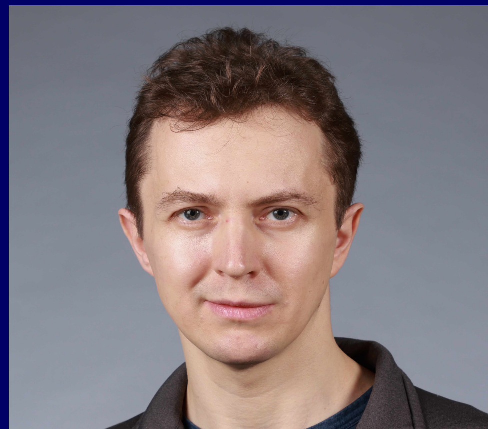
Sublinear time Approximation Algorithms for Max-Cut on Clusterable Graphs (ICALP 2024)

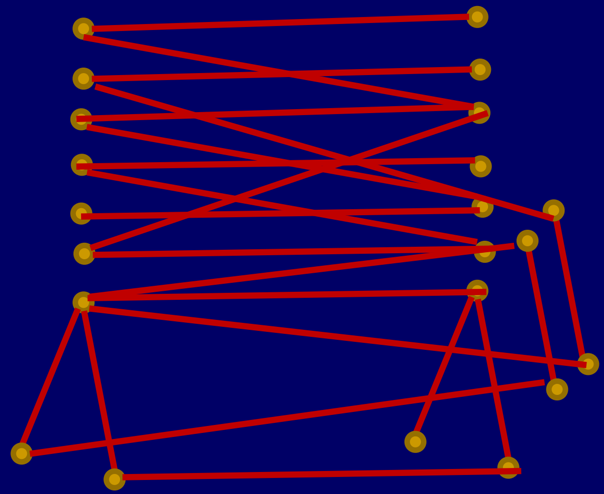
- June 20, Simons Institute
- Workshop on Extroverted Sublinear Algorithms
- Akash Kumar (IIT Bombay)



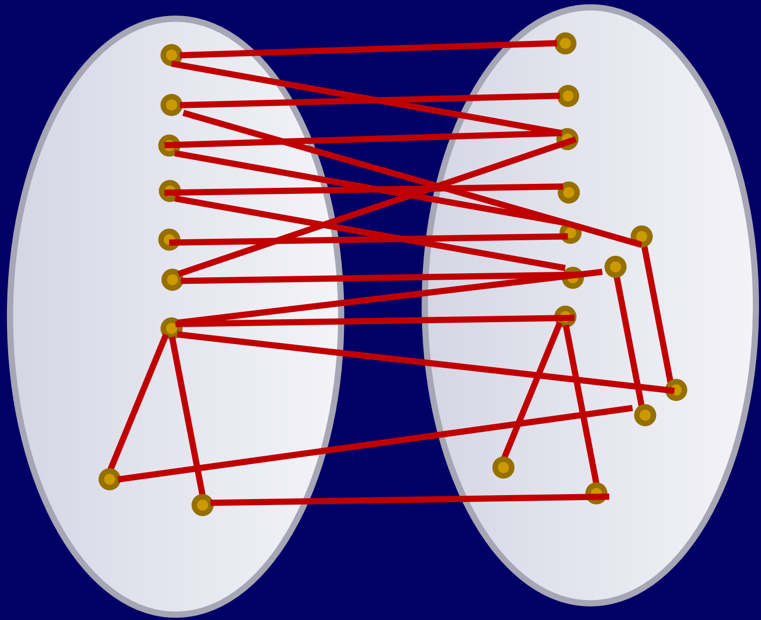
Agastya Jha
(EPFL → U Chicago)

