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Low-Density Parity-Check Codes and Spatial Coupling for Quantitative Group Testing

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Joint work with Mgeni Makambi Mashauri[†] and Alexandre Graell i Amat[‡]

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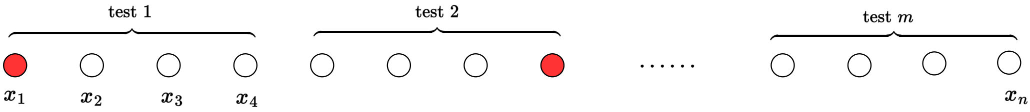
Background: Group Testing

- We have a large population of items
- Very few of them are "defective" (probability of being defective, γ is very small)



- **Goal:** Identify $\boldsymbol{x}$: defective $(x_i = 1)$, non-defective $(x_i = 0)$
- To reduce the number of tests: **test the items in groups** (pooling) [Dorfman1943]
- Rate, $\Omega = \frac{m}{n}$ (smaller is better)
- Adaptive vs non-adaptive test design
- We consider the asymptotic regime: $n \to \infty$

[Dorfman1943] Robert Dorfman, “The Detection of Defective Members of Large Populations,” *The Annals of Mathematical Statistics,*, vol. 14, no. 4, pp. 436–440, 1943.

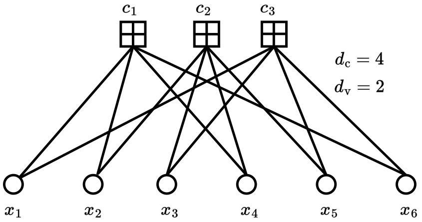
Background: Graphical Representation

- For non-adaptive group testing the pooling can be represented by a test matrix $\boldsymbol{A}$

$$\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

$$\quad x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \quad x_6$$

- The matrix can be represented by a bipartite graph G



- We consider the scenario where the graph is sparse

Non-quantitative vs Quantitative

- **Non-quantitative:** test result, $s_i = 1$ if at least one item is defective otherwise $s_i = 0$ (logical OR)

●	●	○	○	○	●		Non-Quantitative	Quantitative
0	0	1	1	1	0	s_1	0	0
1	1	0	1	0	0	s_2	1	2
0	1	1	0	1	0	s_3	1	1
1	1	0	0	0	1	s_4	1	3

- For **quantitative** group testing, a test result shows the number of defective items

$$s_i = \sum_{j=1}^{n} x_j a_{ij} \quad \rightarrow \quad \boldsymbol{s} = A x$$

Group Testing with Sparse Graphs

- A popular non-quantitative scheme is SAFFRON [Lee2016]
- A variation of SAFFRON uses generalized LDPC (GLDPC) construction with SAFFRON as the signature matrix (component code) [Vem2017]

Signature matrix: $m_\text{b} \times d_\text{c}$

$$\boldsymbol{U} = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

Adjacency matrix: $m_\text{B} \times n$

$$\boldsymbol{B} = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

Test matrix: $m \times n$ $\quad \boldsymbol{A} = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$

$$m = m_\text{b} \times m_\text{B}$$

[Lee2016] K. Lee, R. Pedarsani, and K. Ramchandran, “SAFFRON: A fast, efficient, and robust framework for group testing based on sparse-graph codes,” in *Proc. IEEE Int. Symp. Inf. Theory (ISIT)*, Barcelona, Spain, July 2016.
[Vem2017] A. Vem, N. T. Janakiraman, and K. R. Narayanan, “Group testing using left-and-right-regular sparse-graph codes,” in *CoRR*, vol. abs/1701.07477, 2017.

