Quantized-Constraint Concatenation and The Covering Radius of Constrained Systems

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Joint work with:
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- Constrained codes are often employed in communication and storage systems in order to mitigate the occurrence of data-dependent errors.
- In many channels some patterns are more prone to error than others, and we avoid them by using constrained codes.
- This reduces the number of errors, however the transmitted data may still be corrupted by data-independent errors, requiring additional error-correcting codes.

This is relevant for DNA storage

Examples of constraints

- Homopolymer runs
- GC content
- Local weight constraints

Examples of error types

- Substitution
- Insertions/Deletions
- Burst errors

Banerjee *et al.* ISIT '22, Cai *et al.* T-IT '21, Cai *et al.* ISIT '21, Lu *et al.* IEEE Access '21, Nguyen *et al.* T-IT '21, Press *et al.* PNAS '20, Weber *et al.* IEEE Comm. Lett. '20, (and others).

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- Separate the error-correcting code and the constrained code, and combine them using a concatenation scheme (e.g., concatenation, or reverse concatenation).
  - Many issues need to be resolved (see book draft by Marcus, Roth, and Siegel).
  - In the known schemes, the error-correction capabilities are quite limited: the state-of-the-art method (Gabrys, Siegel and Yaakobi ISIT ’18) allows for a correction of $O(\sqrt{n})$ errors (where n is the block length).

Quantized-Constraint Concatenation (QCC)

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But in QCC:

- We consider the embedding process of information in the constrained media as an irreversible quantization process rather then a coding procedure.
- The constrained word is considered as a corrupted version of the input message, obtained by a quantization procedure.

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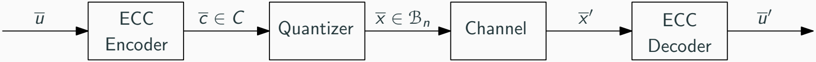
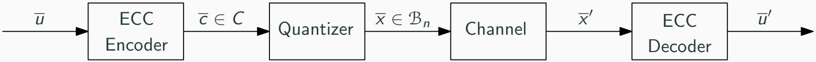


Quantized-Constraint Concatenation – The big picture



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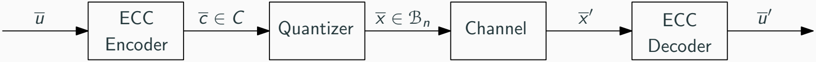

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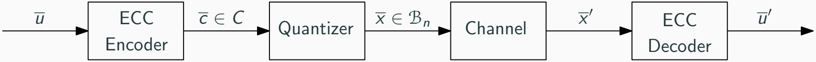

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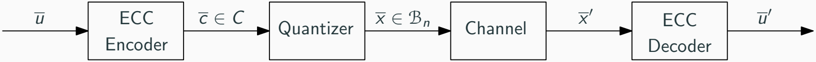

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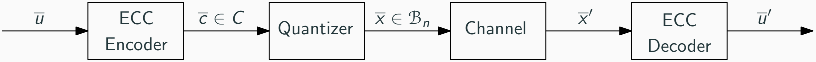

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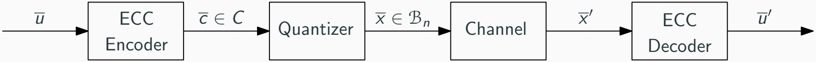
- **Decoding**: Use the decoder for C on $\overline{x}'$ and obtain $\overline{u}'$.

Performance analysis



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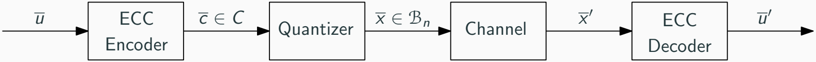

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- If the channel does not introduce more than $t - r$ errors, i.e., $d(\overline{x}, \overline{x}') \leqslant t - r$, then $d(\overline{c}, \overline{x}') \leqslant t$.

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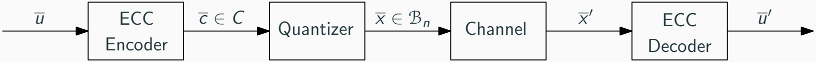

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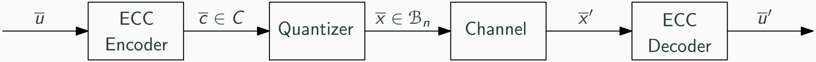

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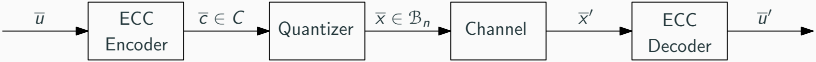

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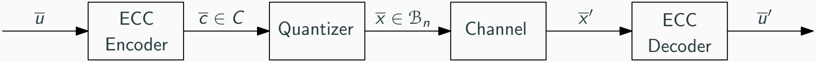

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Assume that $R(\mathcal{B}_n(X))/n$ converges to some number ρ and assume that $(C_n)_{n \in \mathbb{N}}$ is a sequence of codes capable of correcting $\lceil \delta n \rceil$ errors, where $\delta > \rho$.

