Precoded Polar Codes with High Error-Correction Capability

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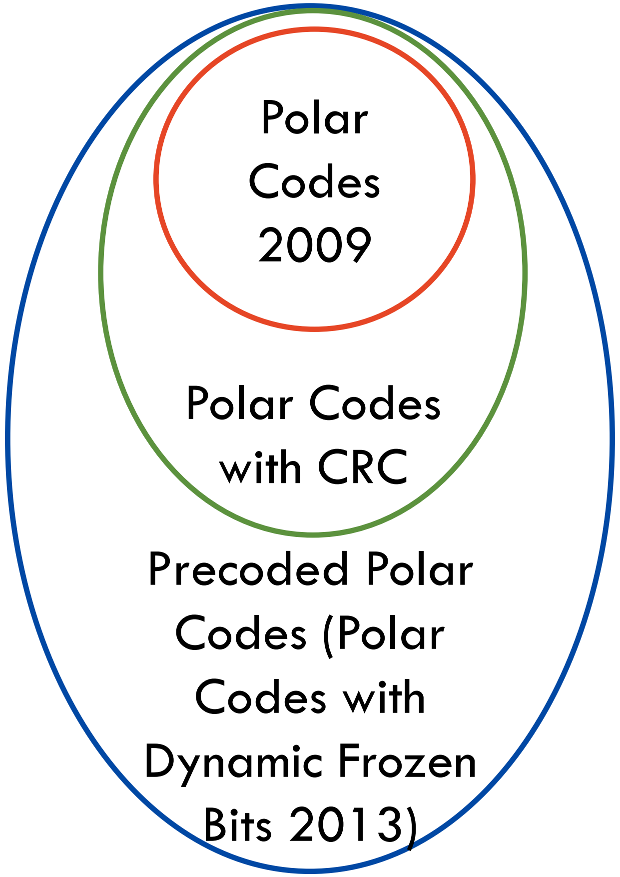
Outline

- Background on polar codes
- Polar code design
  - Design criteria
  - Optimization techniques
  - Search space reduction
- Summary

Polar Codes

- Polar codes [1]
  - Recursive structure based on 2x2 transformation
  - Successive cancellation (SC) decoding
  - Low-complexity encoding and decoding
  - Approach the channel capacity as the code length goes to infinity
- Polar codes with CRC
  - In 5G, polar codes of length up to 1024 are concatenated with CRC codes of lengths 11 and 6
  - Improved FER performance under SC list (SCL) decoding
- Precoded polar codes
  - Also known as pre-transformed polar codes and polar codes with dynamic frozen bits [2]



[1] Arikan E. Channel polarization: A method for constructing capacity-achieving codes for symmetric binary-input memoryless channels // IEEE Transactions on Information Theory.— 2009.—July.— Vol. 55, no. 7.—Pp. 3051–3073.
[2] P. Trifonov and V. Miloslavskaya, "Polar codes with dynamic frozen symbols and their decoding by directed search," 2013 IEEE Information Theory Workshop (ITW), Seville, Spain, 2013, pp. 1-5.

Example. Generator Matrices of (8,4) Polar Codes

- Kernel $T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$
- $T^{\otimes m}$ denotes m -times Kronecker product of matrix T with itself
- Let frozen set $F = \{0,1,2,4,7\} \subset \{0, \dots, 2^m\}$
- (8,3) pure polar code

0 1 2 3 4 5 6 7

$$G_{\text{pure}} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \cdot T^{\otimes 3}$$

- (8,3) precoded polar code

Values are defined by frozen bit expressions

$$G_{\text{prec}} = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \cdot T^{\otimes 3}$$

$$T^{\otimes 3} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

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Polar Code Design as Constrained Optimization Problem

- Code optimization criteria
- Optimization techniques
- Search space size reduction

