Quantum Complexity: Quantum PCP, Area Laws, and Quantum Gravity

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Pseudo Entropy and Holography

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Extreme Universe

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Yukawa Institute for Theoretical Physics, Kyoto University

Contents

Some comment in the light of
[Bouland-Fefferman-Vazirani 2019,..]

This is a quantity which generalizes entanglement
entropy in setups with post-selections.
So it is different from pseudo entanglement
in talks tomorrow
[Aaronson-Bouland-Fefferman-Ghosh-Vazirani-
Zhang-Zhou 2022,..]

Main Reference

arXiv:2005.13801 [Phys.Rev.D 103 (2021) 2, 026005]

with Yoshifumi Nakata (YITP, Kyoto)
Yusuke Taki (YITP, Kyoto)
Kotaro Tamaoka (Nihon U.)
Zixia Wei (Harvard U.)

★Definition/Properties of Pseudo entropy
★Geometric Formula via Holography

Further progress to be mentioned in this talk

arXiv:2011.09648 [Phys.Rev.Lett. 126 (2021) 6, 061604]
arXiv:2106.03118 [Phys.Rev.Res. 3 (2021) 3, 033254]
by Mollabashi-Shiba-Tamaoka-Wei-TT

★Pseudo entropy and Quantum Phase Transition

arXiv:2302.03895 [JHEP 03 (2023) 105] by Kanda-Sato-Suzuki-Wei-TT

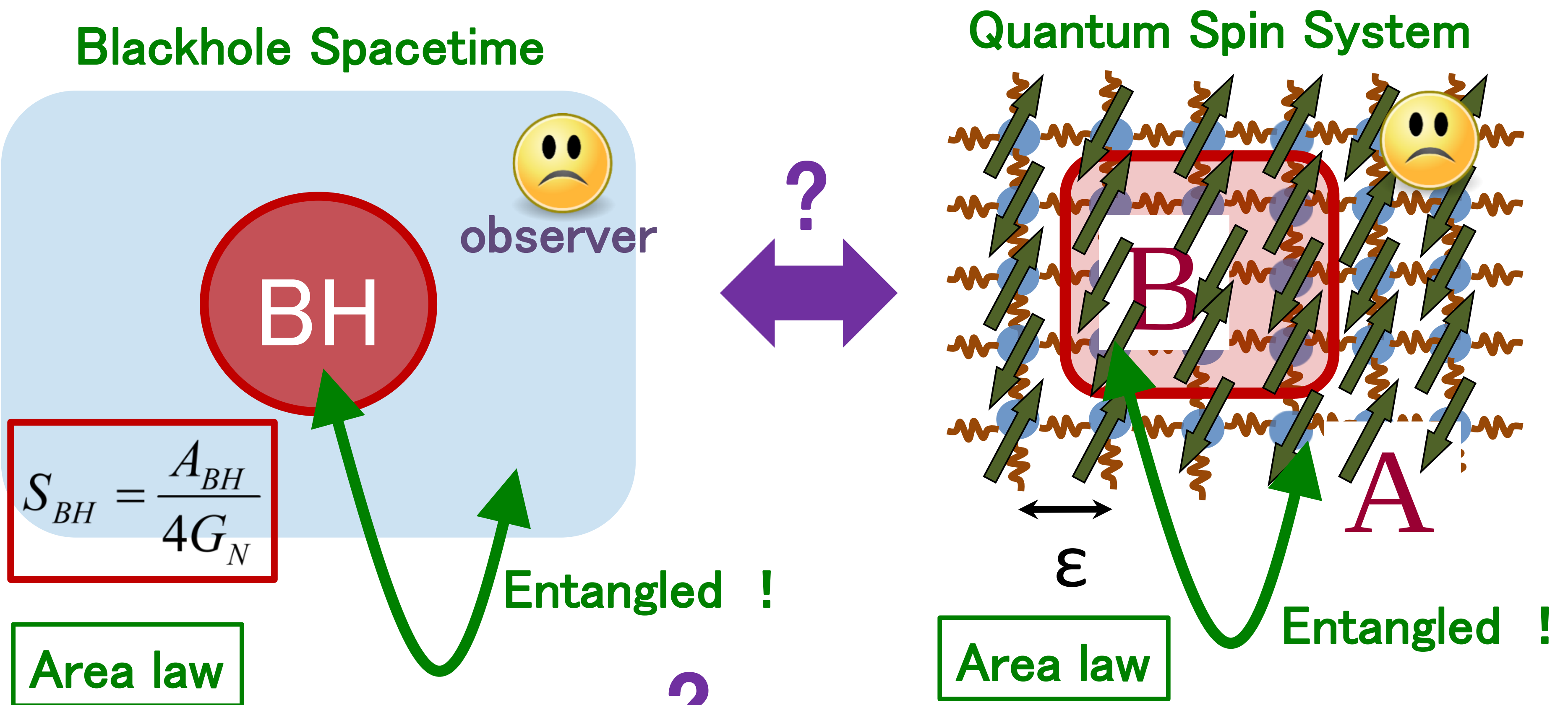
★Pseudo entropy and Entanglement Phase Transition

arXiv: 2307.06531 [JHEP 12 (2023) 123] by Parzygnat-Taki-Wei-TT

★An improved Definition of Pseudo entropy

① Introduction：Entanglement and Holography

Analogy between BH and Qubits



?

BH entropy SBH ⟷ **Entanglement entropy SA**

Spacetime — Matter

Entanglement entropy (EE)

Divide a quantum system into two subsystems A and B:

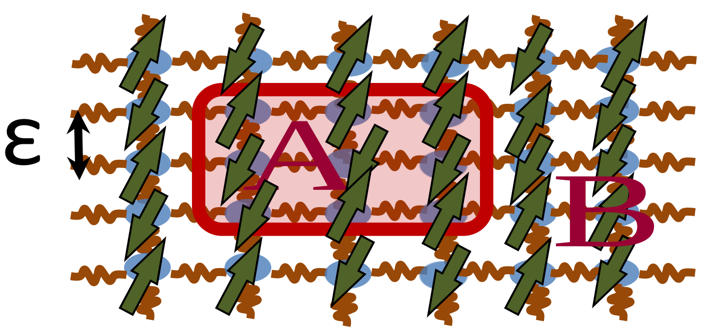
$$H_{tot} = H_A \otimes H_B \quad .$$

Define the reduced density matrix by $\rho_A = \mathrm{Tr}_B |\Psi\rangle\langle\Psi|$.

