arrival Stream

  Immediately follows from our streaming algorithm for **weighted spanner**

- Link Arrival Stream

  It uses **even-odd bucketing**, but needs to deal with more involved structures

Overview of k-CAP in Link Arrival

2-Approximation Algorithm in Link Arrival

Connectivity Augmentation → Cactus Augmentation → Cycle Augmentation

[Galvez, Grandoni, Jabal Ameli, Sornat'19]
[Traub, Zenklusen'23]

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- **Warm-up:** using extra $\log W$ factor in space compared to the unweighted case:

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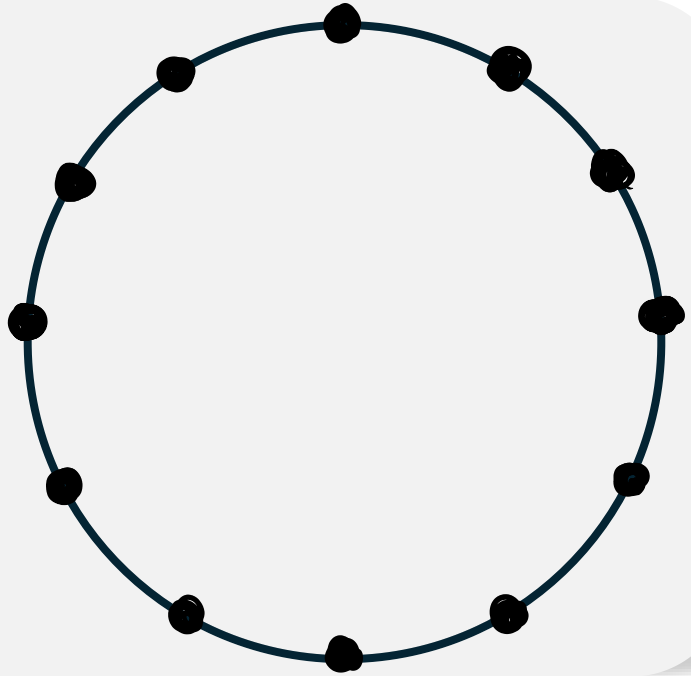

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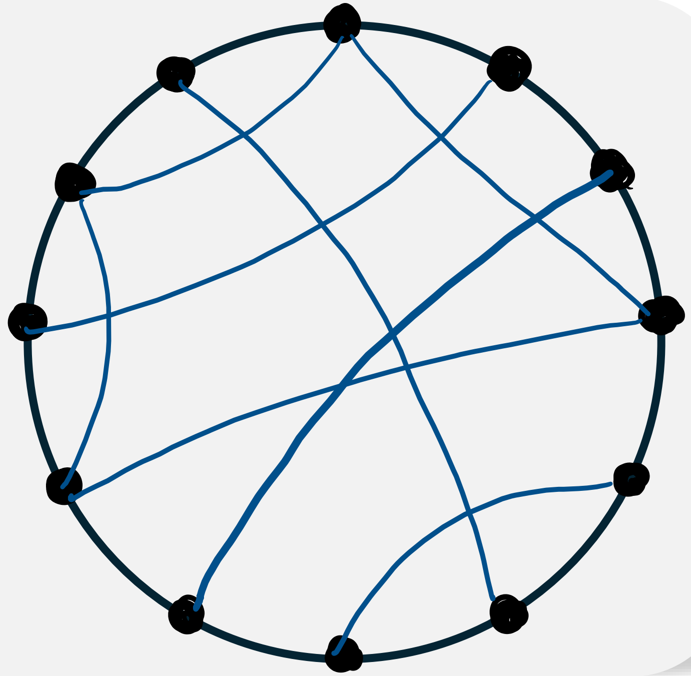

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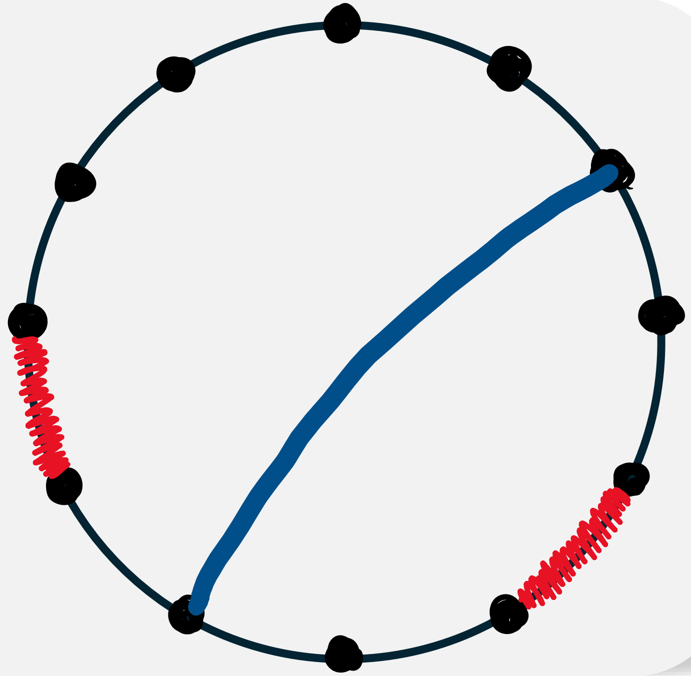