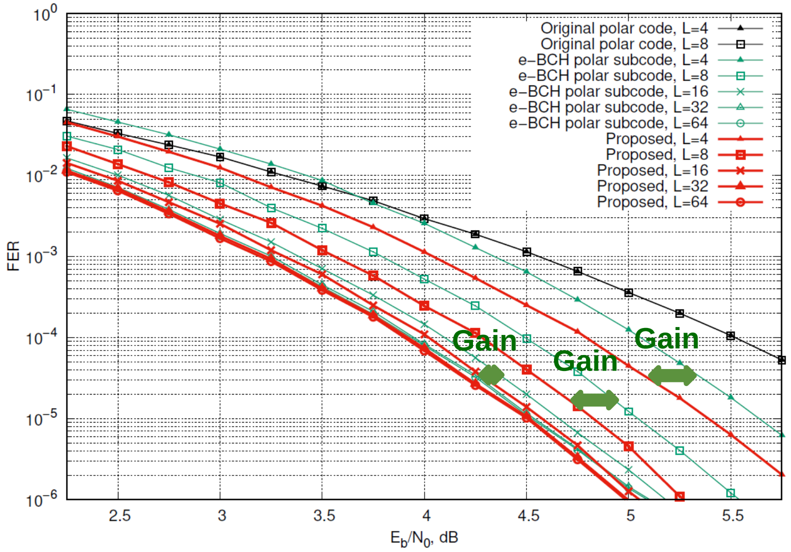
Design Criteria for Polar Codes under SCL Decoding

- SCL decoding FER obtained via simulations
- Successive cancellation (SCL L=1) decoding FER
  - Especially important for SCL with small list size L
  - Analytical approximation requires frozen set and bit-channel error-probabilities
- Maximum-likelihood (ML) decoding FER
  - Especially important for SCL with large list size L
  - Analytical bounds typically require the weight distribution of considered codes
    - Exact number of minimum weight codewords of a pure polar code [1]
    - Exact partial weight distribution of precoded polar codes [2]
    - Approximate weight distribution of polar codes with near-uniformly distributed frozen bit expressions [3]
- Average list size required by SCL to approach the ML performance [4]
  - Analytical bounds require frozen set and bit-channel entropies [4]
  - Fine-tuned version for code design [5]

[1] M. Bardet, V. Dragoi, A. Otmani, and J. Tillich, “Algebraic properties of polar codes from a new polynomial formalism,” in IEEE International Symposium on Information Theory (ISIT), 2016, pp. 230–234.
[2] V. Miloslavskaya, B. Vucetic, and Y. Li, “Computing the partial weight distribution of punctured, shortened, precoded polar codes,” IEEE Transactions on Communications, vol. 70, no. 11, pp. 7146–7159, 2022.
[3] Y. Li, H. Zhang, R. Li, J. Wang, G. Yan, and Z. Ma, “On the weight spectrum of pre-transformed polar codes,” IEEE International Symposium on Information Theory (ISIT), pp. 1224–1229, July 2021.
[4] M. C. Coşkun and H. D. Pfister, “An information-theoretic perspective on successive cancellation list decoding and polar code design,” IEEE Transactions on Information Theory, vol. 68, no. 9, pp. 5779–5791, September 2022.
[5] V. Miloslavskaya, B. Vucetic, and Y. Li, “Frozen Set Design for Precoded Polar Codes”, https://arxiv.org/abs/2311.10047, 2023.

Proposed Precoded Polar Codes of Lengths 32 and 64

- **Joint optimization** of the frozen set and frozen bit expressions
- **Optimality:** the successive cancellation (SC) decoding error probability is minimized under a constraint on the minimum distance
- Any code rate
- The code weight distribution is similar to that of **extended BCH** codes and their subcodes
- The FER performance of extended BCH codes/subcodes but with a lower decoding complexity