Quantitative Group Testing with Sparse Graphs: Prior work

- The test results show the number of defectives
- Best known scheme with sparse graph uses GLDPC [KAR2019]

$$\boldsymbol{U} = \begin{pmatrix} \overbrace{\begin{matrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{matrix}}^{d_\mathsf{c}} \\ \begin{matrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{matrix} \end{pmatrix} \quad t = 1, m_\mathsf{u} = 3$$

- $d_\mathsf{c} = 2^{m_\mathsf{u}} - 1 \rightarrow m_\mathsf{u} = \log_2(d_\mathsf{c} + 1)$
- Tests per subcode $= t\log_2(d_\mathsf{c}+1)+1$
- Rate, $\Omega = \frac{m}{n} = \frac{d_\mathsf{v}}{d_\mathsf{c}}\left(t\left\lceil \log_2(d_\mathsf{c}+1)\right\rceil + 1\right)$

- A t-error-correcting BCH code is used as a component code
- An additional row of ones to identify # of defective items

[KAR2019] E. Karimi, F. Kazemi, A. Heidarzadeh, K. R. Narayanan, and A. Sprintson, “Sparse graph codes for non-adaptive quantitative group testing,” in *Proc. IEEE Inf. Theory Workshop (ITW)*, 2019.

Prior Work

- **Density Evolution**

  For each iteration ℓ

  $q^{(\ell)}$: probability a test sends resolved to item
  $p^{(\ell)}$: probability a defective item is unresolved

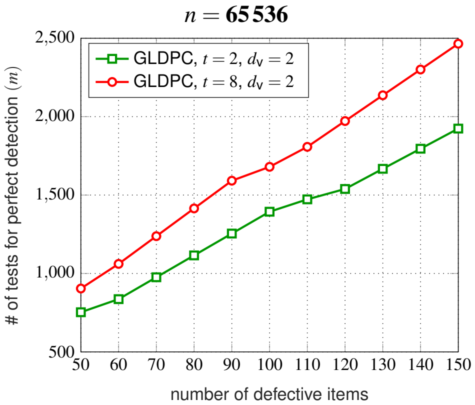
  Test to item:

  $$q^{(\ell)} = \sum_{i=0}^{t-1} \binom{d_c - 1}{i} \left(p^{(\ell-1)}\right)^i \left(1 - p^{(\ell-1)}\right)^{d_c - 1 - i}$$

  Item to test:

  $$p^{(\ell)} = \gamma (1 - q^{(\ell-1)})^{d_v - 1}$$

- Small number of tests for a large population size
- Increasing t improves error correction
- Penalized by increasing number of tests
  $m = n \frac{d_v}{d_c} \left(t \lceil \log_2(d_c + 1) \rceil + 1 \right)$



■ What about using $t = 0$?

Proposed scheme: Group Testing with LDPC

- With $t = 0$ we loose local error correcting capability
- We can observe and utilize two events
  - Syndrome equal zero: $s_i^{(\ell)} = 0$
    Infer all items as 0 (non-defective)
  - Syndrome equals test degree: $s_i^{(\ell)} = d_\mathrm{c}^{(\ell)}$
    Infer all items as 1 (defective)

?	?	?	?	?	?	$\boldsymbol{s}^{(1)}$	$\boldsymbol{d}_\mathrm{c}^{(1)}$
0	0	1	1	1	0	3	3
1	1	0	1	0	0	2	3
0	1	1	0	1	0	2	3
1	0	1	0	0	1	2	3

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- We then peel off *resolved* items (reducing the syndrome accordingly)

?	?	1	1	1	?		
●	○	●	●	●	○	$s^{(1)}$	$d_{\mathrm{c}}^{(1)}$
0	0	1	1	1	0	3	3
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0	1	1	0	1	0	2	3
1	0	1	0	0	1	2	3

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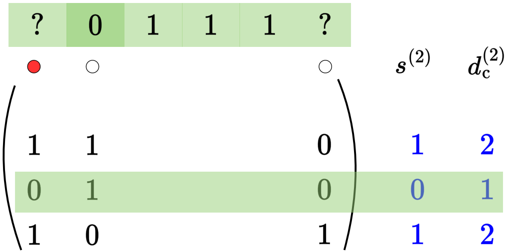
- This is repeated until no item to peel