The entanglement entropy S_A is defined by

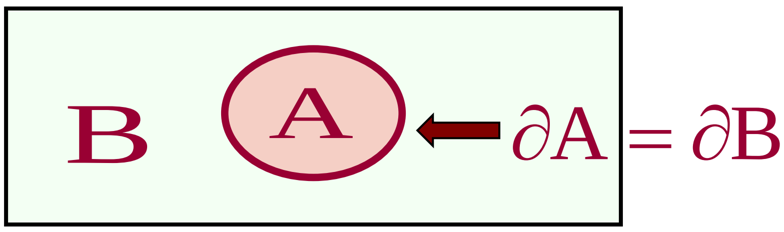
$$S_A = -\mathrm{Tr}_A\, \rho_A \log \rho_A \, .$$ (von-Neumann entropy)

Quantum Many-body Systems



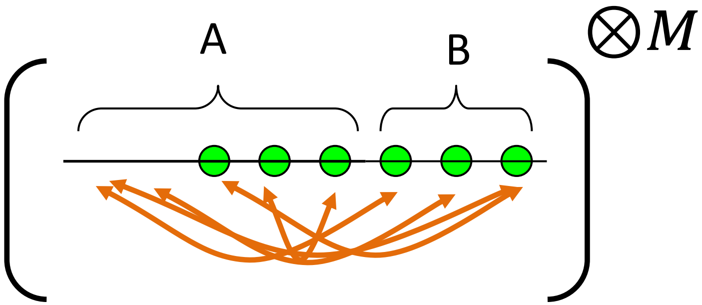
Continuum Limit ε→0

Quantum Field Theories (QFTs)



Entanglement Entropy (EE) from operational viewpoint

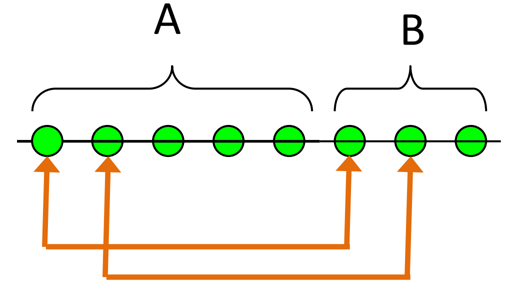
$$(|\Psi\rangle_{AB}\langle\Psi|)^{\otimes M}$$



Entangled in a very complicated way

LOCC

Distillation



N Bell pairs

$$(|\Psi\rangle_{AB}\langle\Psi|)^{\otimes M} \Rightarrow (|\mathrm{Bell}\rangle\langle\mathrm{Bell}|)^{\otimes N}$$

Well-known fact in QI:

$$S_A = \lim_{M\to\infty} \frac{N}{M}$$

$$\rho_A \equiv \mathrm{Tr}_B[|\Psi\rangle_{AB}\langle\Psi|]$$

[Bennett-Bernstein-Popescu-Schumacher 95, Nielsen 98]

Gravitational Holography

BH Entropy Formula

$$S_{BH} = \frac{A_{BH}}{4G_N}$$

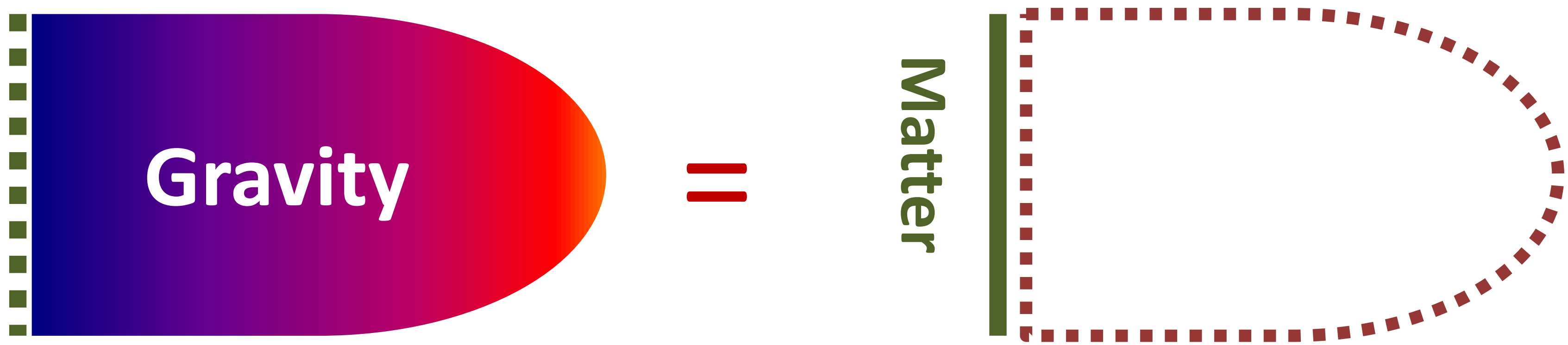
Degrees of freedom in Gravity ∝ Area

['t Hooft 1993, Susskind 1994]

Holography

Boundary of M

Gravity on M = Quantum Matter on ∂M



BH entropy(∝Area)= Thermal Entropy of Matter (∝Volume)

The most well-established holography:

AdS/CFT Correspondence [Maldacena 97]

AdS/CFT

(Quantum) Gravity
on D+1 dim. AdS
(anti de-Sitter space)

=

Gapless quantum system

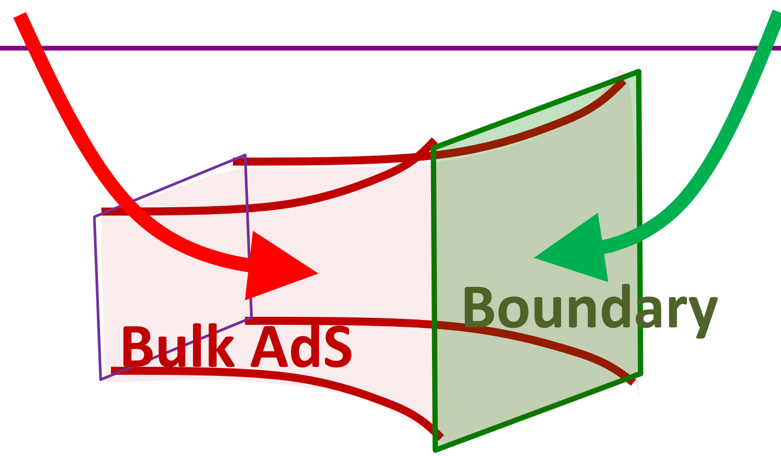
Conformal Field Theory (CFT)
on D dim. Minkowski spacetime