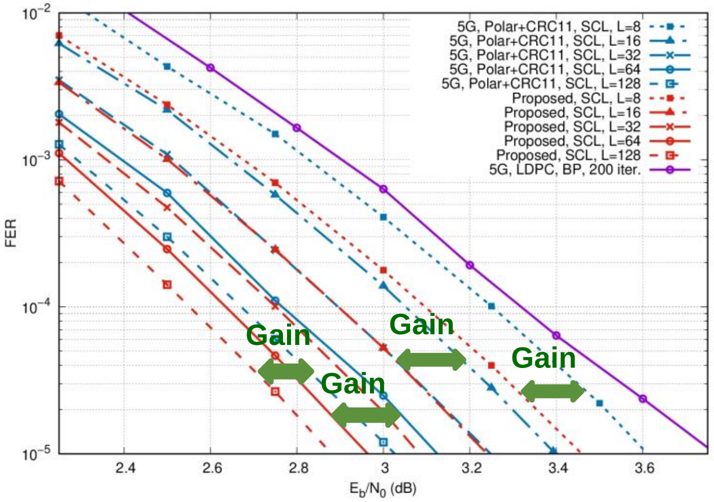


FER performance comparison of (64,25) codes under successive cancellation list (SCL) decoding with various list sizes L

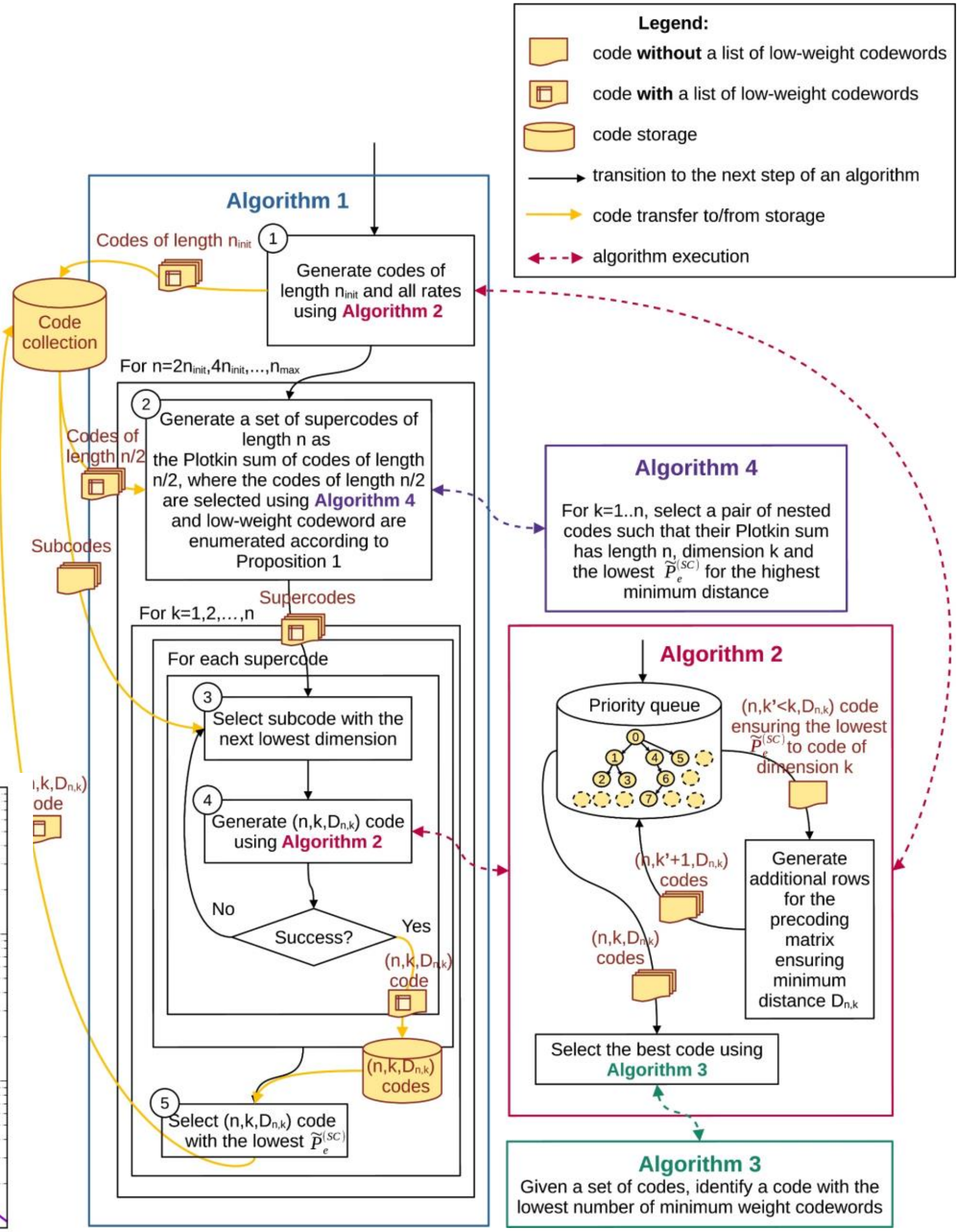
V. Miloslavskaya and B. Vucetic, “Design of short polar codes for SCL decoding,” IEEE Transactions on Communications, vol. 68, no. 11, pp. 6657–6668, November 2020.