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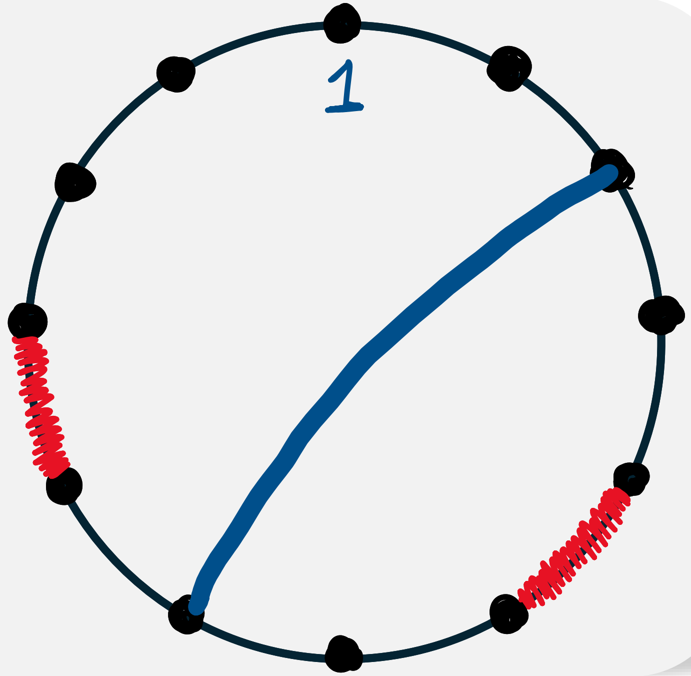

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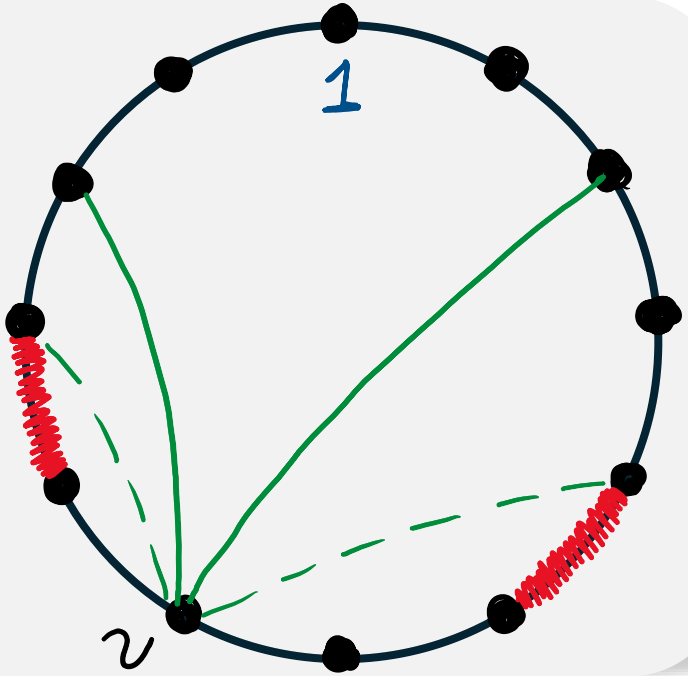
Cycle Augmentation

- 2-cuts corresponds to removal of two edges on the cycle
  - A 2-cut is covered if there exists a crossing link
- Pick an arbitrary root node "**1**"
- Keep incident edges of v whose endpoints are closets to "**1**"
  - Total of $O(n)$ edges are kept.