Classical limit ↓

General relativity
with $\Lambda < 0$

Large N + Strong coupling ↓

Strongly interacting
Quantum Many-body system



Holographic Entanglement Entropy

[Ver. 1] Holographic EE for Static Spacetimes

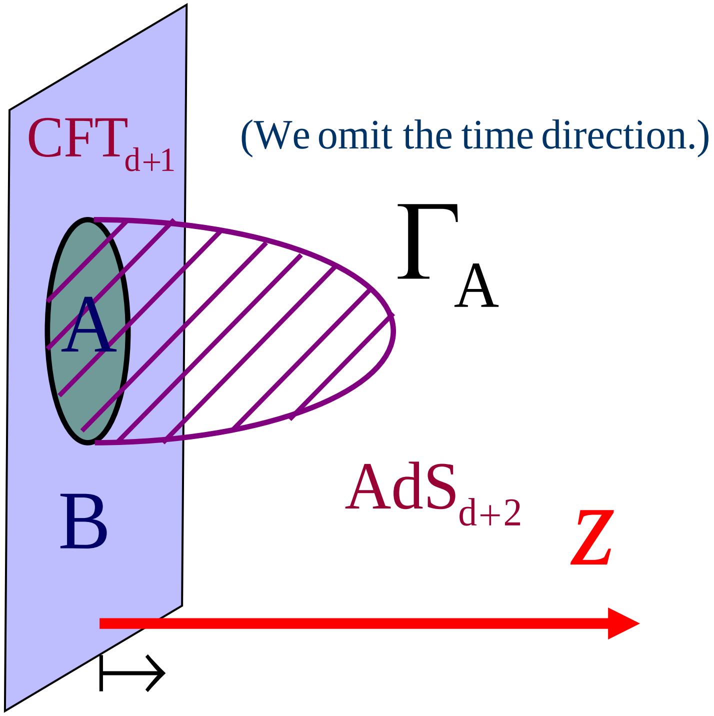
[Ryu-TT 06]

For static asymptotically AdS spacetimes:

$$S_A = \underset{\substack{\partial\Gamma_A = \partial A \\ \Gamma_A \approx A}}{\mathrm{Min}} \left[\frac{\mathrm{Area}(\Gamma_A)}{4G_N} \right]$$

Γ_A is the minimal area surface (codim.=2) on the time slice such that $\partial A = \partial \Gamma_A$ and $A \sim \Gamma_A$.

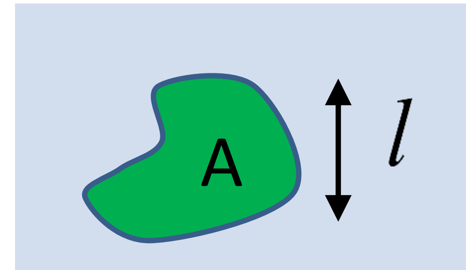
homologous



$z > \varepsilon$ (UV cut off)

General Behavior of HEE(=d+1 dim. CFT EE)

[Ryu–TT 06,⋯]



$$S_A = \frac{\pi^{d/2} R^d}{2G_N^{(d+2)}\Gamma(d/2)}\left[p_1\left(\frac{l}{\varepsilon}\right)^{d-1} + p_3\left(\frac{l}{\varepsilon}\right)^{d-3} + \cdots \right.$$

$$\left. \cdots + \begin{cases} p_{d-1}\left(\frac{l}{\varepsilon}\right) + p_d & (\text{if } d+1 = \text{odd}) \\ p_{d-2}\left(\frac{l}{\varepsilon}\right)^2 + q\log\left(\frac{l}{\varepsilon}\right) & (\text{if } d+1 = \text{even}) \end{cases} \right],$$

where $p_1 = (d-1)^{-1}, p_3 = -(d-2)/[2(d-3)], \ldots.$

$\ldots.. \quad q = (-1)^{(d-1)/2}(d-2)!!/(d-1)!! \quad .$

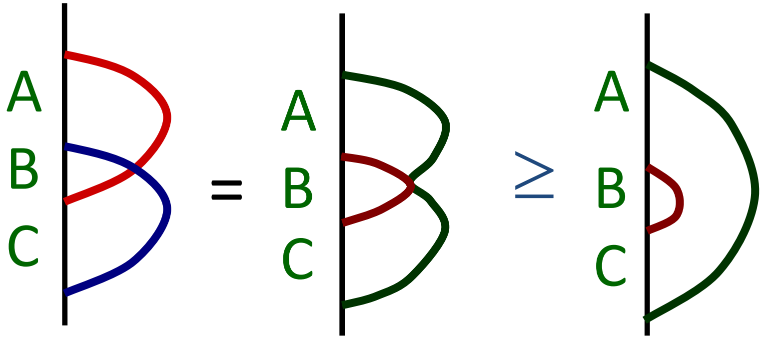
Area law divergence

A universal quantity (**F**) which characterizes **odd dim. CFT**.

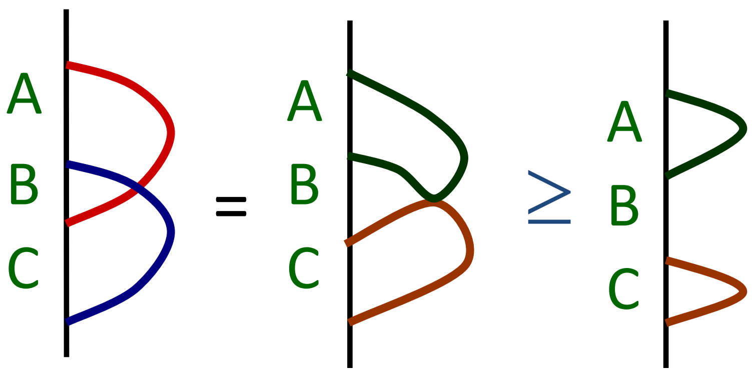
Agrees with **conformal anomaly** (central charge) in **even dim. CFT**

Holographic Proof of Strong Subadditivity

[Headrick−TT 07]



$$\Rightarrow \; S_{AB} + S_{BC} \geq S_{ABC} + S_{B}$$



$$\Rightarrow \; S_{AB} + S_{BC} \geq S_{A} + S_{C}$$

$$(\text{Note}: \quad AB \equiv A \cup B)$$

Algebraic properties in Quantum Information
⇔ **Geometric properties** in Gravity

[Ver. 2] Covariant Holographic Entanglement Entropy

[Hubeny-Rangamani-TT 07]

A generic Lorentzian asymptotic AdS spacetime is dual to a time dependent state $|\Psi(t)\rangle$ in the dual CFT.