Proposed Precoded Polar Codes of Lengths 128 and 256

- **Recursive design** of precoded polar codes **optimized** for successive cancellation list (SCL) decoding
- Inspired by the recursive structure of Reed-Muller codes
- High error-correction capability is achieved by explicitly enumerating low-weight codewords and eliminating them
- The optimization complexity is reduced by introducing a number of **constrains**, e.g., a subcode and a supercode
- Outperform polar codes with various CRC polynomial
- 0.2 dB performance gain compared to **5G polar codes with CRC11**



FER performance comparison of (256,128) codes



V. Miloslavskaya, B. Vucetic, Y. Li, G. Park, and O.-S. Park, "Recursive Design of Precoded Polar Codes for SCL Decoding", IEEE Transactions on Communications, vol. 69, no. 12, pp. 7945–7959, 2021.

Frozen Set Optimization Techniques

- Exhaustive search for very short codes
- Greedy methods
  - Select frozen/unfrozen bit indices one by one
  - Iteratively improve a fixed-cardinality frozen set element by element
- Genetic algorithms
  - Iteratively improve a population of frozen sets according to crossover, mutation and selection rules
- Reinforcement learning techniques
  - Typically used to construct nested frozen sets, specifying polar codes of the same length but different rates
  - Neural networks are trained to predict the FER
- Graph neural networks
- Main drawback: large search space size
  - For the code parameters (N, K), then the total number of frozen sets is $\binom{N}{K}$

Search Space Size Reduction

- Search space size is given by all possible frozen sets
  - For the code parameters (N, K), then the total number of frozen sets is $\binom{N}{K}$
- The most well-known state-of-the-art constraints on the frozen set structure are based on
  - Hamming weight of the binary expansion of indices of frozen/unfrozen bits
    - Bits with low-weight indices are frozen
    - Bits with high-weight indices are non-frozen
  - Partial order
    - Frozen sets designed for SC decoder satisfy partial order
  - Reed-Muller codes, extended BCH codes, other codes

Proposed Triplet-Tuned Frozen Sets

A new frozen set structure that reduces the problem of **frozen set design** to the problem of **selecting three integer numbers**, assuming that a bit-channel reliability sequence is provided

