Homological Quantum Rotor Codes: Logical Qubits form Torsion

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Why **Quantum Rotors**?

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Hardware

Quantum systems in the lab often are not qubits

⇒ Design error correction closer to hardware

⇒ In SC circuits **Josephson junction = quantum rotor**

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Continuous Variable Error Correction

Exploring error correction of infinite dimensional systems

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- ⇒ Design error correction closer to hardware
- ⇒ In SC circuits **Josephson junction = quantum rotor**

Continuous Variable Error Correction

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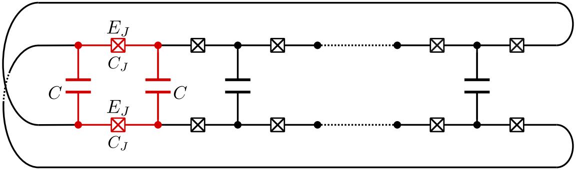
Homology

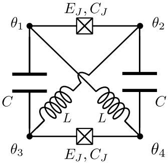
Quantum codes have a close relation to homology

- ⇒ Homology with integer coefficients is a rich playground

Hardware



Protected superconducting qubits[1] are closely related to quantum rotor codes



[1] Kitaev, "Protected qubit based on a superconducting current mirror", 2006
Brooks, Kitaev, Preskill, "Protected gates for superconducting qubits", PRA, 2013

Continuous Variable Error Correction

Doing measurements/operations agreeing with the group structure

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Quantum Oscillators $(x, p \in \mathbb{R}^2)$

- Oscillators into oscillators against discrete errors[2](good)
- Oscillators into oscillators against gaussian noise[3] (no good)

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[3] Vuillot et al "Quantum error correction with the toric GKP code", PRA, 2019

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Quantum Rotors $(\ell, \theta \in \mathbb{Z} \times \mathbb{T})$

- Rotors versions of toric/Haah codes[4]
- $U(1)$ covariant reference frame codes[5]

[2] Lloyd, Slotine, "Analog quantum error correction", PRL, 1998

[3] Vuillot et al "Quantum error correction with the toric GKP code", PRA, 2019

[4] Albert et al "General phase spaces: From discrete variables to rotor and continuum limits", JPA, 2017

[5] Hayden et al "Error Correction of Quantum Reference Frame Information," PRX Quantum, 2021

Modular Error Correction

Doing modular measurements, not agreeing with the group structure

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- Oscillators into oscillators [6]

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[7] GKP, "Encoding a qubit in an oscillator," PRA, 2001

Modular Error Correction

Doing modular measurements, not agreeing with the group structure

- Oscillators into oscillators [6]
- Qubits into oscillators[7]
- Qubits into molecules (including rotors)[8]

[6] Noh et al "Encoding an oscillator into many oscillators" PRL 2020
Hänggli et al "Oscillator-to-Oscillator Codes Do Not Have a Threshold", IEEEtit, 2022

[7] GKP, "Encoding a qubit in an oscillator," PRA, 2001

[8] Albert et al "Robust encoding of a qubit in a molecule," PRX, 2020

Homological Quantum Rotor Codes

- The physical system is a collection of quantum rotors
- “CV” error correction, no modular measurements
- Encodes qubits and quantum rotors

Outline

Hilbert Space $\mathcal{H}_\mathbb{Z}$

- Orthonormal Basis

$$\forall \ell \in \mathbb{Z}, \quad |\ell\rangle \in \mathcal{H}_\mathbb{Z}$$

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- States

$$|\psi\rangle = \sum_{\ell \in \mathbb{Z}} \alpha_\ell |\ell\rangle, \quad \sum_{\ell \in \mathbb{Z}} |\alpha_\ell|^2 = 1$$

Dual Representation

States

$$\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}, \quad |\psi\rangle = \int_{\theta\in\mathbb{T}} \mathrm{d}\theta\, \psi(\theta)\, |\theta\rangle\,, \quad \int_{\theta\in\mathbb{T}} \mathrm{d}\theta\, |\psi(\theta)|^2 = 1$$

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Fourier Series

$$\forall\, |\psi\rangle \in \mathcal{H}_{\mathbb{Z}}, \forall \theta \in \mathbb{T}, \quad \psi(\theta) = \frac{1}{\sqrt{2\pi}} \sum_{\ell\in\mathbb{Z}} \alpha_\ell \mathrm{e}^{i\theta\ell}$$

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Phase States

$$\forall\theta \in \mathbb{T}, \quad |\theta\rangle = \frac{1}{\sqrt{2\pi}} \sum_{\ell\in\mathbb{Z}} \mathrm{e}^{-i\theta\ell}\, |\ell\rangle$$

Generalized Pauli Operators

Pauli X: Jumps

$$\forall m \in \mathbb{Z}, \quad X(m)\,|\ell\rangle = |\ell + m\rangle$$
$$X(m)\,|\theta\rangle = \mathrm{e}^{i\theta m}\,|\theta\rangle$$

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$$\forall \phi \in \mathbb{T}, \quad Z(\phi)\,|\ell\rangle = \mathrm{e}^{i\phi\ell}\,|\ell\rangle$$
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Relations