Michael Kapralov

The Max-Cut Problem



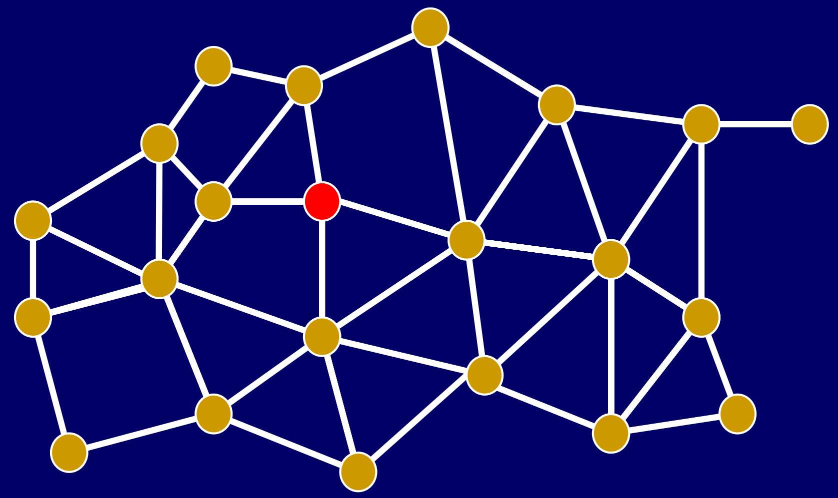
The Max-Cut Problem



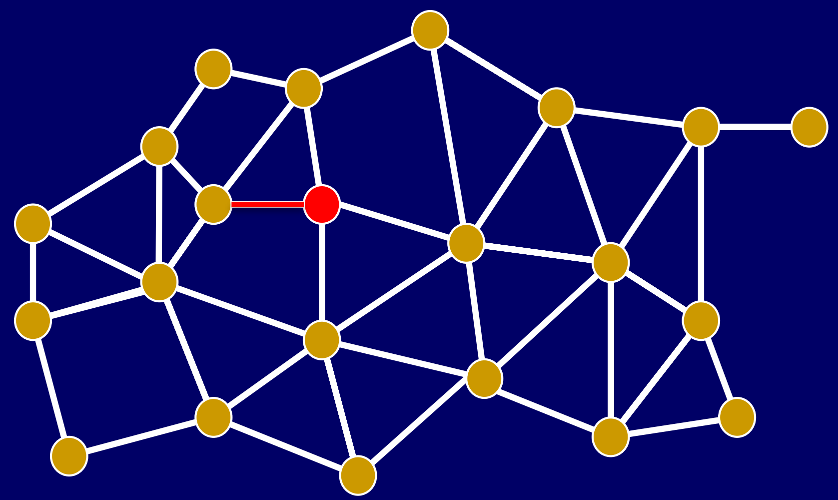
Let MC(G) = fraction of edges cut by the best bipartition.

Query Model:

Vertex Query



Random Neighbor Query



Query Model:

Vertex Query
Random Neighbor Query

Can only “crawl” along the edges of G.

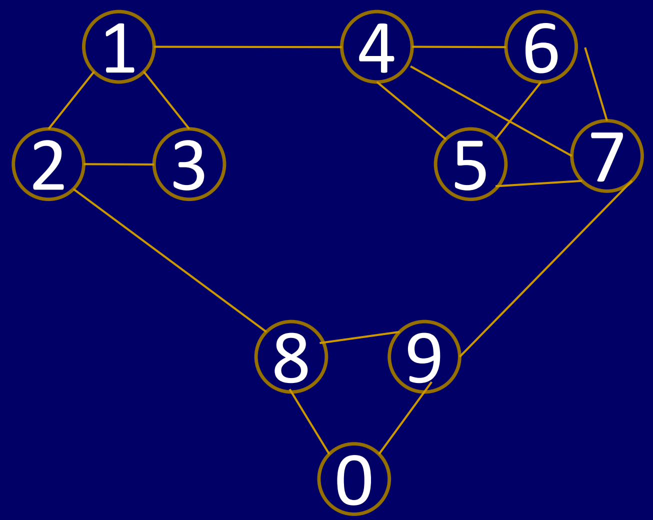
Question:
Decide whether $MC(G) \geq 1 - \gamma$ or $MC(G) \leq ½ + \gamma$

Known results for Sublinear Time Approximation to Max-Cut

- Factor ½ – trivial.
- [CKKMP 18] MC(G) to within a factor ½+ε needs $n^{1/2+\Omega(\varepsilon)}$ time.

Q: Can you beat ½ factor for some interesting graph classes?

- [Peng-Yoshida 23] Yes, on expanders.
- [Jha, K.] Yes, on clusterable graphs.



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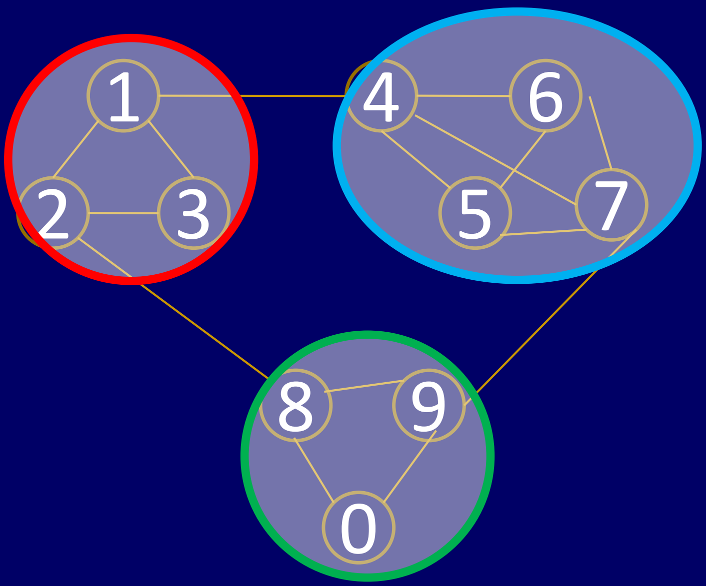
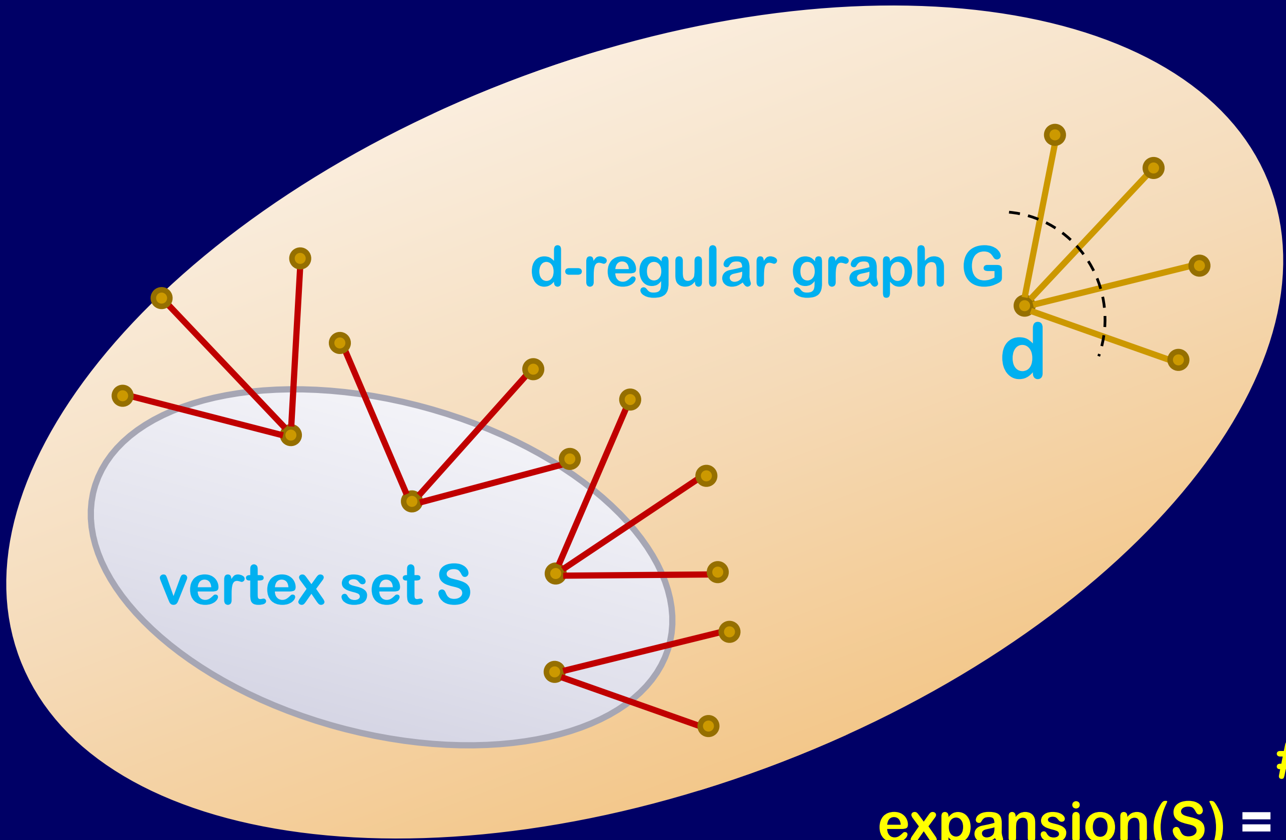