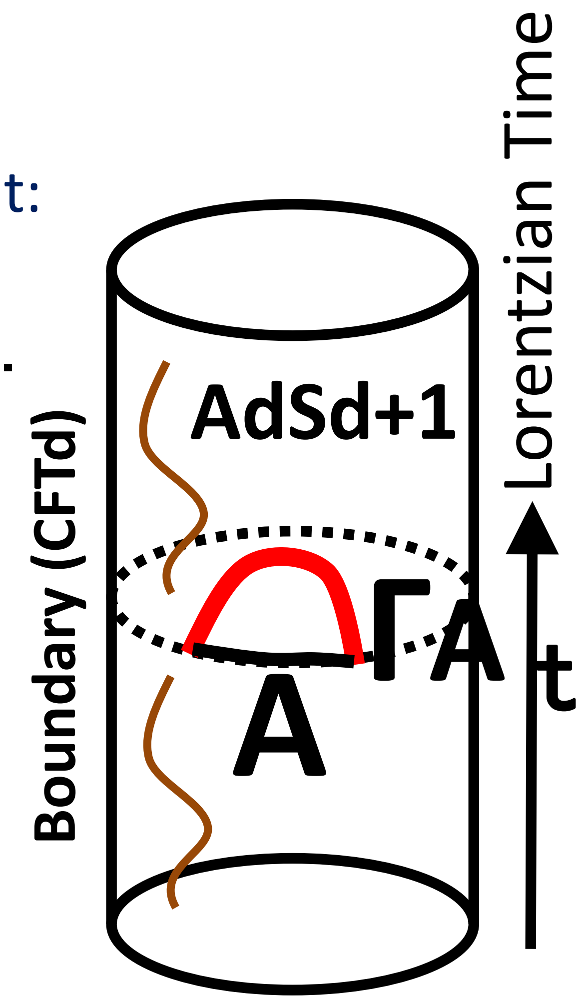
The entanglement entropy gets time-dependent:

$$\rho_A(t) = \mathrm{Tr}_B[|\Psi(t)\rangle\langle\Psi(t)|] \Rightarrow S_A(t).$$

This is computed by the holographic formula:

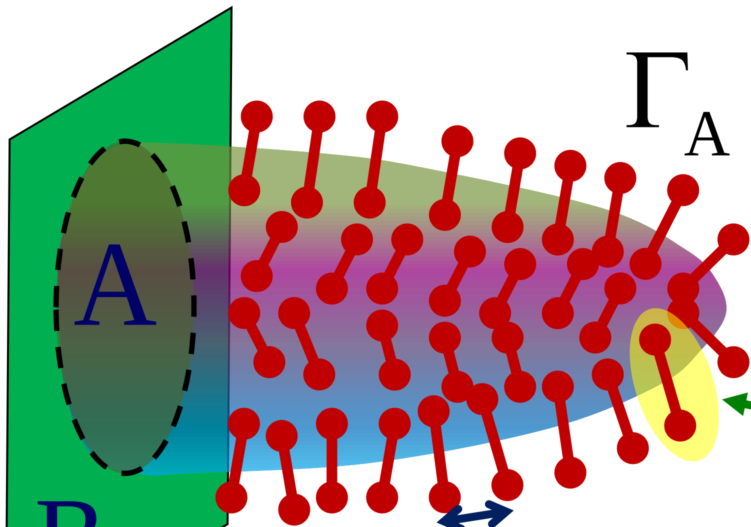
$$S_A(t) = \mathrm{Min}_{\Gamma_A}\mathrm{Ext}_{\Gamma_A}\left[\frac{A(\Gamma_A)}{4G_N}\right]$$

Extremization of the area



② Emergent Space from Entanglement and Complexity

HEE➤ One qubit of entanglement for each Planck length area !



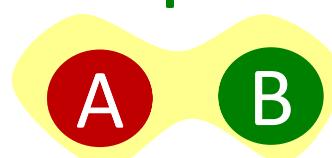
Planck length

$$l_P = \sqrt{\frac{\hbar G}{c^3}} = 1.6\times 10^{-35} m$$

$$S_A = \frac{\mathrm{Area}(\Gamma_A)}{4 l_{pl}^{D-1}}$$

~10^{65} qubits per 1cm^2 !

Bell pair



= Planck scale mini Universe

Spacetime may emerge from entangled Qubits !

➔ Tensor Network (TN) realizes this idea !

Tensor Network (TN)

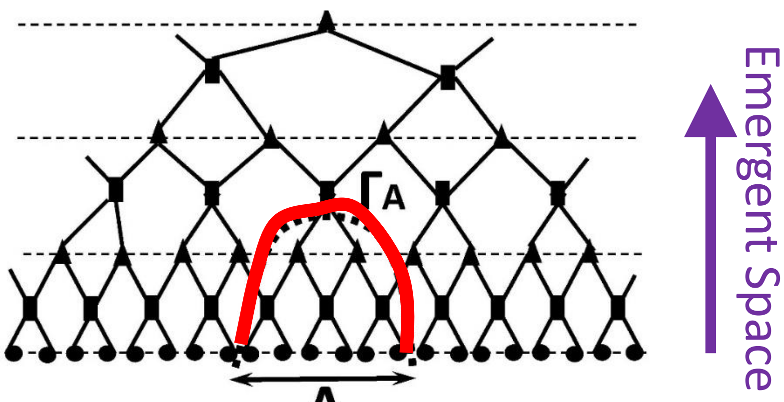
[DMRG: White 92,.. CTM: Nishino-Okunishi 96, PEPS: Verstraete-Cirac 04, …]

TN = Graphical description of quantum states

Quantum State = Network of quantum entanglement

[Ex. 1] MERA TN [Vidal 2005]

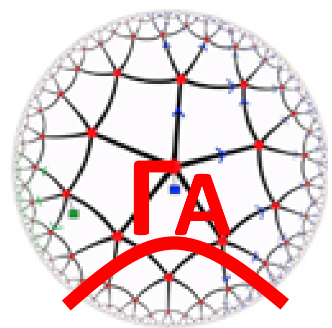
➔ Describe CFT vacuum



S_A = Minimal Cross Section of TN

[Ex. 2] HaPPY model

[Patawski-Yoshida-Harlow-Preskill 2015]



➔ Use quantum error correcting code

[Ex. 3] Random TN model

[Hayden-Nezami-Qi-Thomas -Walter-Yang 2016]

Geometric Structure of Qubits = AdS

[Swingle 2009,…]

[Ex.4 Path-integral Optimization]