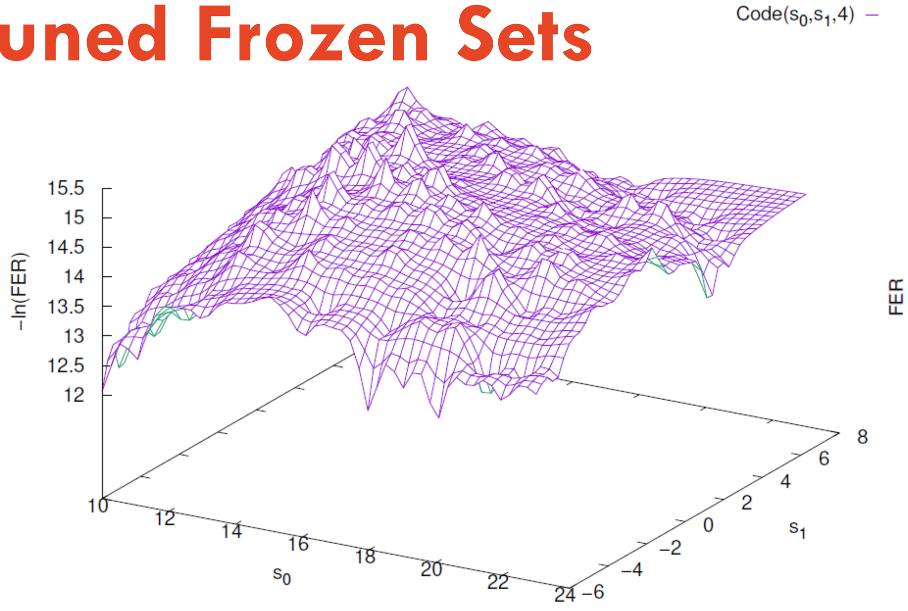
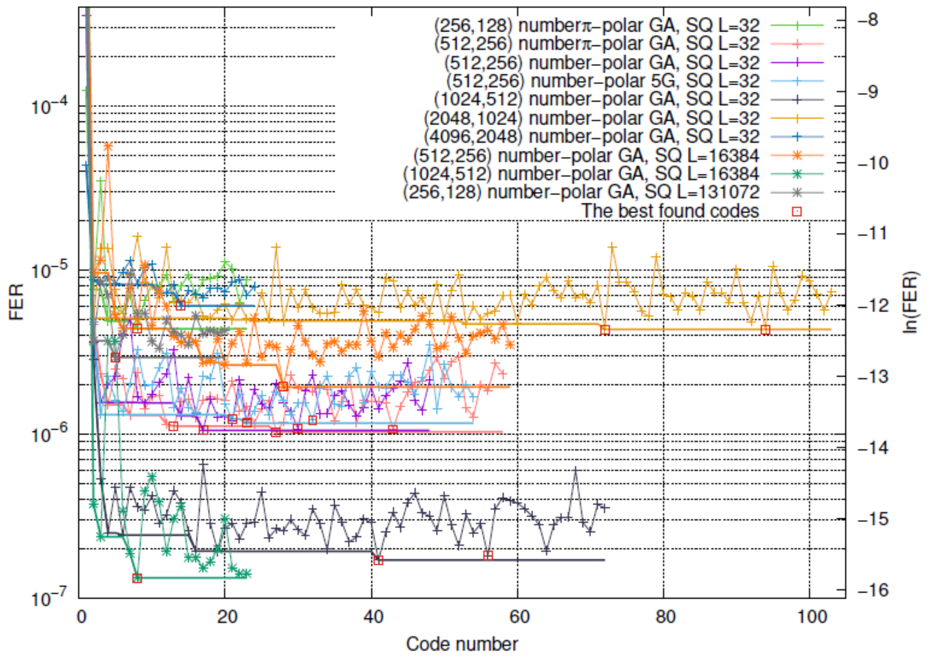


Fig. 3: −ln(FER(Code(s0, s1, 4)))