- $\mathbb{1} = X(0) = Z(0)$
- $X(m_1)X(m_2) = X(m_1 + m_2)$
- $Z(\phi_1)Z(\phi_2) = Z(\phi_1 + \phi_2)$

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$$j \in [n], m \in \mathbb{Z}, \quad X_j(m) = \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes X(m) \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1}$$
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Multi-Rotor Pauli Operators

$$\boldsymbol{m} \in \mathbb{Z}^n, \quad X(\boldsymbol{m}) = \prod_{j=1}^{n} X_j(m_j)$$

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$$X(\boldsymbol{m})Z(\boldsymbol{\phi}) = e^{-i\boldsymbol{\phi}\cdot\boldsymbol{m}^T} Z(\boldsymbol{\phi})X(\boldsymbol{m})$$

Quantum Rotor Code

Definition

Given $H_X \in \mathbb{Z}^{r_X \times n}$ and $H_Z \in \mathbb{Z}^{r_Z \times n}$, such that

$$H_X H_Z^T = 0,$$

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Definition

Given $H_X \in \mathbb{Z}^{r_X \times n}$ and $H_Z \in \mathbb{Z}^{r_Z \times n}$, such that

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define stabilizer generators and the stabilizer group

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- $\mathcal{S} = \langle S_Z(\phi) S_X(\boldsymbol{s}) \mid \forall \phi \in \mathbb{T}^{r_z}, \forall \boldsymbol{s} \in \mathbb{Z}^{r_x} \rangle.$

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The corresponding quantum rotor code is defined as

$$\mathcal{C}^{\mathrm{rot}}(H_X, H_Z) = \{ |\psi\rangle \mid \forall P \in \mathcal{S},\ P|\psi\rangle = |\psi\rangle \}$$

Commutation and Small Example

Stabilizers Commute

$$S_X(\boldsymbol{s})S_Z(\phi) = \mathrm{e}^{-i\phi \cancel{H_Z H_X^T} \boldsymbol{s}^T} S_Z(\phi) S_X(\boldsymbol{s})$$

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4-Rotors Example

$$H_X = \begin{pmatrix} +1 & -1 & 0 & 0 \\ 0 & 0 & +1 & -1 \\ -1 & -1 & +1 & +1 \end{pmatrix} \qquad H_Z = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}$$

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$$\mathcal{S} = \Big\langle X_1(m)X_2^\dagger(m),\ X_3(m)X_4^\dagger(m),\ X_1^\dagger(m)X_2^\dagger(m)X_3(m)X_4(m), \\ Z_1(\phi)Z_2(\phi)Z_3(\phi)Z_4(\phi)\Big\rangle_{m\in\mathbb{Z},\phi\in\mathbb{T}}$$

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Z Constraints

$$\forall \phi,\ |\overline{\psi}\rangle = S_Z(\phi)\,|\overline{\psi}\rangle$$
$$\Rightarrow \sum_{\boldsymbol{\ell}\in\mathbb{Z}^n} \alpha_{\boldsymbol{\ell}}\,|\boldsymbol{\ell}\rangle = \sum_{\boldsymbol{\ell}\in\mathbb{Z}^n} \mathrm{e}^{i\phi H_Z\cdot\boldsymbol{\ell}^T}\alpha_{\boldsymbol{\ell}}\,|\boldsymbol{\ell}\rangle$$
$$\Rightarrow \forall \ell,\ \alpha_{\boldsymbol{\ell}} \neq 0 \Rightarrow \boldsymbol{\ell} \in \ker\left(H_Z\right).$$

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$$\Rightarrow \sum_{\boldsymbol{\ell}\in\mathbb{Z}^n} \alpha_{\boldsymbol{\ell}}\,|\boldsymbol{\ell}\rangle = \sum_{\boldsymbol{\ell}\in\mathbb{Z}^n} \alpha_{\boldsymbol{\ell}}\,|\boldsymbol{\ell} + \boldsymbol{s}H_X\rangle$$
$$\Rightarrow \forall \boldsymbol{\ell}, \boldsymbol{s},\ \alpha_{\boldsymbol{\ell}} = \alpha_{\boldsymbol{\ell}-\boldsymbol{s}H_X} \Rightarrow \ker(H_Z)/\mathrm{im}(H_X).$$

Homology

Chain Complex

$$\mathcal{C}: \quad C_2 \xrightarrow{\partial} C_1 \xrightarrow{\sigma} C_0 \quad \text{with } \sigma \circ \partial = 0$$

Homology

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$$\mathcal{C}: \quad \begin{matrix} C_2 & \xrightarrow{\partial} & C_1 & \xrightarrow{\sigma} & C_0 \\ \parallel & H_X & \parallel & H_Z^T & \parallel \\ \mathbb{Z}^{r_x} & & \mathbb{Z}^n & & \mathbb{Z}^{r_z} \end{matrix} \quad \text{with } \sigma \circ \partial = 0$$

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$$\mathcal{C}: \quad \begin{array}{ccccc} C_2 & \xrightarrow[H_X]{\partial} & C_1 & \xrightarrow[H_Z^T]{\sigma} & C_0 \\ \shortparallel & & \shortparallel & & \shortparallel \\ \mathbb{Z}^{r_x} & & \mathbb{Z}^n & & \mathbb{Z}^{r_z} \\ \shortparallel & & \shortparallel & & \shortparallel \\ \text{stabilizers} & & \text{operators} & & \text{syndrome} \end{array} \quad \text{with } \sigma \circ \partial = 0$$