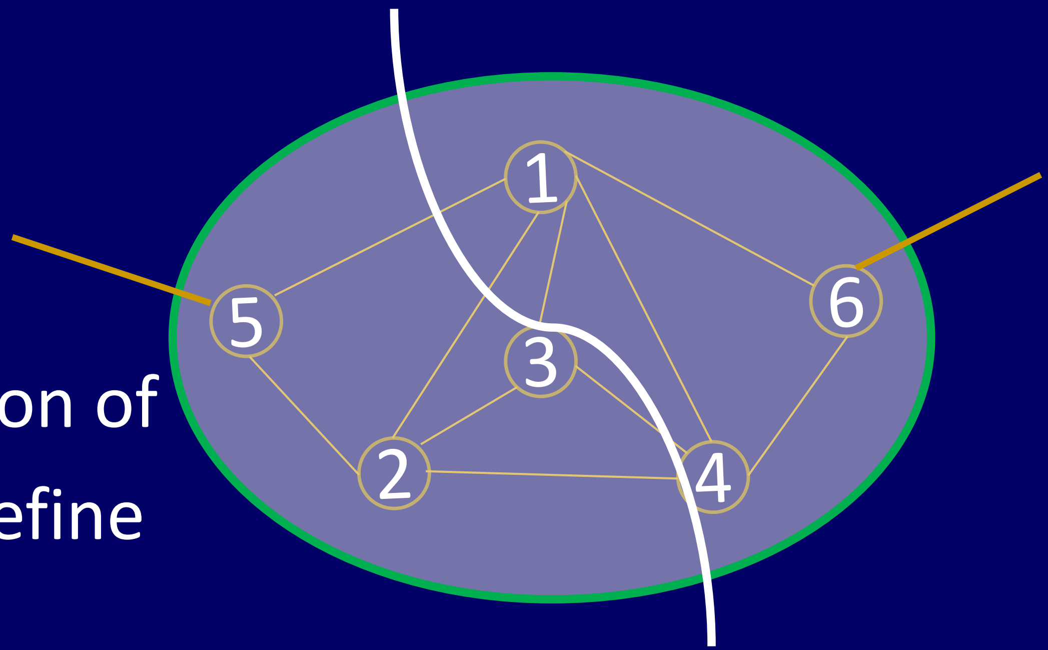
Expanders



$$\text{expansion}(S) = \frac{\#\text{ edges leaving } S}{d\,|S|}$$

$$\text{expansion}(G) = \min_{|S| \leq n/2} \text{expansion}(S)$$

What is a good cluster?



Formalized using the notion of **Inner conductance**. We define

$$\Phi_{in}(S) = \frac{|E(S, C\setminus S)|}{\min(vol(S), vol(C\setminus S))}$$

$$\Phi_{in}(S) = \frac{|E(S, C\setminus S)|}{\min(d|S|, d|C\setminus S|)}$$ for d-regular graphs.