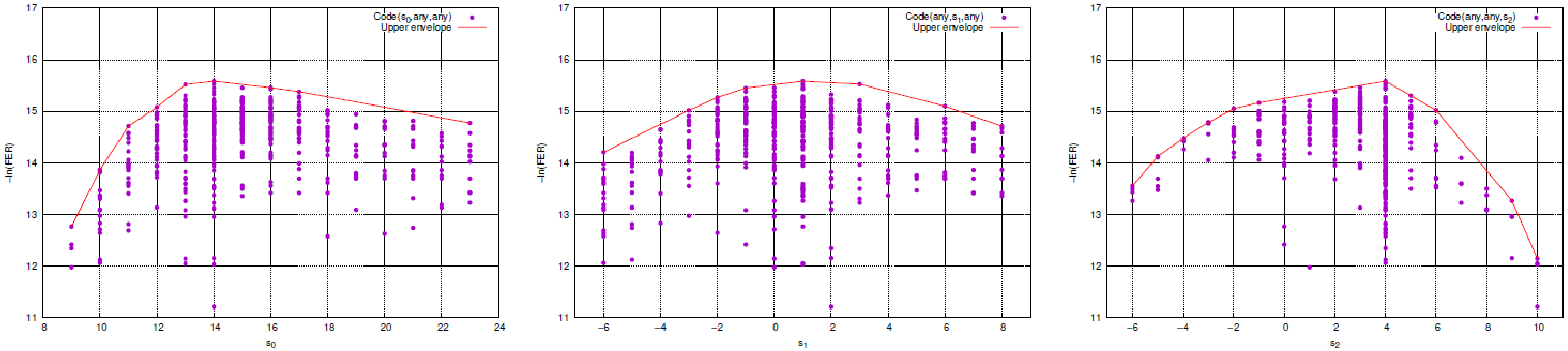
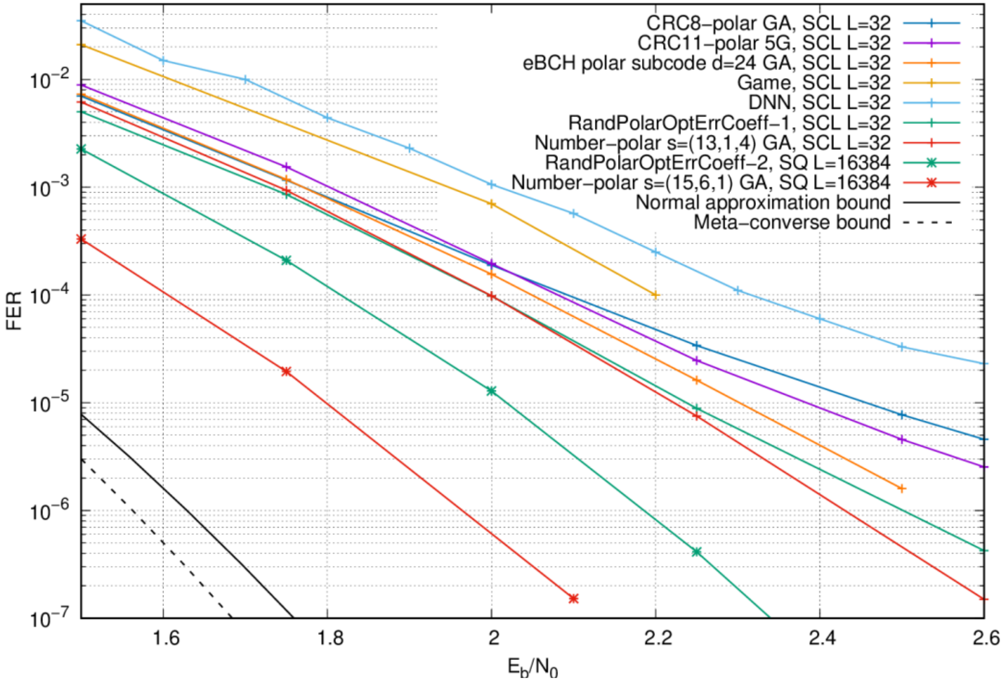