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Homology Group $=$ X Logical Operators

$$\begin{aligned} H_1(\mathcal{C}, \mathbb{Z}) &= \ker \sigma / \mathrm{im} \partial = \ker (H_Z) / \mathrm{im} (H_X) \\ &= F \oplus T \\ &= \mathbb{Z}^k \oplus \left(\mathbb{Z}_{d_1} \oplus \cdots \oplus \mathbb{Z}_{d_{k'}} \right) = \mathcal{L}_X \end{aligned}$$

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$$\forall \boldsymbol{m} \in \mathcal{L}_X,\ \overline{X}(\boldsymbol{m}) = X(\boldsymbol{m} L_X + \boldsymbol{s} H_X), \quad L_X \in \mathbb{Z}^{(k+k')\times n}$$

Example

$$H_X = \begin{pmatrix} +1 & -1 & 0 & 0 \\ 0 & 0 & +1 & -1 \\ -1 & -1 & +1 & +1 \end{pmatrix} \qquad H_Z = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}$$

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$$\boldsymbol{x} = \begin{pmatrix} 0 & -1 & +1 & 0 \end{pmatrix} \in \ker(H_Z)$$
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$$\boldsymbol{e}H_X = d\boldsymbol{w},\ \boldsymbol{w} \notin \mathrm{im}(H_X) \Rightarrow \mathbb{Z}_d \subset \mathcal{L}_X$$

Cohomology with $\mathbb{T}$ Coefficients

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where

$$\begin{array}{ccccc} C_j^* & = & \mathrm{Hom}(C_j, \mathbb{T}) & & \\ \partial_j^* & : & C_{j-1}^* & \longrightarrow & C_j^* \\ & & \phi & \mapsto & \phi \circ \partial \end{array}$$

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$$\mathrm{Hom}(\mathbb{Z}, \mathbb{T}) \simeq \mathbb{T}, \quad \partial^* = H_X^T, \quad \sigma^* = H_Z$$

Our Case

Cochain Complex

$$\mathcal{C}^* : \quad C_2^* \xleftarrow[H_X^T]{\partial^*} C_1^* \xleftarrow[H_Z]{\sigma^*} C_0^*$$

$$C_2^* = \mathbb{T}^{r_x} = \text{syndrome}, \quad C_1^* = \mathbb{T}^{n} = \text{operators}, \quad C_0^* = \mathbb{T}^{r_z} = \text{stabilizers}$$

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Cohomology Group $=$ Z Logical Operators

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H^1(\mathcal{C}, \mathbb{T}) &= \ker \partial^* / \mathrm{im} \sigma^* = \ker (H_X) / \mathrm{im} (H_Z) \\
&= \mathbb{T}^k \oplus \left(\mathbb{Z}^*_{d_1} \oplus \cdots \oplus \mathbb{Z}^*_{d_{k'}} \right) \\
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Our Case

Cochain Complex

$$\mathcal{C}^*: \quad \begin{array}{ccccc} C_2^* & \overset{\partial^*}{\underset{H_X^T}{\longleftarrow}} & C_1^* & \overset{\sigma^*}{\underset{H_Z}{\longleftarrow}} & C_0^* \\ \| & & \| & & \| \\ \mathbb{T}^{r_x} & & \mathbb{T}^{n} & & \mathbb{T}^{r_z} \\ \| & & \| & & \| \\ \text{syndrome} & & \text{operators} & & \text{stabilizers} \end{array}$$

Cohomology Group $=$ Z Logical Operators

$$\begin{aligned} H^1(\mathcal{C}, \mathbb{T}) &= \ker \partial^* / \mathrm{im} \sigma^* = \ker(H_X) / \mathrm{im}(H_Z) \\ &= \mathbb{T}^k \oplus \left(\mathbb{Z}_{d_1}^* \oplus \cdots \oplus \mathbb{Z}_{d_{k'}}^* \right) \\ &= \mathcal{L}_Z \end{aligned}$$

$$\forall \phi \in \mathcal{L}_Z,\ \overline{Z}(\phi) = Z(\phi L_Z + \nu H_Z), \quad L_Z \in \mathbb{Z}^{(k+k')\times n}$$

Example

$$H_X = \begin{pmatrix} +1 & -1 & 0 & 0 \\ 0 & 0 & +1 & -1 \\ -1 & -1 & +1 & +1 \end{pmatrix} \qquad H_Z = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}$$

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A Logical Qubit

$$\overline{X} = X\left(\begin{pmatrix} 0 & -1 & +1 & 0 \end{pmatrix}\right), \qquad \overline{Z} = Z\left(\pi \begin{pmatrix} 1 & 1 & 0 & 0 \end{pmatrix}\right)$$

Noise Models

Pauli Noise

$$\forall m \in \mathbb{Z},\ \mathbb{P}\left(X(m)\right) = N_X \exp\left(-\beta_X V_X(m)\right),$$
$$\forall \phi \in \mathbb{T},\ \mathbb{P}\left(Z(\phi)\right) = N_Z \exp\left(-\beta_Z V_Z(\phi)\right).$$