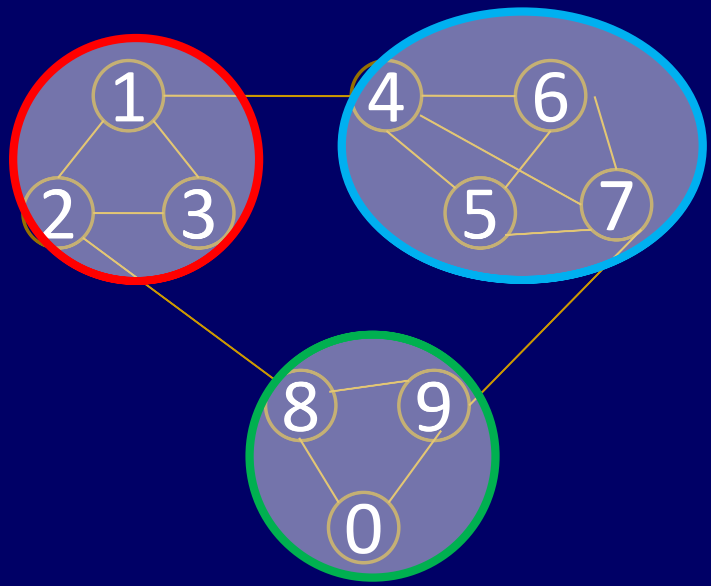
$C \subseteq V$ is a good cluster if all sets $S \subseteq C$ have $\Phi_{in}(S) > \Phi$

Clusterable Graphs

G is (k, Φ, ε) clusterable if you can partition V(G) into k disjoint subsets $C_1, C_2, .., C_k$ (of roughly same size) such that

1) $\Phi_{in}(C_i) > \Phi$
2) $\Phi_{out}(C_i) < \varepsilon$

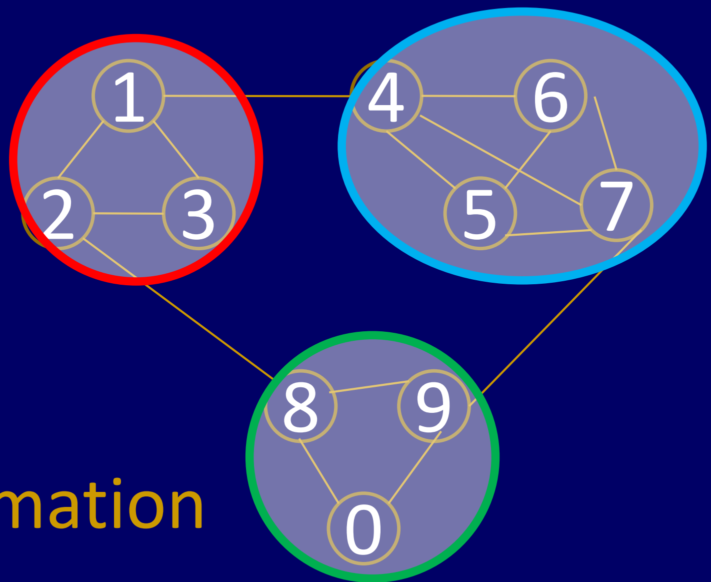
Here, $\Phi_{out}(S) = \frac{|E(S,V\setminus S)|}{d|S|}$



Clusterable Graphs

G is $(k, \Omega(1), \varepsilon)$ clusterable if you can partition $V(G)$ into k disjoint subsets $C_1, C_2, .., C_k$ (of roughly same size) such that

1) $\Phi_{in}(C_i) > \Omega(1)$
2) $\Phi_{out}(C_i) < \varepsilon$



Here, $\Phi_{out}(S) = \frac{|E(S, V \setminus S)|}{d|S|}$

Goal: Get a better than ½ approximation for MC(G) on this graph class.

The k = 1 case [Peng-Yoshida 23]

Decide if $MC(G) \geq 1 - \gamma$ or $MC(G) \leq ½ + \gamma$ on an expander.

Notation: $p_{v,l}$ = end-pt distrib. of step lazy r/ walks from v.

$l = C \log n/\Phi^2$

PY idea:

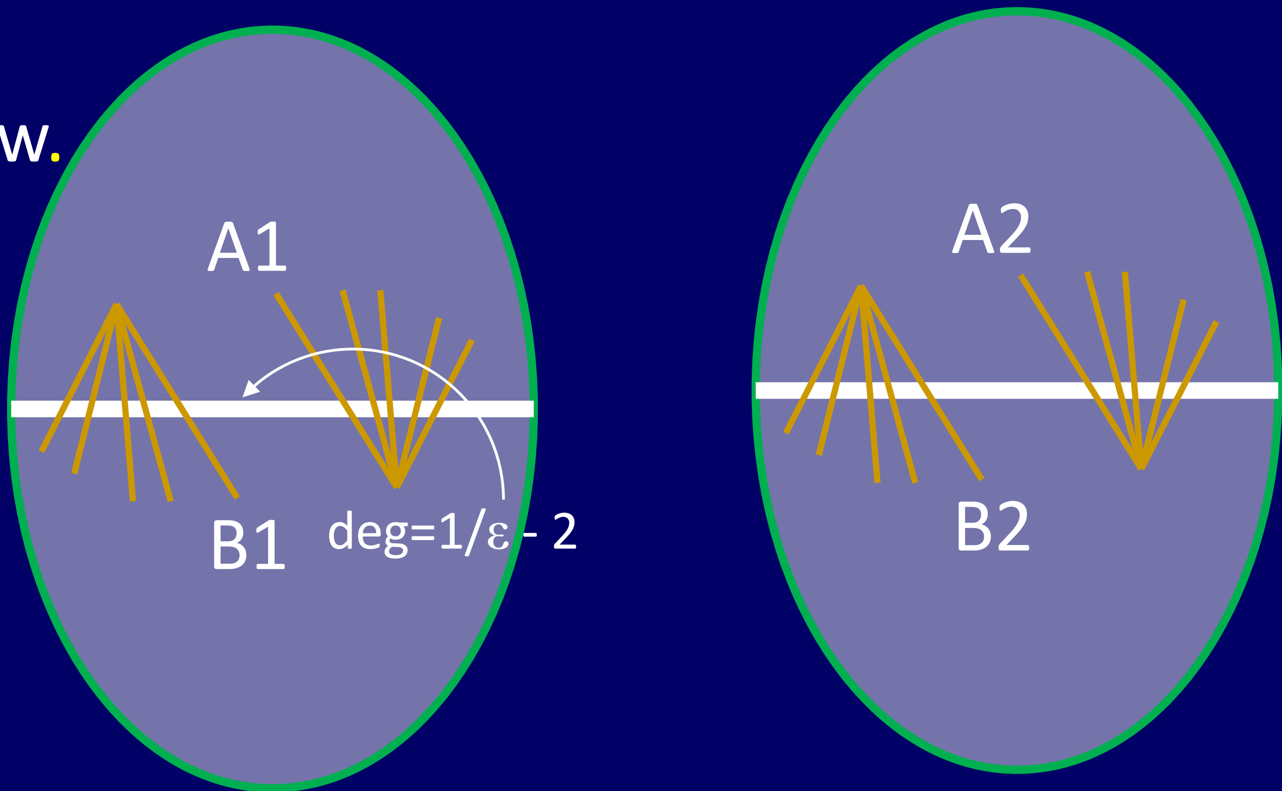
- Pick $v \sim V(G)$ and consider $W \sim p_{v,l}$.
- Remove all loops on the walk W and obtain W'.
- Let $p_{v,l}^{(e)}$ = distrib on walks with length $|W'|$ being even.
- Let $p_{v,l}^{(o)}$ = distrib on walks with length $|W'|$ being odd.
- Return $MC(G)$ large iff $\|p_{v,l}^{(e)} - p_{v,l}^{(o)}\|_2^2$ large.