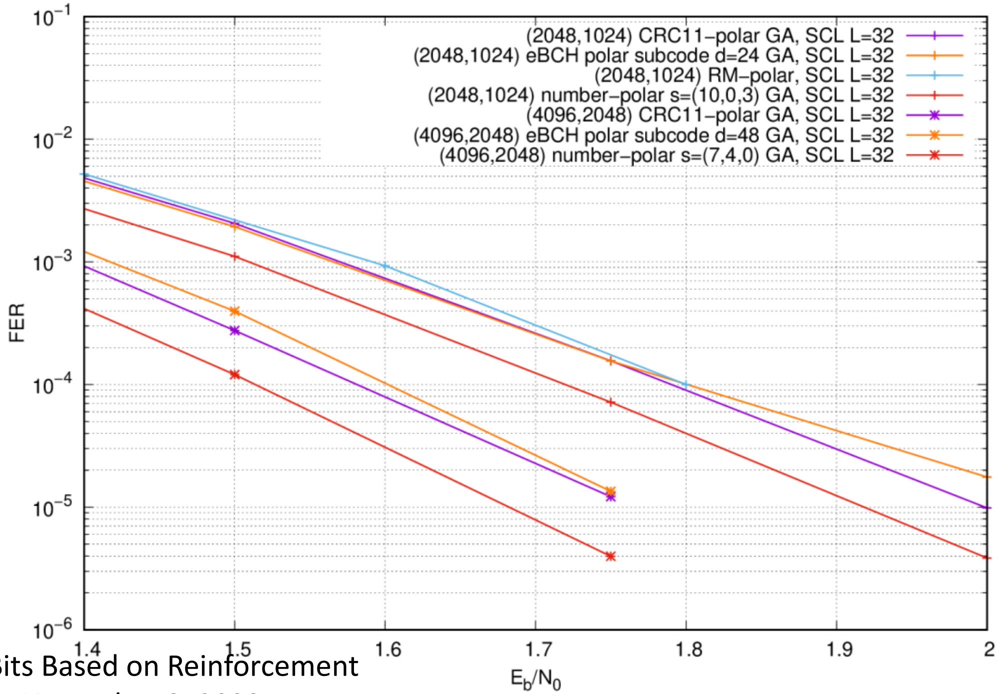
Fig. 2: The projections of −ln(FER(Code(s_0, s_1, s_2))) for n = 1024, k = 512 and L = 32

V. Miloslavskaya, Y. Li and B. Vucetic, "Design of Compactly Specified Polar Codes with Dynamic Frozen Bits Based on Reinforcement Learning", IEEE Transactions on Communications, Early Access, Date of publication in IEEE Xplore is November 8, 2023.

Precoded Polar Codes with Triplet-Tuned Frozen Sets

Code name	L in SCL
(256,128) number-polar s=(7,1,0) GA	32
(512,256) numberπ-polar s=(11,3,0) GA	32
(512,256) number-polar s=(10,3,1) GA	32
(2048,1024) number-polar s=(10,0,3) GA	32
(4096,2048) number-polar s=(7,4,0) GA	32
(512,256) number-polar s=(14,2,0) GA	16384
(1024,512) number-polar s=(15,6,1) GA	16384





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Summary

- Criteria for precoded polar code design under SCL decoding
  - Analytical bounds on the ML FER, SC FER, and SCL list size
- Short-length precoded polar code design
  - Joint optimization of the frozen set and frozen bit expressions [1], [2]
- Long-length precoded polar code design
  - Frozen set is optimized for given near-uniformly distributed frozen bit expressions or other pre-determined frozen bit expressions
  - The problem of frozen set design can be reduced to the problem of selecting three integer numbers [3]

[1] V. Miloslavskaya and B. Vucetic, "Design of short polar codes for SCL decoding," IEEE Transactions on Communications, vol. 68, no. 11, pp. 6657–6668, 2020.
[2] V. Miloslavskaya, B. Vucetic, Y. Li, et al., "Recursive Design of Precoded Polar Codes for SCL Decoding", IEEE Trans. on Com., vol. 69, no. 12, pp. 7945–7959, 2021.
[3] V. Miloslavskaya, Y. Li and B. Vucetic, "Design of Compactly Specified Polar Codes with Dynamic Frozen Bits Based on Reinforcement Learning", IEEE Transactions on Communications, Early Access, Date of publication in IEEE Xplore is November 8, 2023.