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Possible Choice

$$V_Z(\phi) = \sin^2\left(\frac{\phi}{2}\right) \qquad \beta_Z = \frac{1}{\sigma^2}$$
$$V_X(m) = |m| \qquad \beta_X = -\log p$$

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Weight Function

$$W_Z(\boldsymbol{\phi}) = \sum_{j=1}^{n} V_Z(\phi_j) = \sum_{j=1}^{n} \sin^2\left(\frac{\phi_j}{2}\right)$$
$$W_X(\boldsymbol{m}) = \sum_{j=1}^{n} V_X(m_j) = ||\boldsymbol{m}||_1$$

Distances

X Distance

$$d_X = \min_{\boldsymbol{m} \neq \boldsymbol{0}} \min_{\boldsymbol{s} \in \mathbb{Z}^{r_X}} W_X(\boldsymbol{m} L_X + \boldsymbol{s} H_X)$$

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Z Distances

$$\delta_Z = \min_{\phi\neq 0} \min_{\boldsymbol{\nu}\in\mathbb{T}^{r_Z}} \frac{W_Z(\phi L_Z + \boldsymbol{\nu}H_Z)}{W_Z(\phi)}$$

X Bound

X Distance

Given a quantum rotor code $\mathcal{C}^{\mathrm{rot}}(H_X, H_Z)$, denote as d_X^p the X distance of the corresponding qupit code $\mathcal{C}^p(H_X, H_Z)$, then

$$d_X \geq \max_{p \in P} d_X^p,$$

where P is the set of qupit dimensions for which there exists a logical X of minimal weight in $\mathcal{C}^{\mathrm{rot}}$ non trivial in $\mathcal{C}^p$.

Spreading Z Operators

$$H_X = \begin{pmatrix} +1 & -1 & 0 & 0 \\ 0 & 0 & +1 & -1 \\ -1 & -1 & +1 & +1 \end{pmatrix} \qquad H_Z = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}$$

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$$\begin{aligned} \boldsymbol{z} &= \begin{pmatrix} \pi & \pi & 0 & 0 \end{pmatrix} - \frac{\pi}{2} H_Z \\ &= \begin{pmatrix} \frac{\pi}{2} & \frac{\pi}{2} & -\frac{\pi}{2} & -\frac{\pi}{2} \end{pmatrix} \end{aligned}$$

Z Bound and Disjointness

Given $\mathcal{C}^{\mathrm{rot}}(H_X, H_Z)$,

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Z Bound and Disjointness

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$$\delta_Z \geq \frac{N_X D_X \sin^2\left(\frac{\alpha}{2D_X}\right)}{\sin^2(\frac{\alpha}{2})}.$$

Code Parameters

A homological quantum rotor code, $\mathcal{C}^{\mathrm{rot}}(H_X, H_Z)$, is described by the parameters

$$[\![n, (k, d_1 \cdot d_2 \cdot \ldots \cdot d_{k'}), (d_X, \delta_Z)]\!]_{\mathrm{rot}},$$

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if it is defined on n quantum rotors, encodes k logical rotors and k' logical qudits of dimensions $d_1, \ldots, d_{k'}$ and has X-distance d_X and Z-distance δ_Z.

Codes from Cellular Homology in 2D

$$\begin{array}{ccccccc} \mathcal{C}: & C_2 & \xrightarrow{\partial} & C_1 & \xrightarrow{\sigma} & C_0 & \text{with } \sigma \circ \partial = 0 \\ & \shortparallel & & \shortparallel & & \shortparallel & \\ & \mathbb{Z}^F & & \mathbb{Z}^E & & \mathbb{Z}^V & \\ & \shortparallel & & \shortparallel & & \shortparallel & \\ & \text{faces} & & \text{edges} & & \text{vertices} & \end{array}$$

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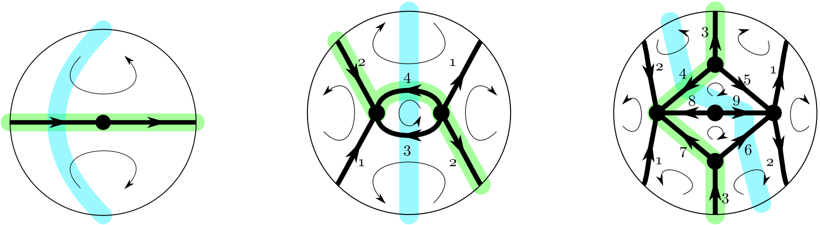
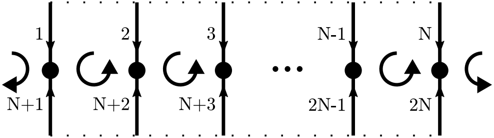
Example: Projective Plane