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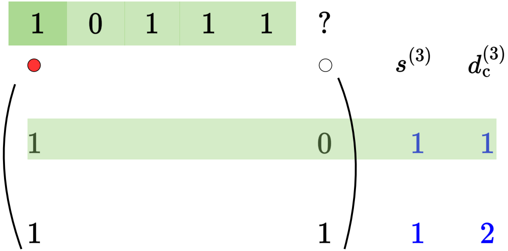

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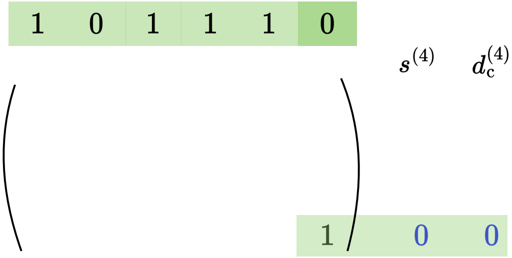

Density Evolution

- $p_1^{(\ell)}$: probability that a message from a defective is *unresolved*
- $q_0^{(\ell)}$: **probability that a message to a non-defective is *resolved***
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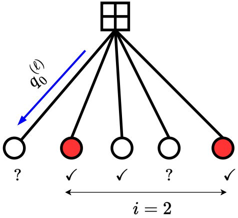

From test to item

$$q_0^{(\ell)} = \sum_{i=0}^{d_c-1} \binom{d_c - 1}{i} \gamma^i (1-\gamma)^{d_c-1-i} \left(1 - p_1^{(\ell-1)}\right)^i$$

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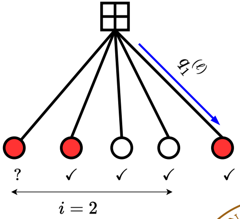

Density Evolution

- $p_1^{(\ell)}$: **probability that a message from a defective is *unresolved***
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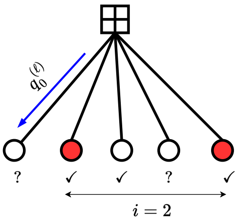

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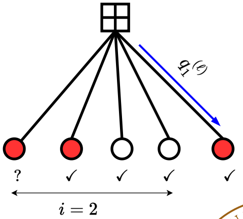


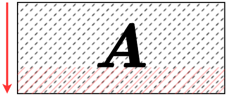
From item to test

$$p_0^{(\ell)} = \left(1-q_0^{(\ell-1)}\right)^{d_v-1}$$

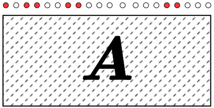
$$p_1^{(\ell)} = \left(1-q_1^{(\ell-1)}\right)^{d_v-1} .$$

Performance Comparison

- We consider two scenarios
  - **Fixing the proportion of defective items γ and changing the rate $\Omega = \frac{m}{n}$**