The k > 1 case.

Decide if MC(G) ≥ 1 – γ or MC(G) ≤ ½ + γ when G is (k, $\Omega(1)$, ε)-clusterable.

Assume $\gamma \leq \varepsilon$.

PY idea fails now.

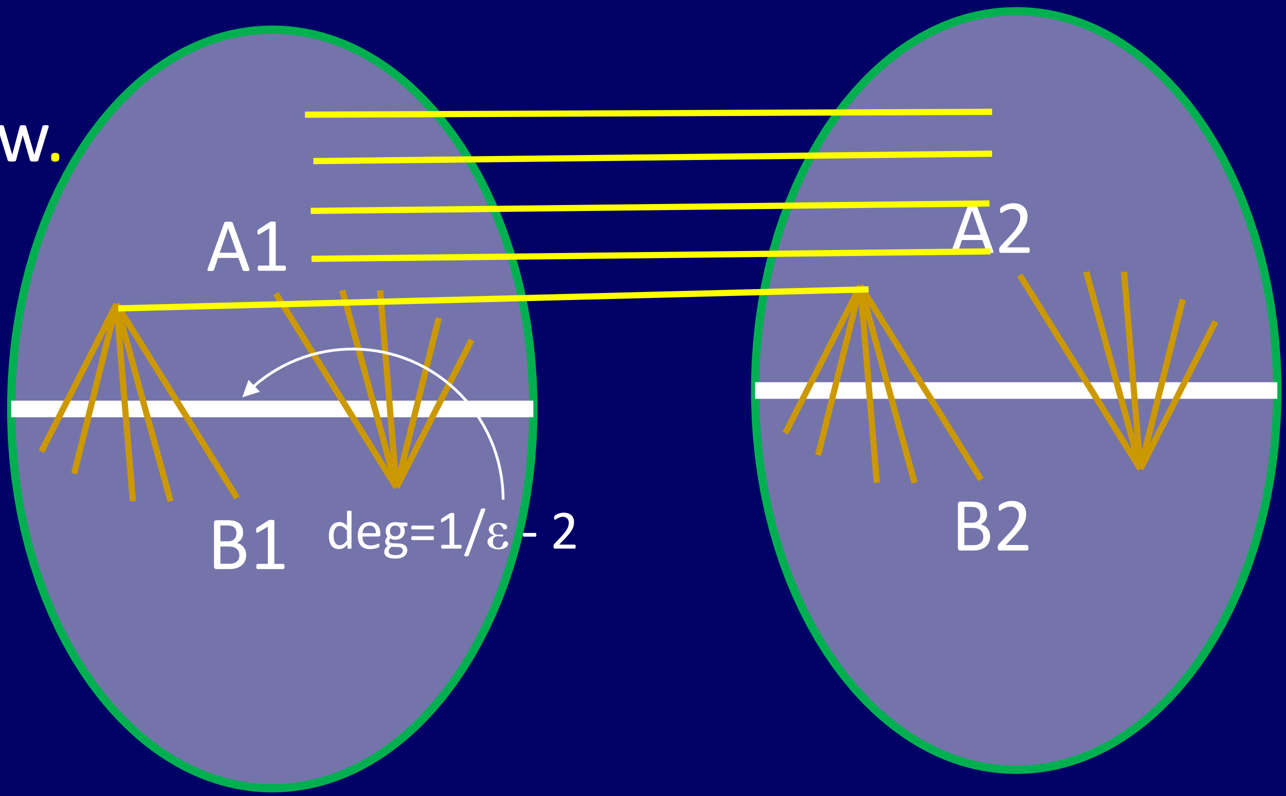


The $k > 1$ case.

Decide if $MC(G) \geq 1 - \gamma$ or $MC(G) \leq ½ + \gamma$ when G is $(k, \Omega(1), \varepsilon)$-clusterable.

Assume $\gamma \leq 10\varepsilon$.

PY idea fails now.