Projective Plane (Co)Homology

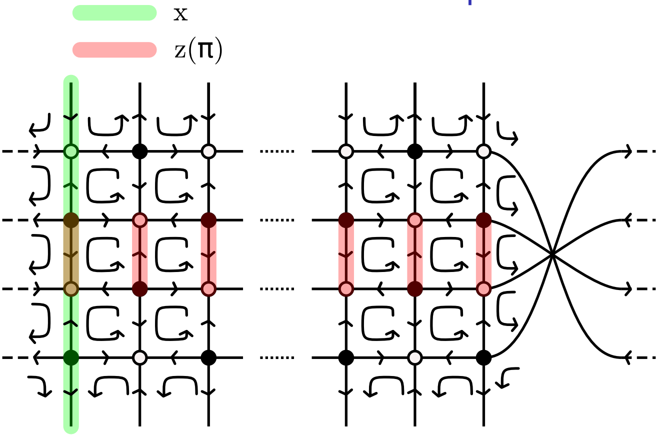
Coefficients	Homology			Cohomology		
	$C_2 \xrightarrow{\partial} C_1 \xrightarrow{\sigma} C_0$			$C_2^* \xleftarrow{\partial^*} C_1^* \xleftarrow{\sigma^*} C_0^*$		
$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}_2$	0	$\mathbb{Z}$
$\mathbb{T}$	$\mathbb{Z}_2$	0	$\mathbb{T}$	0	$\mathbb{Z}_2$	$\mathbb{T}$
$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
$\mathbb{Z}_3$	0	0	$\mathbb{Z}_3$	0	0	$\mathbb{Z}_3$
$\mathbb{R}$	0	0	$\mathbb{R}$	0	0	$\mathbb{R}$

Thin Möbius Strip



$$[\![2N,(0,2),(2,N)]\!]_{\mathrm{rot}}$$

$w \times N$ Möbius Strip



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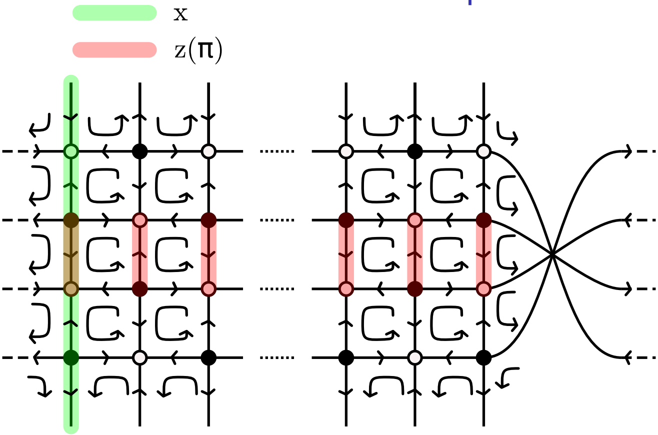
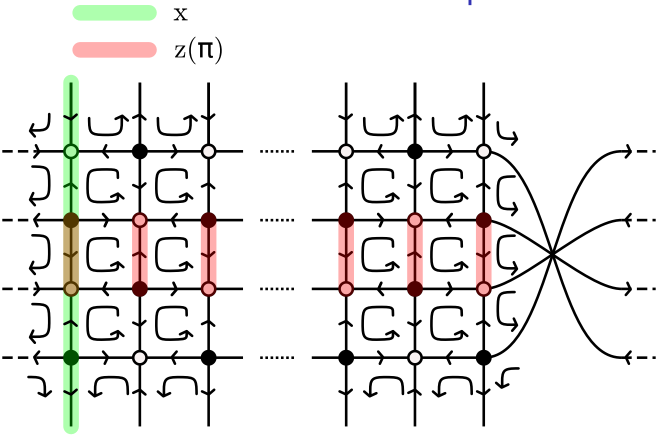


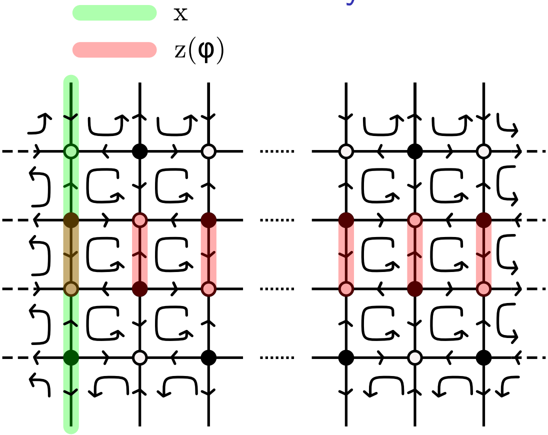
$$\left[\!\left[2Nw - N, (0,2), \left(w, \Theta\left(\frac{N}{w}\right)\right)\right]\!\right]_{\mathrm{rot}}$$

$w \times N$ Möbius Strip



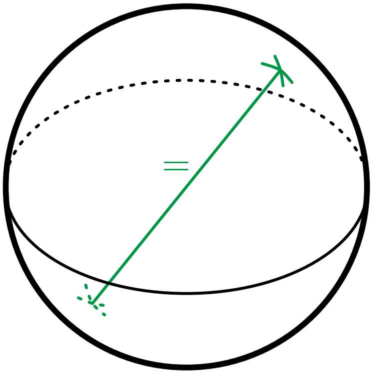
$$\left[\!\left[2Nw - N, (0,2), \left(w, \Theta\left(\frac{N}{w}\right)\right)\right]\!\right]_{\text{rot}} \qquad (\text{pick } N = w^2)$$