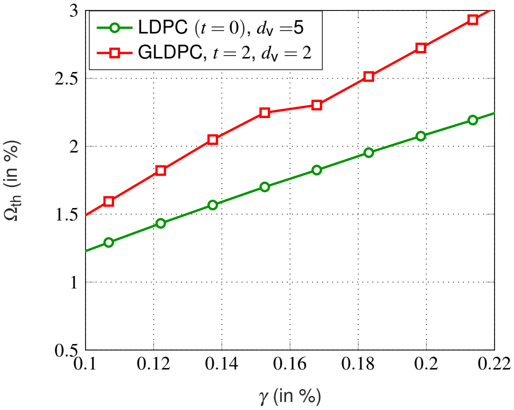


  - Same as in previous work [KAR2019]
  - **Fixing the rate Ω and changing γ**



  - A new perspective considering $\boldsymbol{A}$ (code) as fixed

Minimum rate required for a fixed γ



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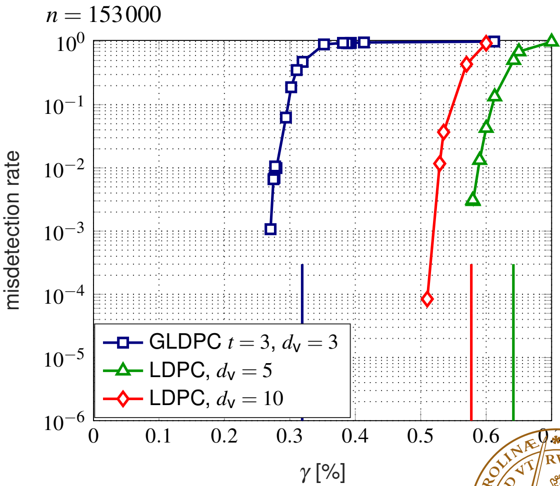
Performance Comparison: Fixed Rate, $\Omega = 5\%$

Table: GLDPC Based

t	d_v	γ_{th}
	2	0.2487
1	3	0.3708
	4	0.3510
	2	0.3983
2	3	0.3372
	4	0.2884
	2	0.3784
3	3	0.3189
	4	0.2441
	2	0.3418
5	3	0.2686
	4	0.2014

Table: LDPC Based

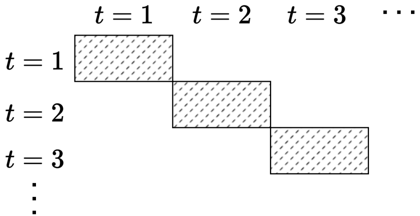
d_v	γ_{th}
3	0.4555
4	0.5982
5	0.6416
6	0.6464
7	0.6353
10	0.5773

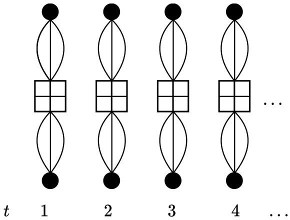


- Can we get consistently better with increasing d_v? What about spatial coupling?

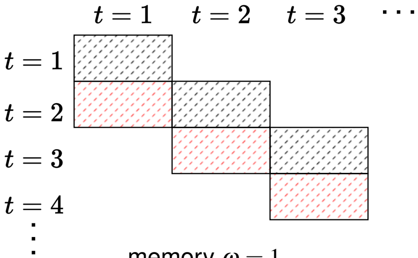
Group Testing with Spatial Coupling

- **Classical approach**: test each block of items separately

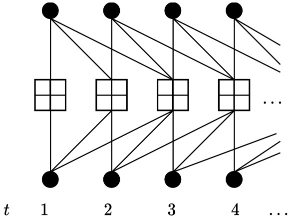




- **Spatial Coupling**: Interconnect blocks (motivated by results in coding theory)



memory $\omega = 1$



Group Testing with Spatial Coupling

- The chain is terminated after length L
- Density evolution

$$q_{0,\tau}^{(\ell)} = \frac{1}{\omega+1}\sum_{j=0}^{\omega}\sum_{i=0}^{d_c-1} \text{Bino}(d_c-1,i,\gamma)\left(1-p_{1,\tau-j}^{(\ell-1)}\right)^{i}$$

$$q_{1,\tau}^{(\ell)} = \frac{1}{\omega+1}\sum_{j=0}^{\omega}\sum_{i=0}^{d_c-1} \text{Bino}(d_c-1,i,\gamma)\left(1-p_{0,\tau-j}^{(\ell-1)}\right)^{d_c-1-i}$$

$$p_{0,\tau}^{(\ell)} = \frac{1}{\omega+1}\sum_{j=0}^{\omega}\left(1-q_{0,\tau+j}^{(\ell-1)}\right)^{d_v-1}$$

$$p_{1,\tau}^{(\ell)} = \frac{1}{\omega+1}\sum_{j=0}^{\omega}\left(1-q_{1,\tau+j}^{(\ell-1)}\right)^{d_v-1} .$$