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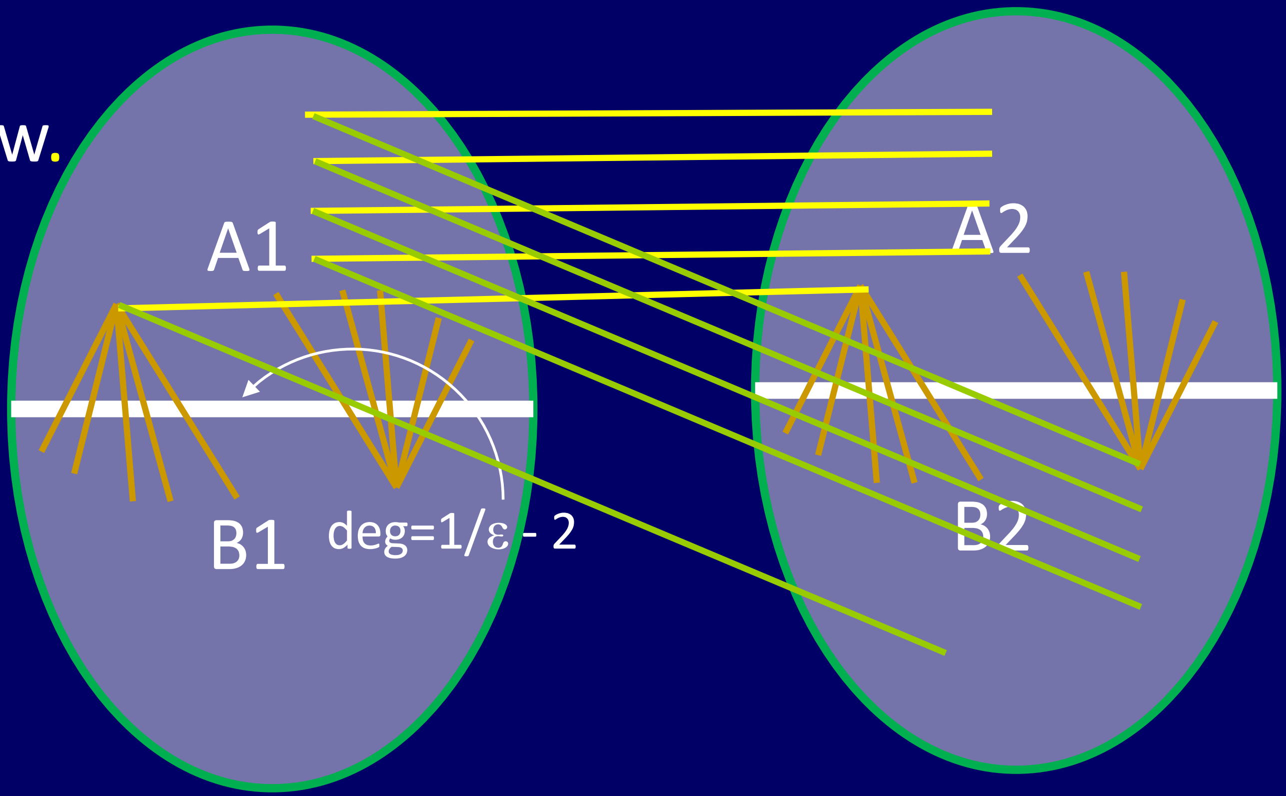

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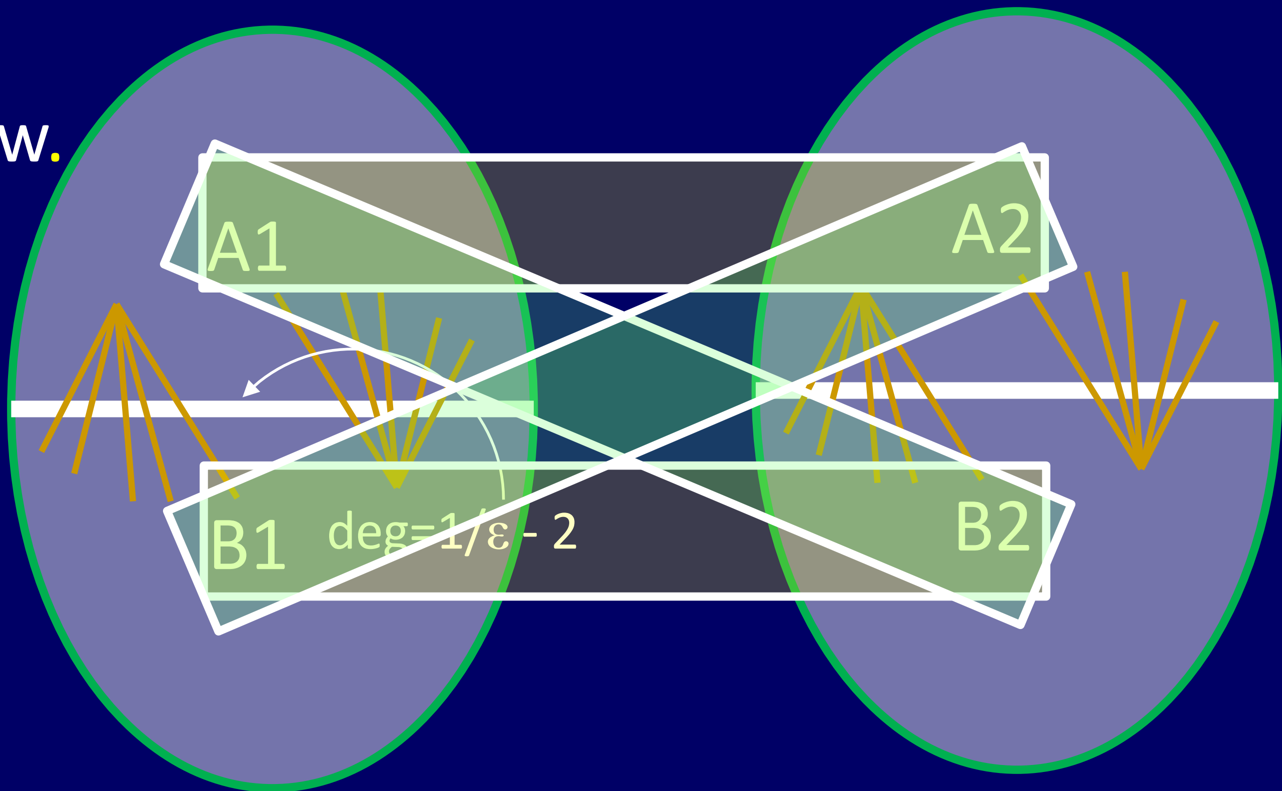