$w \times N$ Cylinder



$$\left[\!\left[2Nw - N, (0,2), \left(w, \Theta\left(\frac{N}{w}\right)\right)\right]\!\right]_{\mathrm{rot}} \qquad (\text{pick } N = w^2)$$

Real Projective Space



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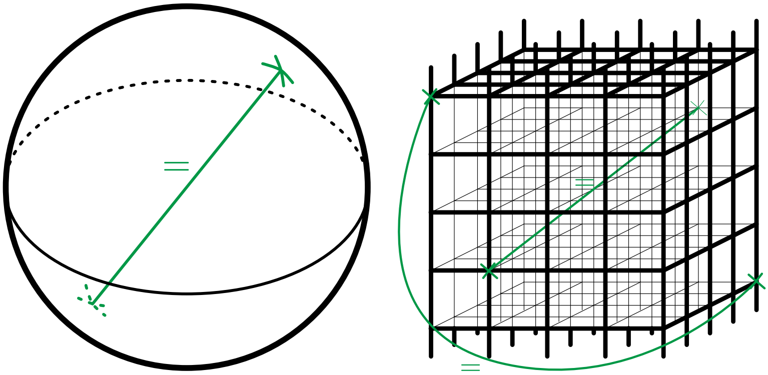

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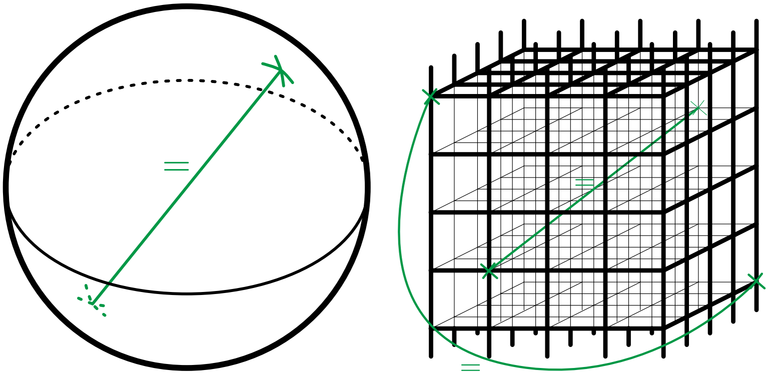

$$[\![3N^3 - N^2, (0, 2), (N, N)]\!]_{\mathrm{rot}}$$

Construction from Product of Chain Complexes

$$\mathcal{C}: \quad \mathbb{Z}^{m_C} \xrightarrow{\partial^{\mathcal{C}}} \mathbb{Z}^{n_C} \qquad \Bigg| \qquad \mathcal{D}: \quad \mathbb{Z}^{n_D} \xrightarrow{\partial^{\mathcal{D}}} \mathbb{Z}^{m_D}$$

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$$\mathcal{C} \otimes \mathcal{D}: \quad \mathbb{Z}^{m_C n_D} \xrightarrow{H_X} \mathbb{Z}^{n_C n_D + m_C m_D} \xrightarrow{H_Z^T} \mathbb{Z}^{n_C m_D}$$

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$$H_X = \left(\partial^{\mathcal{C}} \otimes \mathbb{1}_{n_D} \quad -\mathbb{1}_{m_C} \otimes \partial^{\mathcal{D}}\right) \qquad H_Z = \left(\mathbb{1}_{n_C} \otimes \partial^{\mathcal{D}\,T} \quad \partial^{\mathcal{C}\,T} \otimes \mathbb{1}_{m_D}\right)$$

Künneth Theorem

$$\mathcal{C}: \quad C_1 \xrightarrow{\partial^{\mathcal{C}}} C_0 \quad \Big| \quad \mathcal{D}: \quad D_1 \xrightarrow{\partial^{\mathcal{D}}} D_0$$

Homology Group

$$\begin{aligned} H_1(\mathcal{C}\otimes\mathcal{D}) \simeq & H_1(\mathcal{C})\otimes H_0(\mathcal{D}) \\ & \oplus H_0(\mathcal{C})\otimes H_1(\mathcal{D}) \\ & \oplus \mathrm{Tor}\left(H_0(\mathcal{C}), H_0(\mathcal{D})\right) \end{aligned}$$

Free+Free

$$H_1(\mathcal{C}\otimes\mathcal{D}) = \mathbb{Z}^{k_C k_D}$$

Repetition code + good LDPC
$\Rightarrow [\![n, (\sqrt[3]{n}, 0), (\sqrt[3]{n}, \sqrt[3]{n})]\!]_{\mathrm{rot}}$

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Torsion+Free

$$H_1(\mathcal{C} \otimes \mathcal{D}) = \left(\mathbb{Z}_{d_1} \oplus \cdots \oplus \mathbb{Z}_{d_{k'_C}}\right)^{k_D}$$

Sign-twisted repetition code + good LDPC
$\Rightarrow [\![n, (0, 2^{\sqrt[3]{n}}), (\sqrt[3]{n}, \sqrt[3]{n})]\!]_{\mathrm{rot}}$

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Torsion+Torsion

$$H_1(\mathcal{C}\otimes\mathcal{D}) = \bigoplus_{i\in[k'_C], j\in[k'_D]} \mathbb{Z}_{\mathrm{gcd}(d_i,\tilde{d}_j)}$$