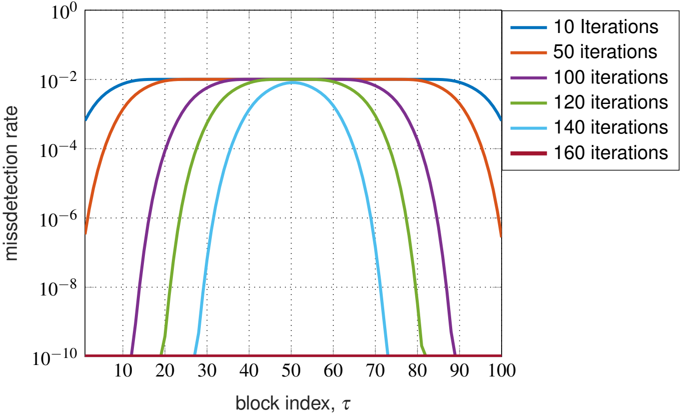
- $p_{0,\tau}^{(\ell)} = p_{0,\tau}^{(\ell)} = 0$ for $\tau < 0$ and $\tau > L$
- Rate becomes

$$\Omega_{SC} = \Omega\left(1+\frac{\omega}{L}\right)$$

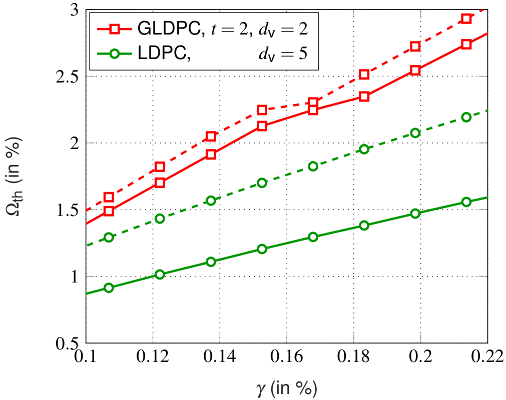
- The rate increase vanishes as L increases

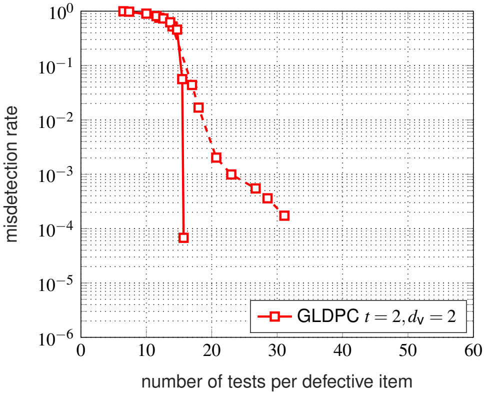
Spatial Coupling: Wave Effect

- Tests at boundary have lower degree
  - The nodes at the boundary can be resolved with higher probability
  - The effect spreads within the chain as a wave



Spatial coupling: Performance Changing Rate





- coupled (solid) uncoupled (dashed)
- memory $\omega = 5$
- $n = 153\,000$
- Low error floor with coupling

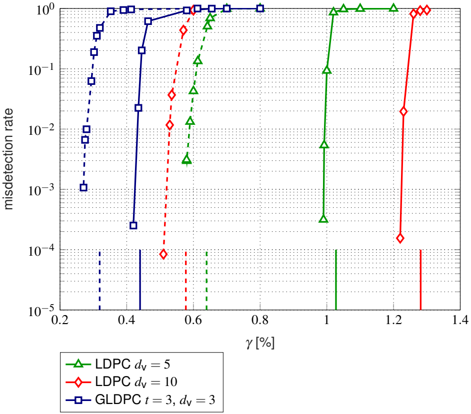
Spatial coupling: Performance for Fixed Rate