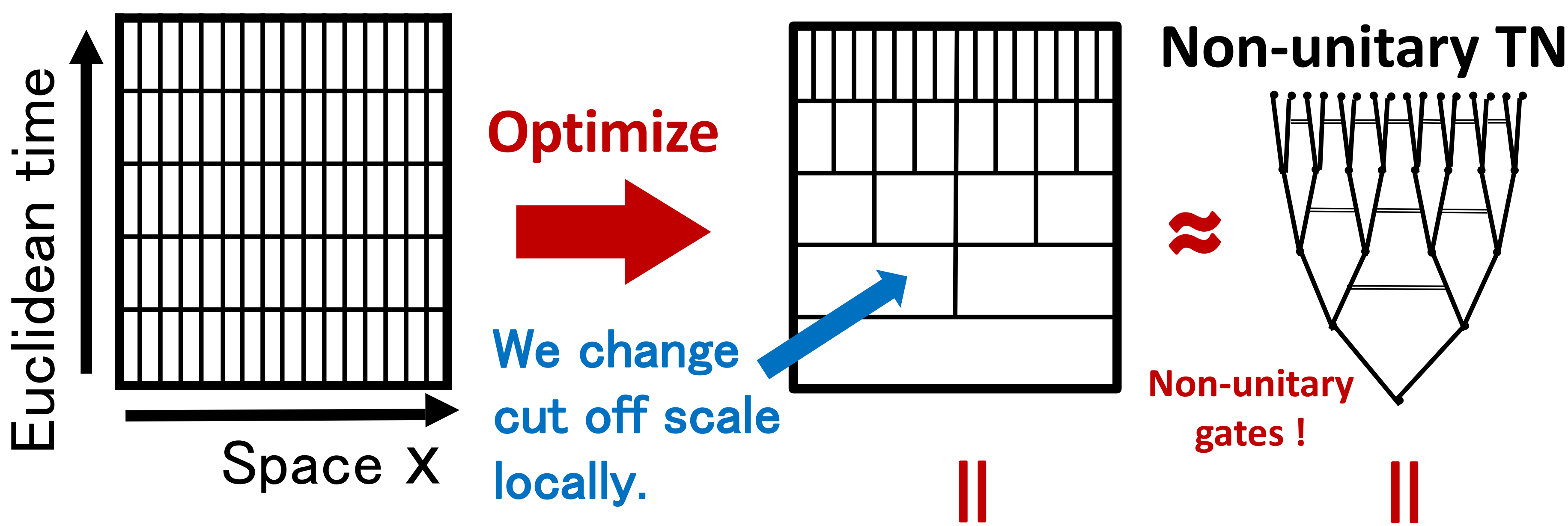
[Caputa-Kundu-Miyaji-Watanabe-TT 2017]
[Boruch-Caputa-Ge-TT 2021]

- For AdS/CFT we need continuum theory as TN
- Time slice of AdS must be Euclidean (non-unitary)

⇒ **Feynman Path-integral !**

Basic Principle

Minimize the computational cost of (discretized) path-integral.

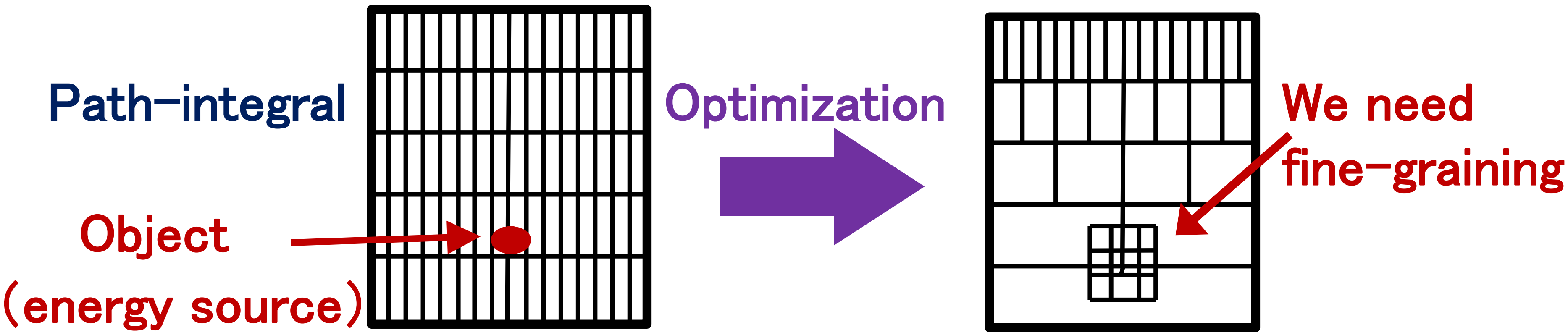


$$\lim_{T\to\infty} e^{-T\cdot H}|\psi\rangle = |\psi_0\rangle$$

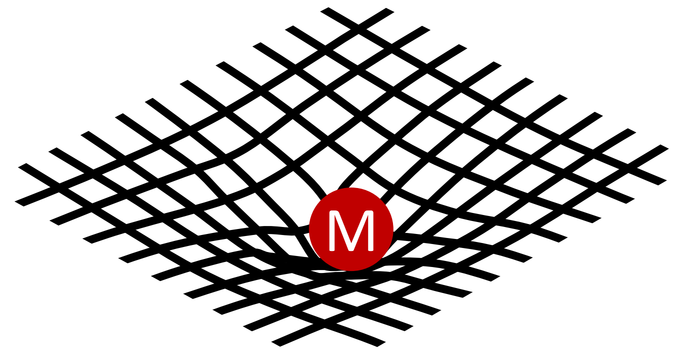
ground state

A time slice of AdS emerges

Upshot : Minimizing computational costs leads to gravity



Energetic source (=information source) distorts the spacetime
→ The essence of general relativity !



e^{-itH} $e^{-\beta H}$

Comment

Emergent space in AdS/CFT seems to be related to non-unitary gates.

→ Relevant complexity class might be like PostBQP.

[Thanks to Tomoyuki Morimae and Ryu Hayakawa for suggestions]

③ Pseudo Entropy

<u>Question: Ver 3. Holographic Entropy Formula ?</u>

Minimal areas in ***Euclidean time dependent*** asymptotically AdS spaces

= What kind of QI quantity (Entropy ?) in CFT ?

➡ **The answer is Pseudo Entropy !**

(3-1) Definition of Pseudo (Renyi) Entropy

Consider two quantum states $|\psi\rangle$ and $|\varphi\rangle$, and define the *transition matrix*:

$$\tau^{\psi|\varphi} = \frac{|\psi\rangle\langle\varphi|}{\langle\varphi|\psi\rangle}.$$

We decompose the Hilbert space as $H_{tot} = H_A \otimes H_B$.

and introduce the reduced transition matrix:

$$\tau_A^{\psi|\varphi} = \mathrm{Tr}_B\left[\tau^{\psi|\varphi}\right]$$

Pseudo Entropy

$$S\left(\tau_A^{\psi|\varphi}\right) = -\mathrm{Tr}\left[\tau_A^{\psi|\varphi}\log\tau_A^{\psi|\varphi}\right].$$

Renyi Pseudo Entropy

$$S^{(n)}\left(\tau_A^{\psi|\varphi}\right) = \frac{1}{1-n}\log\mathrm{Tr}\left[\left(\tau_A^{\psi|\varphi}\right)^n\right].$$

(3-2) Basic Properties of Pseudo Entropy (PE)

- In general, $\tau_A^{\psi|\varphi}$ is not Hermitian. Thus PE is complex valued.

- If either $|\psi\rangle$ or $|\varphi\rangle$ has no entanglement (i.e. direct product state), then
$$S^{(n)}\left(\tau_A^{\psi|\varphi}\right) = 0.$$

- We can show $$S^{(n)}\left(\tau_A^{\psi|\varphi}\right) = \left[S^{(n)}\left(\tau_A^{\varphi|\psi}\right)\right]^{\dagger}.$$

- We can show $$S^{(n)}\left(\tau_A^{\psi|\varphi}\right) = S^{(n)}\left(\tau_B^{\psi|\varphi}\right).$$ →"SA=SB"

- If $|\psi\rangle = |\varphi\rangle$, then $S^{(n)}\left(\tau_A^{\psi|\varphi}\right)$ = Renyi entropy.