The k > 1 case.

Consider the following intuition.

- Perform a two-step non-lazy walk in G.
- If G had c clusters with large induced cut-value, the new graph has k+c sparse cuts.

How about doing even 2t length walks and tracking (k+c)-th eigenvalue of L^{2t}?

Main Theorem

[Jha, K.]

There exists an algorithm which when given as input a graph (k, Φ, ε) clusterable G (d-regular) runs in time

$$O^*(n^{1/2+O(\varepsilon)})$$

and returns a high/low cut-value verdict which is correct with probability at least 2/3.

O* hides terms poly(d,k, log n) terms.

Recapping our approach

Understand spectra of these instances.

Show that spectra is appreciably different in the two cases.

Obtain access to the spectral info via random walks.

Note:

If $MC(G) \geq 1 - \gamma$, then $c \geq 2k/3$ clusters have induced cut-value at least $1 - O(\gamma)$.

If $MC(G) \leq ½ + \gamma$, then $c \geq 2k/3$ clusters have induced cut-value at most $½ + O(\gamma)$.

The Algorithm (Inspired by [CKKMP 18])

1. Assign $c = 2k/3$.
2. Assign $s = poly(k)\ n^{O(\varepsilon)}$.
3. Assign $t = \frac{C \log n}{\Phi^2}$.
4. Sample set S of s vertices.
5. Compute Gram Matrix $(M^t S)^T (M^t S)$ approximately.
6. If $\frac{n}{s}\ \nu_{k+c}(M^t S)^T (M^t S) \geq n^{-\varepsilon}$, return High cut-value
7. Else return Low cut-value

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Fix (i,j). Can compute entry (i,j) to an additive $n^{-1-\varepsilon}$ approximation in time $n^{1/2+\varepsilon}$ using collision counting.

Analysis (a very high level overview)

Claim 1: If $MC(G) \geq 1 - \varepsilon$, then $v_{k+c}(M^t)^T (M^t) \geq (1 - O(\varepsilon))^{2t}$.

Note: $(1 - \varepsilon)^{2t} \geq \exp(-4t\,\varepsilon) \geq \exp\left(-4\,\frac{C \log n}{\Phi^2}\,\varepsilon\right) \geq n^{-C'\varepsilon}$

Claim 2: If $MC(G) \leq ½ + \varepsilon$, then $v_{k+c}(M^t)^T (M^t) \leq (1 - O(\Phi^2))^{2t}$.

Note: $(1 - \Phi^2)^{2t} \leq \exp(-2t\,\Phi^2) \leq \exp\left(-2\,\frac{C \log n}{\Phi^2}\,\Phi^2\right) \leq n^{-2C}$

Analysis (a very high level overview)

Claim 1: If $MC(G) \geq 1 - \varepsilon$, then $v_{k+c}(M^t)^T (M^t) \geq (1 - O(\varepsilon))^{2t}$.

Claim 2: If $MC(G) \leq ½ + \varepsilon$, then $v_{k+c}(M^t)^T (M^t) \leq (1 - O(\Phi^2))^{2t}$.

Intuition for Claim 1

1. By easy direction of Higher order Cheeger, $v_k(M) \geq 1–2\varepsilon$.
2. If $MC(G) \geq 1 - \varepsilon$ then $c \geq 2k/3$. This means c eigenvalues of M are at most $-1 + 10\varepsilon$.
3. This implies Claim 1.

Analysis (a very high level overview)

Claim 1: If $MC(G) \geq 1 - \varepsilon$, then $\nu_{k+c}(M^t)^T (M^t) \geq (1 - O(\varepsilon))^{2t}$.

Claim 2: If $MC(G) \leq ½ + \varepsilon$, then $\nu_{k+c}(M^t)^T (M^t) \leq (1 - O(\Phi^2))^{2t}$.

Intuition for Claim 2

1. A new spectral lemma: In a spectral expander, $\nu_{n-1} >> -1$. (Proof later)
2. In an expander G, if MC(G) close to ½, then $\nu_n >> -1$.
3. This implies Claim 2.

Analysis (a very high level overview)

Claim 1: If $MC(G) \geq 1 - \varepsilon$, then $v_{k+c}(M^t)^T (M^t) \geq (1 - O(\varepsilon))^{2t}$.

Claim 2: If $MC(G) \leq ½ + \varepsilon$, then $v_{k+c}(M^t)^T (M^t) \leq (1 - O(\Phi^2))^{2t}$.

Finish up with Matrix Bernstein.