Matrix with Torsion

Pick $H \in \{0,1\}^{(n-k)\times n}$ full rank parity check matrix of binary code $\mathcal{C}_{\mathrm{b}}$. Define

$$M = H^T H \pmod 2 \in \{0,1\}^{n\times n}.$$

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If M is full rank (over $\mathbb{Z}$) then you only have torsion for codewords of $\mathcal{C}_{\mathrm{b}}$

$$\forall \boldsymbol{x} \in \mathcal{C}_{\mathrm{b}},\ M\boldsymbol{x} = 2\boldsymbol{w}, \quad \boldsymbol{w} \notin \mathrm{im}(M)$$

Hamiltonian for the Code

Given $\mathcal{C}^{\mathrm{rot}}(H_X, H_Z)$ we can define the following Hamiltonian

$$H_{\mathrm{code}} = -\sum_{j=1}^{r_X} \cos\left(\boldsymbol{h}_j^X \cdot \hat{\boldsymbol{\theta}}\right) + \sum \left(\boldsymbol{h}_j^Z \cdot \hat{\boldsymbol{\ell}}\right)^2$$

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The groundspace of H_{code} is the code. **Can it be realized?**

Superconducting Circuits

Circuit elements

- Josephson junction $\rightarrow\ -\cos(\hat{\theta}_1 - \hat{\theta}_2)$
- Isolated large capacitance $\rightarrow \sim \left(\hat{\ell}_1 + \hat{\ell}_2\right)^2$

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No-go for JJ based subsystem rotor codes

Rotor subsystem code with only $X_i X_j^\dagger$-type X-gauge generators and any Z-gauge generators can **only encode logical rotors**.

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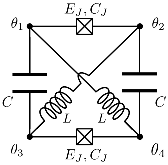
No-go for JJ based subsystem rotor codes

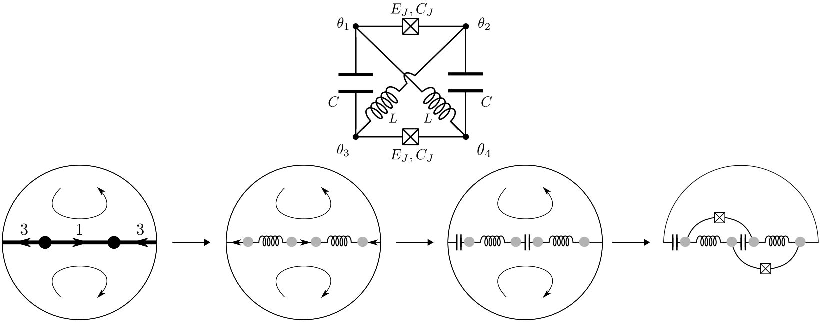
Rotor subsystem code with only $X_i X_j^\dagger$-type X-gauge generators and any Z-gauge generators can **only encode logical rotors**.

$\Rightarrow$ Need a perturbative approach

$0 - \pi$ Qubit

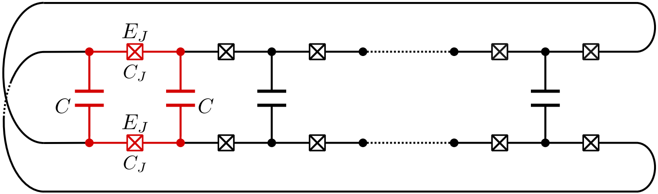


$0 - \pi$ Qubit



$$H_{\text{code}} = -\cos\left(2\hat{\theta}_3 - 2\hat{\theta}_1\right) + \left(\hat{\ell}_1 + \hat{\ell}_3\right)^2$$

Kitaev's Current-Mirror/Möbius Strip Qubit



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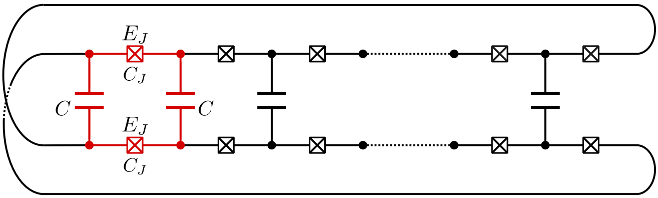
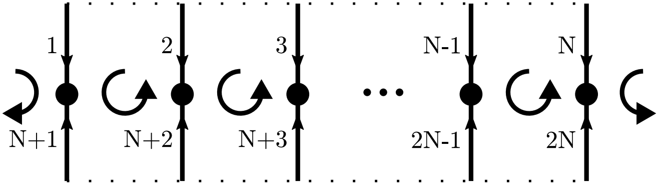