Comment: In quantum theory, transition matrices arise when we consider *post-selection*.

$$\frac{\langle\varphi|O_A|\psi\rangle}{\langle\varphi|\psi\rangle} = \mathrm{Tr}[O_A \tau_A^{\psi|\varphi}]$$

Final state after post-selection

Initial State

This quantity is called **weak value** and is complex valued in general.

[Aharanov-Albert-Vaidman 1988,…]

Thus, **pseudo entropy =weak value of "modular operator"**:

= Area Operator

$$S\left(\tau_A^{\psi|\varphi}\right) = \frac{\langle\varphi|H_A|\psi\rangle}{\langle\varphi|\psi\rangle}.$$

H_A=-logτ_A

(3-3) Holographic Pseudo Entropy (Ver.3 formula)

Holographic Pseudo Entropy (HPE) Formula

$$S\left(\tau_A^{\psi|\varphi}\right) = \mathrm{Min}_{\Gamma_A}\left[\frac{A(\Gamma_A)}{4G_N}\right]$$

Basic Propertie

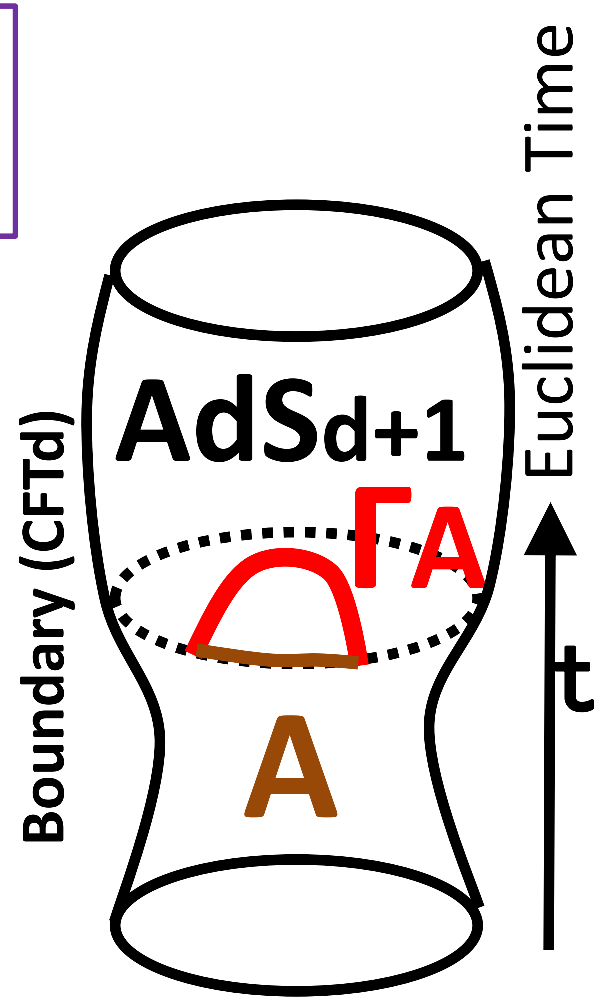
(i) If ρ_A *is pure*, $S\left(\tau_A^{\psi|\varphi}\right)$ =0.

(ii) If ψ *or* φ *is not entangled*,

$S\left(\tau_A^{\psi|\varphi}\right)$ =0.

→This follows from AdS/BCFT [TT 2011]

(iii) $S\left(\tau_A^{\psi|\varphi}\right) = S\left(\tau_B^{\psi|\varphi}\right)$. **"SA=SB"**



(3-4) Pseudo Entropy as Entanglement Distillation

In the special case where we can diagonalize $|\psi\rangle$ and $|\varphi\rangle$ at the same time, PE has a nice interpretation:

$$|\psi\rangle = \cos\theta_1|00\rangle + \sin\theta_1|11\rangle, \qquad |\varphi\rangle = \cos\theta_2|00\rangle + \sin\theta_2|11\rangle$$

$$\Rightarrow S\left(\tau_A^{\psi|\varphi}\right) = -\frac{\cos\theta_1\cos\theta_2}{\cos(\theta_1-\theta_2)}\cdot\log\frac{\cos\theta_1\cos\theta_2}{\cos(\theta_1-\theta_2)} - \frac{\sin\theta_1\sin\theta_2}{\sin(\theta_1-\theta_2)}\cdot\log\frac{\sin\theta_1\sin\theta_2}{\sin(\theta_1-\theta_2)}$$

In this case, we can show

$$S\left(\tau_A^{\psi|\varphi}\right) \approx \sum_k p_k \cdot \text{ \#of Bell Pairs in } |\mathrm{Bell_k}\rangle$$
$$\approx \text{\#of Bell Pairs in intermediate states}$$

$$\frac{\langle\varphi|\sum_k|\mathrm{Bell}_k\rangle\langle\mathrm{Bell}_k|\ |\psi\rangle}{\langle\varphi|\ |\psi\rangle} = p_k$$

(3-5) SVD entropy [Parzygnat-Taki-Wei-TT 2023]

Motivation: Improve PE so that (i) it become real and non-negative and (ii) it has a better LOCC interpretation.

SVD entropy

$$S_{SVD}\left(\tau_A^{\psi|\varphi}\right) = -\mathrm{Tr}\left[|\tau_A^{\psi|\varphi}|\cdot \log|\tau_A^{\psi|\varphi}|\right].$$

here, $|\tau_A^{\psi|\varphi}| \equiv \sqrt{\tau_A^{\dagger\psi|\varphi}\tau_A^{\psi|\varphi}}$

- This is always non-negative and is bounded by log dim H_A.

- From quantum information theoretic viewpoint, this is the number of Bell pairs distilled from the intermediate state:

$$\tau_A^{\psi|\varphi} = \mathrm{U}\cdot\Lambda\cdot\mathrm{V}, \qquad \frac{\langle\phi|\mathrm{V}^\dagger \ \sum_k|\mathrm{EPR}_k\rangle\langle\mathrm{EPR}_k| \ \mathrm{U}^\dagger\ |\psi\rangle}{\langle\phi|\mathrm{V}^\dagger\mathrm{U}^\dagger\ |\psi\rangle} = p_k, \quad \sum_k p_k = 1$$

$$S_{SVD} \approx \sum_k p_k \cdot \#\text{of Bell Pairs in}\ |\mathrm{EPR}_k\rangle$$

④ Pseudo Entropy and Quantum Phase Transitions

[Mollabashi–Shiba–Tamaoka–Wei–TT 20, 21]

(4-1) Basic Properties of Pseudo entropy in QFTs

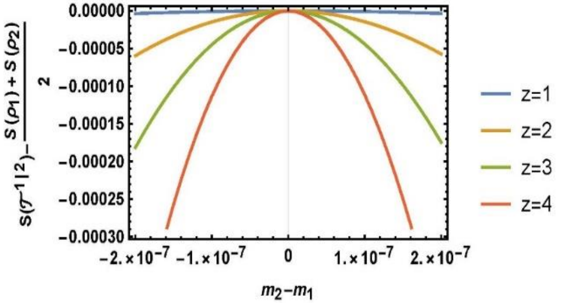
[1] Area law

$$S_A \sim \frac{\mathrm{Area}(\partial A)}{\varepsilon^{d-1}} + (\text{subleading terms}),$$

[2] The difference

$$\Delta S = \mathrm{Re}\left[S\left(\tau_A^{1|2}\right)\right] - \left(S(\rho_A^1) + S(\rho_A^2)\right)/2$$

is negative if $|\psi_1\rangle$ and $|\psi_2\rangle$ are in a same phase.