Table: γ_{th} for $\Omega = 5\%$ with GLDPC Code-Based

t	d_{v}	$\omega = 0$	$\omega = 1$	$\omega = 5$	$\omega = 10$
1	3	0.3708	0.4166	0.4166	0.4166
	4	0.3510	0.4395	0.4425	0.4425
3	3	0.3189	0.4257	0.4379	0.4395
	4	0.2441	0.3662	0.4028	0.4028
5	3	0.2686	0.3784	0.4089	0.4089
	4	0.2014	0.3159	0.3769	0.3769

Table: γ_{th} for $\Omega = 5\%$ with LDPC Code-Based

d_{v}	$\omega = 0$	$\omega = 1$	$\omega = 5$	$\omega = 10$
4	0.5982	0.8423	0.8540	0.8540
5	0.6416	0.9682	1.0274	1.0250
6	0.6464	1.0044	1.1325	1.1327
10	0.5773	0.9188	1.2814	1.2816



- $n = 153\,000$, $L = 200$, $\omega = 5$
- solid(coupled) dashed(uncoupled)

Can We Prove Threshold Saturation?

- Finding a density evolution recursion that follows the conditions of a vector admissible system.

$$q_0^{(\ell)} = \sum_{i=0}^{d_c-1} \binom{d_c-1}{i} \gamma^i (1-\gamma)^{d_c-1-i} \left(1-p_1^{(\ell-1)}\right)^i$$

$$q_1^{(\ell)} = \sum_{i=0}^{d_c-1} \binom{d_c-1}{i} \gamma^i (1-\gamma)^{d_c-1-i} \left(1-p_0^{(\ell-1)}\right)^{d_c-1-i}$$

$$p_0^{(\ell)} = \left(1-q_0^{(\ell-1)}\right)^{d_v-1}$$

$$p_1^{(\ell)} = \left(1-q_1^{(\ell-1)}\right)^{d_v-1} .$$

With $x_0^{(\ell)} = 1-q_0^{(\ell)}$, $x_1^{(\ell)} = 1-q_1^{(\ell)}$, $y_0^{(\ell)} = (1-\gamma)q_0^{(\ell)}$, $y_1^{(\ell)} = \gamma p_1^{(\ell)}$
we can write:

$$x_0^{(\ell)} = 1-\left(1-y_1^{(\ell-1)}\right)^{d_c-1}$$

$$x_1^{(\ell)} = 1-\left(1-y_0^{(\ell-1)}\right)^{d_c-1}$$

$$y_0^{(\ell)} = (1-\gamma)\left(x_0^{(\ell-1)}\right)^{d_v-1}$$

$$y_1^{(\ell)} = \gamma\left(x_1^{(\ell-1)}\right)^{d_v-1} .$$

With spatial coupling:

$$x_{0,\tau}^{(\ell)} = 1-\frac{1}{\omega+1}\sum_{j=0}^{\omega}\left(1-y_{1,\tau-j}^{(\ell-1)}\right)^{d_c-1}$$

$$x_{1,\tau}^{(\ell)} = 1-\frac{1}{\omega+1}\sum_{j=0}^{\omega}\left(1-y_{0,\tau-j}^{(\ell-1)}\right)^{d_c-1}$$

$$y_{0,\tau}^{(\ell)} = (1-\gamma)\frac{1}{\omega+1}\sum_{j=0}^{\omega}\left(x_{0,\tau+j}^{(\ell-1)}\right)^{d_v-1}$$

$$y_{1,\tau}^{(\ell)} = \gamma\frac{1}{\omega+1}\sum_{j=0}^{\omega}\left(x_{1,\tau+j}^{(\ell-1)}\right)^{d_v-1} .$$

First approach: finding maximum γ for a fixed Ω

- **Vector admissible system:** [YED2012] a recursion $(\mathbf{f}, \boldsymbol{g})$ with

$$\boldsymbol{x}^{(\ell)} = \mathbf{f}\left(\boldsymbol{g}(\boldsymbol{x}^{(\ell-1)}); \varepsilon\right) \;, \quad \boldsymbol{x}^{(0)} = \mathbf{1} \;, \; \varepsilon \in [0,1]$$

  where $\mathbf{f}(\boldsymbol{x}) = [f_1(\boldsymbol{x}), \cdots, f_d(\boldsymbol{x})]$ and $\boldsymbol{g}(\boldsymbol{x}) = [g_1(\boldsymbol{x}), \cdots, g_d(\boldsymbol{x})]$ are twice continuously differentiable and strictly increasing in all arguments.