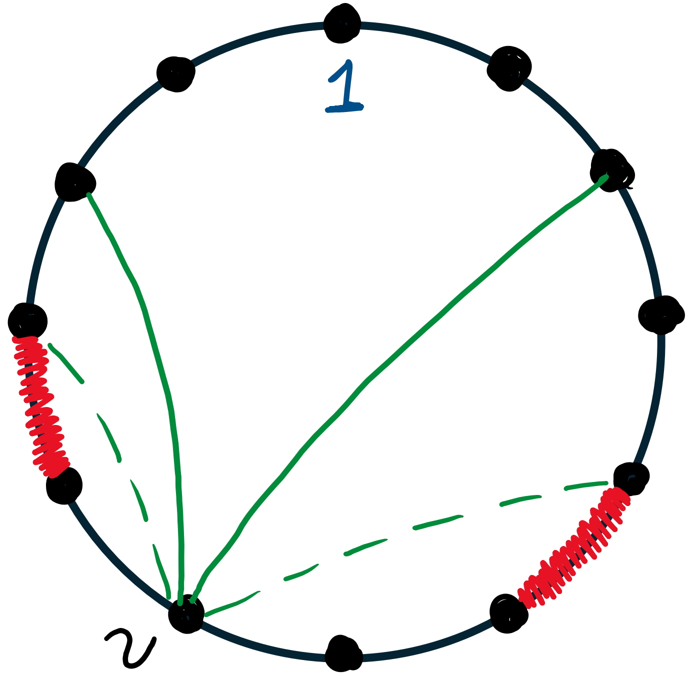


Overview of k-CAP in Link Arrival

2-Approximation Algorithm in Link Arrival

Connectivity Augmentation → **Cactus Augmentation** → **Cycle Augmentation**

- **Warm-up:** using extra $\log W$ factor in space compared to the unweighted case:
  - ❑ keep at most 2 edges per vertex for each weight class
- **Weighted:** similar **even-odd bucketing** idea works
  - ❑ shrink "3-edge-connected" components
  - ❑ not Cycle Augmentation anymore
  - ❑ still has nice structures to exploit



Algorithms Overview

General Network Design in Streams

General Network Design

k-Edge Connected Spanning Subgraph (k-ECSS)

- Augmentation Framework [Williamson et al.'93]
  - ❑ Increase the connectivity one by one in each pass till get to k
  - ❑ Using our algorithm for CAP in link arrival, we get an algorithm that runs
    - in k passes,
    - uses $O(nk + n \cdot \log n)$ space, and
    - **Approx. factor**: trivial bound is $O(k)$ but a more careful analysis gives $O(\log k)$

Approximation Bound

- The integrality gap of the standard LP relaxation of the augmentation problem is 2.
- In ℓ-th phase (i.e., ℓ-CAP), the optimal fractional solution has cost $\leq \frac{\mathrm{OPT}_{k-\mathrm{ECSS}}}{\ell}$

General Network Design (contd.)

SNDP: A collection of k spanner-like structure is a coreset for the problem

- Analysis relies on
  - **Reverse** Augmentation Framework [Goemans et al.'94], and
  - Our algorithm for CAP in edge arrival streams
- Guarantee
  - In one pass, and
  - using $O(kn^{1+1/t})$,
  - achieves $O(t \cdot \log k)$-approximation

Summary:

"Network Design in streaming and its connection to weighted spanners"

Questions

- Close the gaps (for SNDP and k-ECSS)!
- Close the gaps for unweighted k-CAP in link arrival streams:
  - UB: $(2+\epsilon)$-approx. in $\tilde{O}(n)$ space
  - LB: < 1.409-approx. not possible in $\tilde{O}(n)$ space
- Dynamic streaming?

Problem	Pass	Approx.	Space	Stream
k-CAP	1	$2+\epsilon$ $2-\epsilon$	$O(\frac{n}{\epsilon}\log n)$ $\Omega(n^2)$ bits	link arrival
		$O(t)$	$\tilde{O}(kn + n^{1+\frac{1}{t}})$ $\Omega(kn + n^{1+\frac{1}{t}})$ bits	fully streaming
STAP	1	$O(t)$	$\tilde{O}(n^{1+\frac{1}{t}})$ $\Omega(n^{1+\frac{1}{t}})$ bits	fully streaming link arrival
Spanner	1	$O(t)$	$\tilde{O}(n^{1+\frac{1}{t}})$ $\Omega(n^{1+\frac{1}{t}})$ bits	edge arrival
SNDP	1	$O(t\log k)$ $O(t)$	$\tilde{O}(kn^{1+\frac{1}{t}})$ $\Omega(n^{1+\frac{1}{t}})$ bits	edge arrival
k-ECSS	k	$O(\log k)$	$O(kn\log n)$	edge arrival

Thank you!