Requires proving bounds on $||M^t 1_x||_2^2$. (Adapt [GKLMS 21])

$$||M^t 1_x||_2^2 = ||M^t \Sigma \beta_i(x) u_i||_2^2$$
$$= ||\Sigma \lambda_i^t \beta_i(x) u_i||_2^2$$
$$= \Sigma_{i \in Close} \lambda_i^{2t} \beta_i(x)^2 + \Sigma_{i \in Far} \lambda_i^{2t} \beta_i(x)^2$$

That deferred Claim:

Claim: If $\lambda_{n-1} \geq 2 - \varepsilon$, then $\lambda_2 \leq O(\varepsilon)$.

This means, in a spectral expander you have

$$\lambda_{n-1} \leq 2 - \Omega(1)$$

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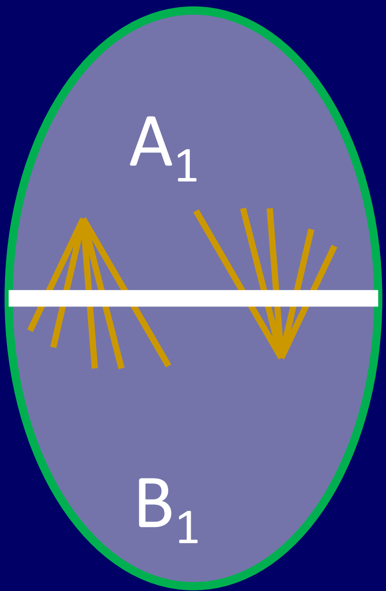
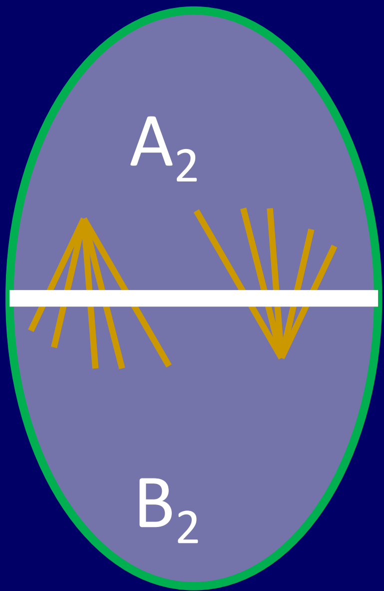
Proof Overview:





$$x_i = \begin{cases} -1/\sqrt{n}, & i \in A_1 \\ 1/\sqrt{n}, & i \in B_1 \end{cases}$$

$$y_i = \begin{cases} -1/\sqrt{n}, & i \in A_2 \\ 1/\sqrt{n}, & i \in B_2 \end{cases}$$

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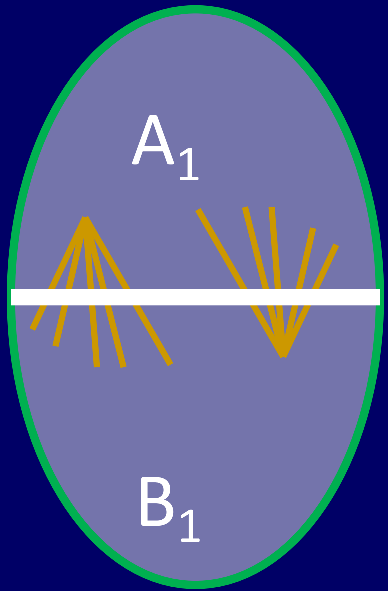

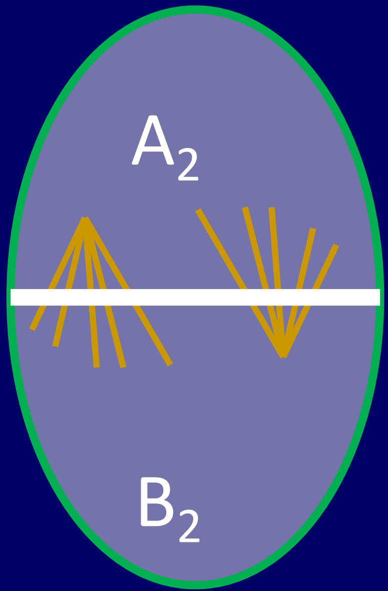

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Claim: If $\lambda_{n-1} \geq 2 - \varepsilon$, then $\lambda_2 \leq O(\varepsilon)$.

Proof Overview:

Take any vector z with $R(z) \geq 2 - \varepsilon$.

Let $z' = |z|$.

You have $R(z') \leq \varepsilon$.

Produce two orthogonal vectors x, y with R(x), R(y) small.

Choose x = all 1's.

~~Can I pick~~ $y = z'$ for some z with $R(z) \geq 2 - \varepsilon$?

That deferred Claim:

Claim: If $\lambda_{n-1} \geq 2 - \varepsilon$, then $\lambda_2 \leq O(\varepsilon)$.

Choose y = z' – (component along all 1's).

Annoyance: *R(y) might shoot up! (Call such z a bad vector)*

Would like to show there exists a "good" vector z.

Existence of a good vector:

If y is bad, then z' is highly correlated with all 1's.

Thus, z' has large l_1-norm.

Suppose v_{n-1} and v_n are both bad.

Choose $z = \frac{1}{\sqrt{2}}(v_{n-1} + v_n)$. Can show z' has small l_1-norm

$\Rightarrow$ The resulting vector y is good!

$y = z'$ is bad if

- $R(y)$ is large, or
- $||y||_2$ is small

Open Problems

1. Generalize better than ½ sublinear time approximation to a larger class of graphs?
2. Lower Bounds for Testing 3-colorability on expanders. Eg decide whether $\mathrm{val}(G) \geq 1 - \gamma$ or $\mathrm{val}(G) \leq \frac{2}{3} + \gamma$
3. Characterizing approximation resistance for approximating 2-CSPs on expanders in sublinear time?

Thank you!


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