- With $\varepsilon = \gamma$ we get from density evolution equations:

$$\mathbf{f}_\gamma(x_0, x_1; \gamma) = \left[(1-\gamma)\cdot x_0^{d_\mathsf{v}-1}, \quad \gamma \cdot x_1^{d_\mathsf{v}-1}\right]$$
$$\mathbf{g}_\gamma(y_0, y_1) = \left[1-(1-y_1)^{d_\mathsf{c}-1}, \quad 1-(1-y_0)^{d_\mathsf{c}-1}\right].$$

- **Problem:** $(1-\gamma)$ and γ cannot both increase $\Rightarrow$ conditions not fulfilled

[YED2012] A. Yedla, Y.-Y. Jian, P. S. Nguyen, and H. D. Pfister, "A simple proof of threshold saturation for coupled vector recursions," in *Proc. IEEE Inf. Theory Workshop (ITW)*, 2012.

Second approach: finding minimum Ω for a fixed γ

- **Vector admissible system:** [YED2012] a recursion $(\mathbf{f}, \boldsymbol{g})$ with

$$\boldsymbol{x}^{(\ell)} = \mathbf{f}\left(\boldsymbol{g}(\boldsymbol{x}^{(\ell-1)}); \varepsilon\right) \;, \quad \boldsymbol{x}^{(0)} = \mathbf{1} \;,\; \varepsilon \in [0,1]$$

where $\mathbf{f}(\boldsymbol{x}) = [f_1(\boldsymbol{x}), \cdots, f_d(\boldsymbol{x})]$ and $\boldsymbol{g}(\boldsymbol{x}) = [g_1(\boldsymbol{x}), \cdots, g_d(\boldsymbol{x})]$ are twice continuously differentiable and strictly increasing in all arguments.

- Setting $\varepsilon = 1 - \frac{1}{d_\mathsf{c}}$ we get from density evolution equations:

$$\mathbf{f}(y_0, y_1; \varepsilon) = \left[1 - (1-y_1)^{\frac{\varepsilon}{1-\varepsilon}}, \quad 1 - (1-y_0)^{\frac{\varepsilon}{1-\varepsilon}}\right]$$

$$\boldsymbol{g}(x_0, x_1) = \left[(1-\gamma) \cdot x_0^{d_\mathsf{v}-1}, \quad \gamma \cdot x_1^{d_\mathsf{v}-1}\right]$$

- Threshold saturation occurs

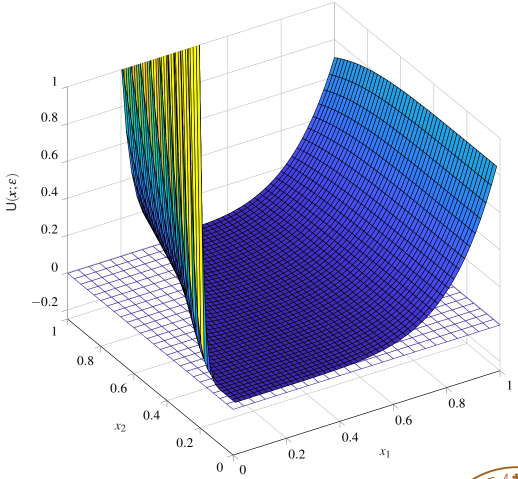
- The potential function is then given as

$$\mathsf{U}(\boldsymbol{x}; \varepsilon) = \int_0^1 \left((z(\lambda) - \mathbf{f}(\boldsymbol{g}(z(\lambda)); \varepsilon)) \boldsymbol{D}\boldsymbol{g}'(z(\lambda)) \right) \cdot z'(\lambda) \mathrm{d}\lambda$$