PE in a 2 dim. free scalar when we change its mass.

➡ **What happen if they belong to different phases ?**
Can ΔS be positive ?

(4-2) Quantum Ising Chain with a transverse magnetic field

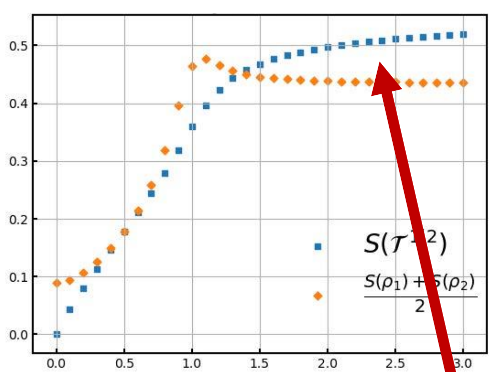
$$H = -J \sum_{i=0}^{N-1} \sigma_i^z \sigma_{i+1}^z - h \sum_{i=0}^{N-1} \sigma_i^x ,$$

Ψ1→ vacuum of H(J1)
Ψ2→ vacuum of H(J2)
(We always set h=1)

J<1 Paramagnetic Phase
J>1 Ferromagnetic Phase

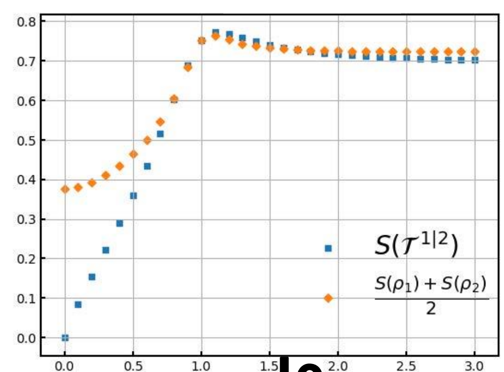
N=16, NA=8

J1=1/2



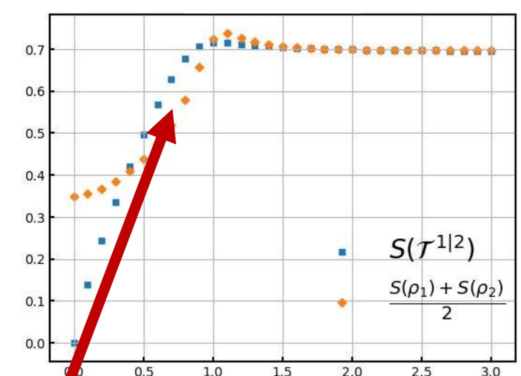
J2

J1=1



J2

J1=2

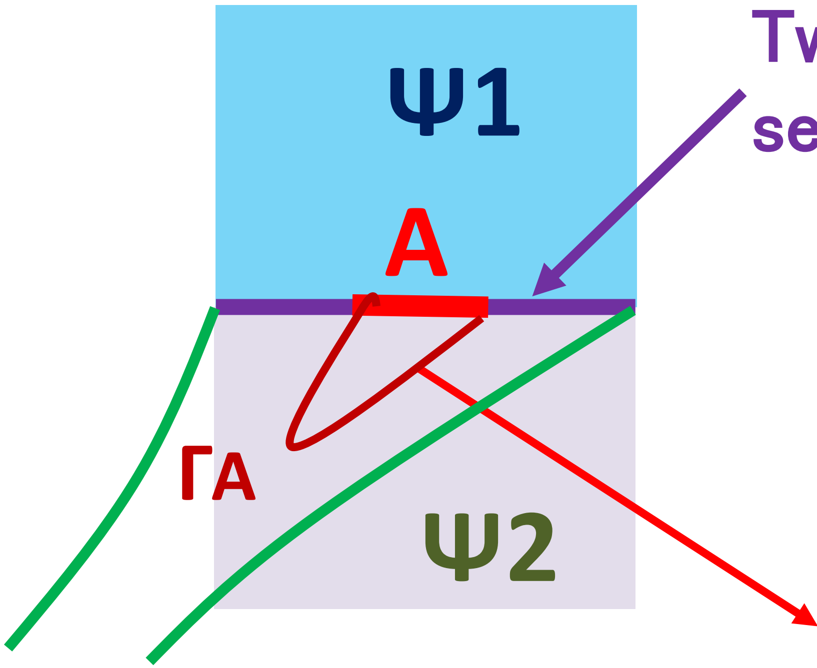


J2

$$\Delta S = S\left(\tau_A^{1|2}\right) + S\left(\tau_A^{1|2}\right) - S(\rho_A^1) - S(\rho_A^2) > 0$$

We find
when Ψ1 and Ψ2 are in different phases !

Heuristic Interpretation



AdS Dual of
Gapless Interface

Two gapped phases are separated by a gapless phase.
CFT !

PE is enhanced ! $\Delta S > 0$

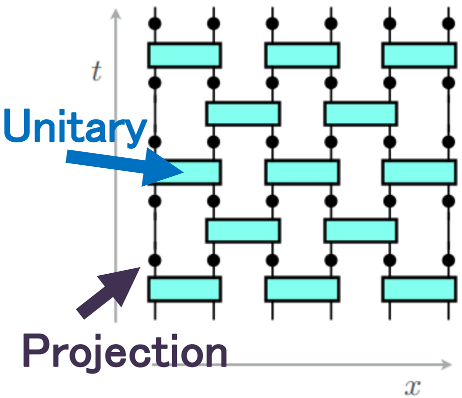
The gapless interface (edge state) also occurs in topological orders.
➔Topological pseudo entropy [Nishioka-Taki-TT 2021].