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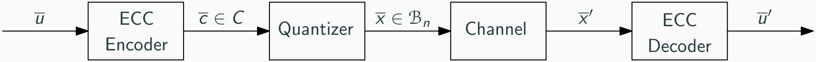

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If $\rho < \frac{1}{2}(1 - \frac{1}{q})$ it is possible to correct $\Theta(n)$ errors with a non-vanishing rate.

The Covering Radius of a Constrained System

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Remark

Typically, $Y = \Sigma^{\mathbb{Z}}$, hence, $\mathcal{B}_n(Y) = \Sigma^n$ for all n and $R(\mathcal{B}_n(X), \mathcal{B}_n(Y))$ is the usual covering radius of $\mathcal{B}_n(X)$.

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Example

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Taking limits: $R(X_{0,k}, Y) = \liminf_{n\to\infty} \dfrac{R(\mathcal{B}_n(X_{0,k}), \{0,1\}^n)}{n} = \dfrac{1}{k+1}$.

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- For $k = 1$ we have $R(X_{0,1}, \{0,1\}^{\mathbb{Z}}) = \frac{1}{2}$.
- Consider the system $X_{\mathrm{rep}} = \{\overline{0}, \overline{1}\}$. A simple calculation shows that $R(X_{\mathrm{rep}}, \{0,1\}^{\mathbb{Z}})$ is also $\frac{1}{2}$.

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- For $k = 1$ we have $R(X_{0,1}, \{0,1\}^{\mathbb{Z}}) = \frac{1}{2}$.
- Consider the system $X_{\mathrm{rep}} = \{\overline{0}, \overline{1}\}$. A simple calculation shows that $R(X_{\mathrm{rep}}, \{0,1\}^{\mathbb{Z}})$ is also $\frac{1}{2}$.
- We have two systems, one which has strictly positive capacity ($\mathrm{Cap}(X_{0,1}) \approx 0.694$) and the other with zero capacity ($\mathrm{Cap}(X_{\mathrm{rep}}) = 0$), with the same covering radius!

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We need an alternative definition for the covering radius which ignores such rare patterns.

The Essential Covering Radius

The essential covering radius

A Trade-off Between Quantization-Error and Rate

Question: What happens to the covering radius if we allow to drop an $\varepsilon \in (0,1)$ fraction of the words in $\mathcal{B}_n(Y)$ to be covered?

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Definition

Let X and Y be constrained systems, μ be an invariant ergodic measure on Y. For $\varepsilon \in (0,1)$ we define $R_\varepsilon(\mathcal{B}_n(X), \mathcal{B}_n(Y), \mu_n)$ by:

$$\min\left\{ r \in \mathbb{N} \;\middle|\; \mu_n\left(\mathcal{B}_n(Y) \cap \left(\bigcup_{\overline{x} \in \mathcal{B}_n(X)} \mathrm{Ball}_r(\overline{x})\right)\right) \geqslant 1 - \varepsilon \right\}.$$

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Remark

In the typical case, where Y is the trivial (non) constrained system, taking μ to be the i.i.d uniform measure, $R_\varepsilon(\mathcal{B}_n(X), \mathcal{B}_n(Y), \mu_n)$ is the minimal radius for covering a fraction of $(1-\varepsilon)$ of Σ^n.

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An Asymptotic Definition

For a fixed $\varepsilon \in (0,1)$ define

$$R_\varepsilon(X, Y, \mu) \triangleq \liminf_{n\to\infty} \frac{R_\varepsilon(\mathcal{B}_n(X), \mathcal{B}_n(Y), \mu_n)}{n}.$$

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Taking the uncovered-fraction of Y to 0 we define the essential covering radius of X with respect to (Y, μ) as

$$R_0(X, Y, \mu) \triangleq \lim_{\varepsilon\to 0} R_\varepsilon(X, Y, \mu).$$

Do we get improved results?

The Case of $(0,k)$-RLL

We revisit the case where $Y = \{0,1\}^{\mathbb{Z}}$ is non-constrained and $X_{0,k}$ is the $(0,k)$-RLL system. Let μ be the Ber($\frac{1}{2}$) i.i.d measure on Y.

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Theorem

$$R_0(X_{0,k}, Y, \mu) = \frac{1}{2(2^{k+1}-1)} \ll \frac{1}{k+1} = R(X_{0,k}, Y).$$

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In The Context of QCC

For a sequence of ECCs capable of correcting δn errors:

- Using the combinatorial covering radius – it is possible to correct up to $(\delta - \frac{1}{k+1})n$ errors.
- Using the essential covering radius – with vanishing loss of rate, it is possible to correct $(\delta - \frac{1}{2(2^{k+1}-1)})n$ errors!

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The Essential Covering Radius

- We find an equivalent characterization of the essential covering radius using ergodic theory.
- The ergodic-theoretic definition is useful for establishing bounds on the essential covering radii of constrained systems.

What's next?

The covering radius of a constrained system is a new and interesting parameter due to its applications for error-correcting constrained codes, but also as a mathematical figure of merit.

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- The algorithmic aspect of QCC - developing quantization algorithms.

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Possible Directions for The Future

- The algorithmic aspect of QCC - developing quantization algorithms.
- Studying the covering radii of well-known constrained systems.
- Providing general bounds and methods to study the covering radius for studying constrained systems.

Thank you for your attention!