Potential function

$$\mathsf{U}(\boldsymbol{x};\varepsilon) = (1-p)x_1^{d_\mathsf{v}-1}\left((1-\varepsilon)\frac{1-\left(1-px_2^{d_\mathsf{v}-1}\right)^{\frac{1}{1-\varepsilon}}}{px_2^{d_\mathsf{v}-1}} + \frac{(d_\mathsf{v}-1)}{d_\mathsf{v}}x_1 - 1\right)$$

$$+px_2^{d_\mathsf{v}-1}\left((1-\varepsilon)\frac{1-\left(1-(1-p)x_2^{d_\mathsf{v}-1}\right)^{\frac{1}{1-\varepsilon}}}{(1-p)x_1^{d_\mathsf{v}-1}} + \frac{(d_\mathsf{v}-1)}{d_\mathsf{v}}x_2 - 1\right)$$

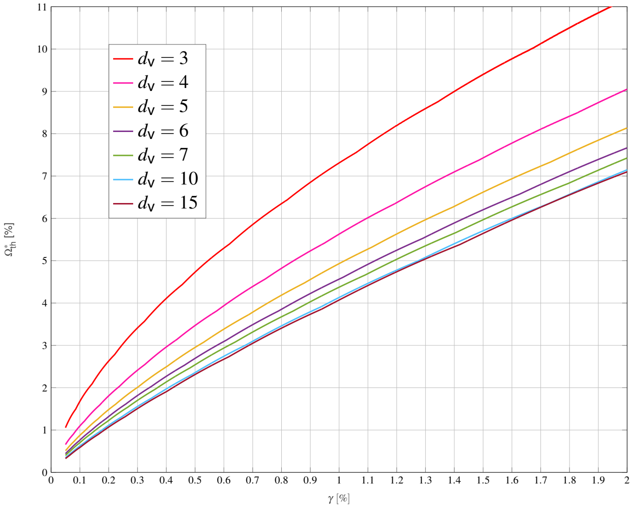


Potential threshold:

$$\varepsilon^* = \sup\{\varepsilon \in [0,1] \mid \min_{\boldsymbol{x}} \mathsf{U}(\boldsymbol{x};\varepsilon) \geq 0\}\,.$$

$d_\mathsf{v} = 6$, $\gamma = 1\%$ with $\varepsilon^* = 0.9924$. $\mathsf{U}(\boldsymbol{x};\varepsilon)$ is above the $z = 0$ plane since $\varepsilon = 0.9667 < \varepsilon^*$.

Potential thresholds



$$\Omega^*_{\mathsf{th}} = \frac{d_{\mathsf{v}}}{d_{\mathsf{c}}} = d_{\mathsf{v}}(1-\varepsilon^*)\,.$$

$$\varepsilon^* = \sup\{\varepsilon \in [0,1] \,|\, \min_{\boldsymbol{x}} \mathsf{U}(\boldsymbol{x};\varepsilon) \geq 0\}\,.$$

The minimum rate Ω^*_{th} for a fixed γ computed from the potential threshold ε^*.

Conclusions and Outlook

Conclusions

- Using a simple LDPC code significantly outperforms a GLDPC construction with t-error-correcting component code
- With spatial coupling we can improve the performance of both schemes
- We can measure the performance by two different approaches
  - Fixing the proportion γ and determining minimum rate Ω
  - Fixing the rate, Ω and determining the maximum γ
- Threshold saturation: with coupling the BP decoder achieves the potential threshold

Outlook

- Bundling of tests: non-binary messages can improve performance
- Looking at soft message passing