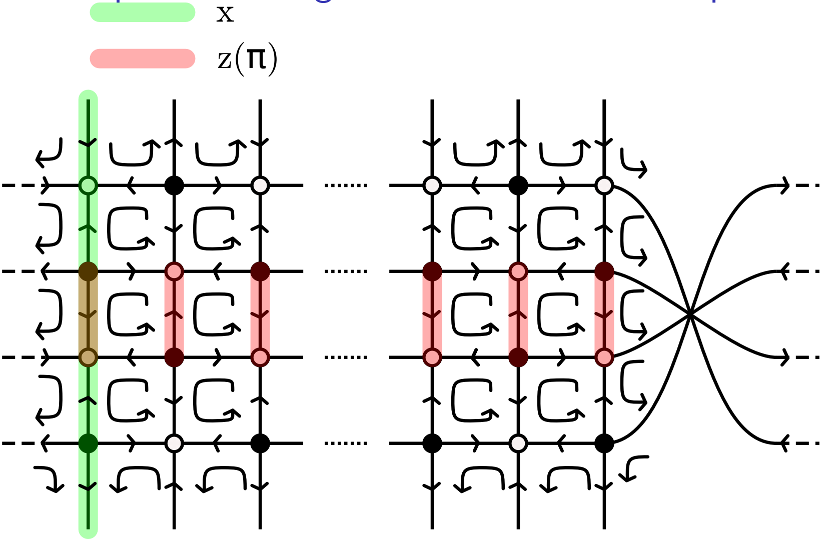
Summary

- Defined Homological Quantum Rotor Codes
- Logical rotors or logical qudits without modular constraints
- X-distance straightforward, Z-distance more tricky
- Can construct codes with at least $\sqrt[3]{n}$-distance
- Describe 0-π type protected superconducting qubits

Future Directions

- Superconducting circuits for any homological rotor code?
- Explore 3D codes (toric/Haah)
- Rotor code $\rightarrow$ number-phase code $\rightarrow$ multimode cat code?
- Systolic freedom and the relation with torsion?
- Active realizations?

Spread-out Logicals for the Möbius Strip



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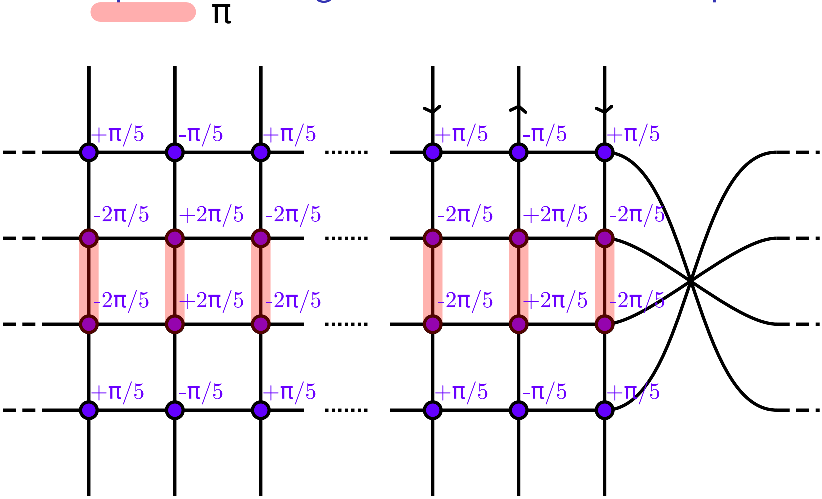

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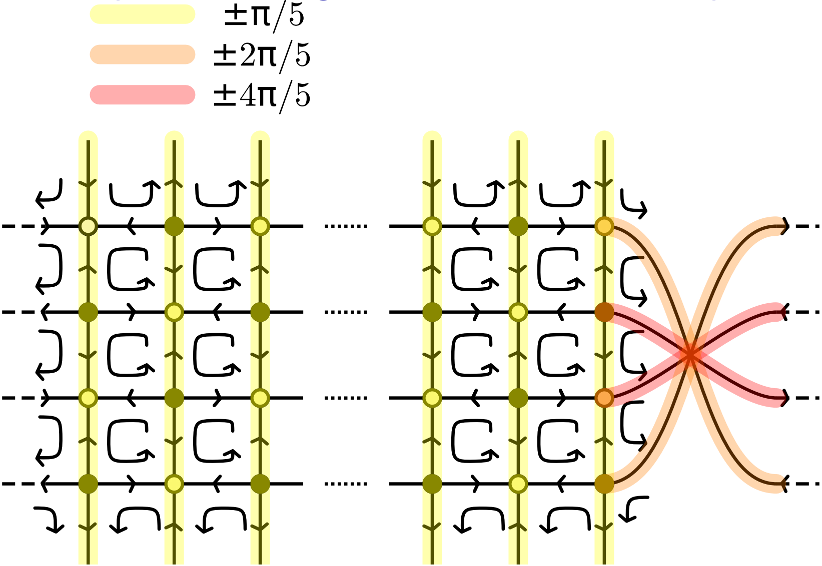

Hamming Code Examples

$$H = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 \end{pmatrix} \qquad H_1(H) = \mathbb{Z}^4$$

$$G = \begin{pmatrix} 1 & 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 1 & -1 & 0 & -1 \end{pmatrix} \quad E = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Square Hamming Code Parity Check Matrix

$$H^T H \pmod 2 = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{pmatrix} \qquad H_0 = \mathbb{Z}_2^3 \oplus \mathbb{Z}_4$$

$$G'_C = \begin{pmatrix} -1 & 0 & 0 & 1 & 0 & 1 & -1 \\ -1 & 0 & 0 & 0 & 1 & 1 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 & -1 \\ -1 & 1 & -1 & 1 & 1 & 1 & -1 \end{pmatrix} \quad E'_C = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$