⑤ Pseudo Entropy and Entanglement Phase Transition

[Kanda–Sato–Suzuki–Wei–TT 2023] [Kanda–Kawamoto–Suzuki–Wei–TT 2023]

Entanglement Phase Transition [Skinner–Ruhman–Nahum, Li–Chen–Fisher 2018]

(Measurement Induced Phase Transition)

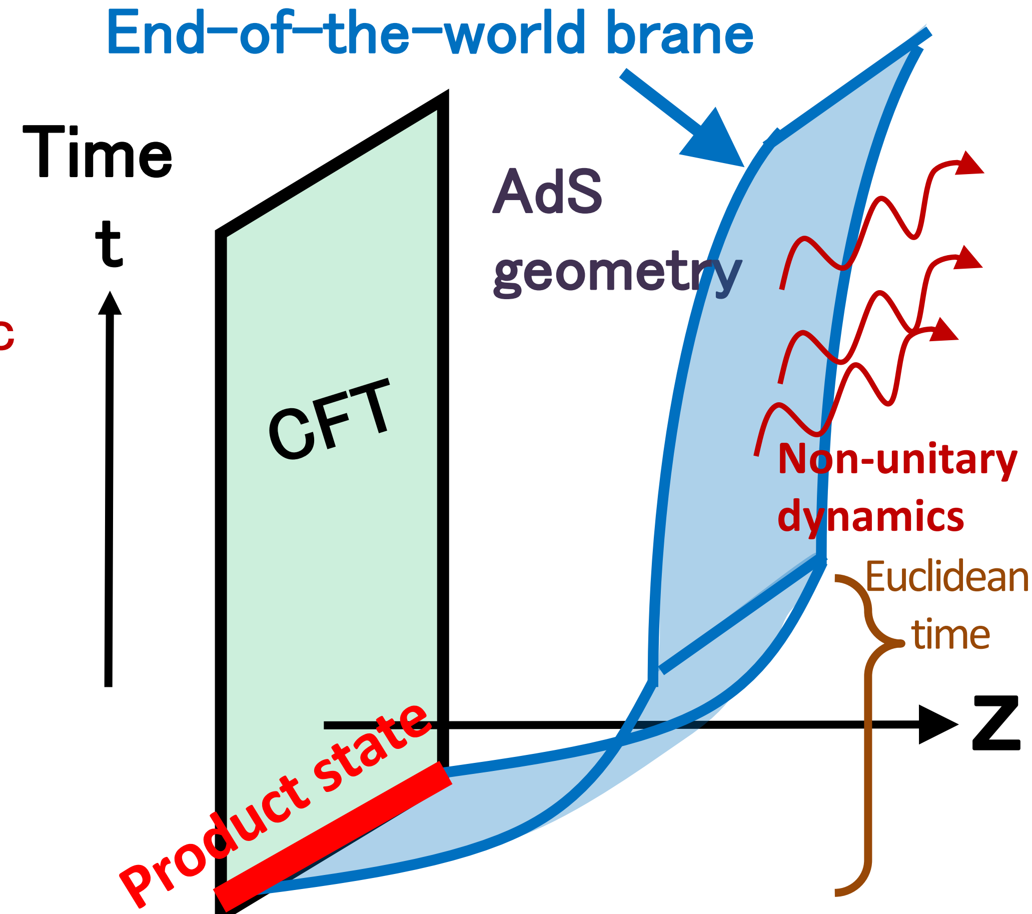


$(i)\quad p < p_* : \quad S_A \propto t,$

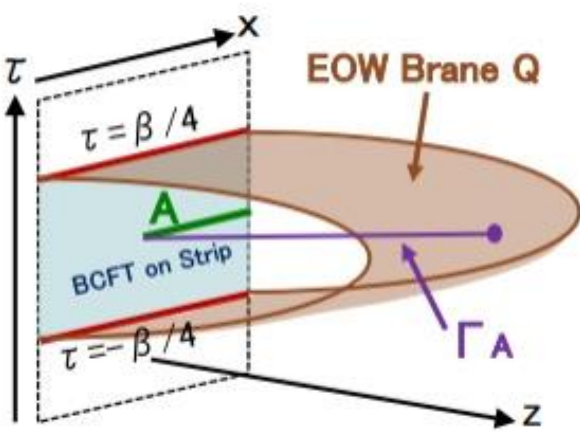
$(ii)\quad p = p_* : \quad S_A \propto \log t,$

$(iii)\quad p > p_* : \quad S_A = \text{finite},$

Holographic Model ?



Double Wick Rotation and Time Evolution



τ =it

Imaginary valued scalar field on EOW brane ➔ Disscipation

$$\rho(t) = e^{-\left(\frac{\beta}{4}+it\right)H} \left| B(\varphi_0 + \Delta\varphi) \right\rangle \left\langle B(\varphi_0) \right| e^{-\left(\frac{\beta}{4}-it\right)H}$$

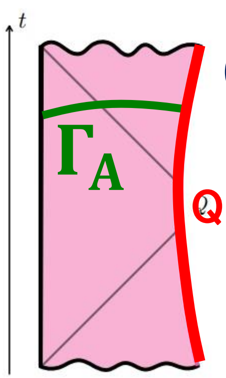
|B>=Boundary state ～ Direct product state

A= a half space

$\Delta\phi < \Delta\phi_*$

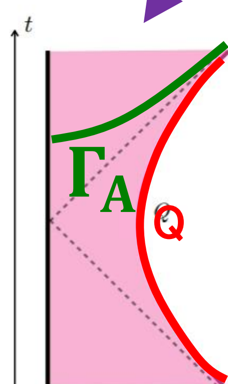
$\Delta\phi = \Delta\phi_*$

$\Delta\phi > \Delta\phi_*$



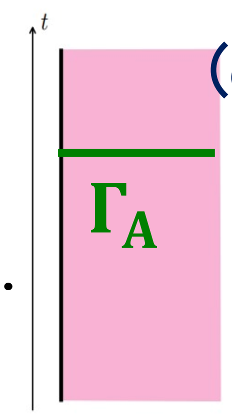
(a) BTZ phase

$$S_A \approx \frac{c}{6} \cdot \frac{\pi}{2\alpha} \cdot t$$



(b) Poincare AdS phase

$$S_A \approx \frac{c}{3} \cdot \log t.$$



(c) Thermal AdS phase

$$S_A \approx \text{const.}$$

⑥ Conclusions

Pseudo entropy (PE) is a generalization of entanglement entropy.

- PE depends on both the initial and final state.
- PE is in general complex valued.
- PE for diagonalizable states measures the amount of quantum entanglement in the intermediate states.
- In AdS/CFT, PE is equal to the minimal surface area.
  ➡ Emergence of space from PE
- PE can detect the difference of quantum phases.
  ➡ New quantum order parameter

Future directions

- Quantum information meaning of the complex values of PE ?

  Gravity formula suggests that if EE is useful in QI, so does PE…

  We might think “pseudo pseudo entanglement”.

- Implications to quantum gravity including de Sitter spaces ?

  ▶Imaginary part of PE= Emergent time in holography ?

  [Doi-Harper-Mollabashi-Taki-TT. PRL 130(2023)031601]

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Thank you very much ！