Floquet codes from anyon condensation: a tutorial on Floquet codes

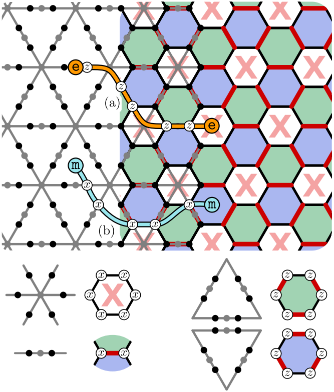
Benjamin J. Brown

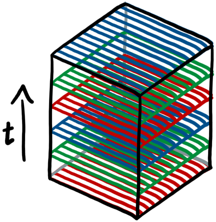
rx	gx	bx
ry	gy	by
rz	gz	bz

based on work in
Markus S. Kesselring *et al.* arXiv:2212.00042

Floquet codes

Hastings and Haah proposed Floquet codes, such as the honeycomb code, where syndromes are read out using a sequence of two-body measurements.





Hastings and Haah, Quantum (2021)
Kitaev, Ann. Phys. (2006)

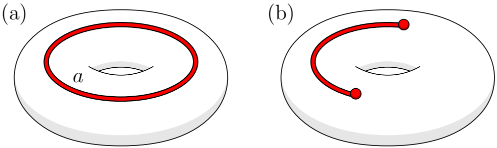
Outline - Floquet codes from anyon condensation

- Anyons and the vacuum
- Color code and its anyons
- Anyon condensation using the color code boson table
- Constructing Floquet code boundaries with condensation
- Generalising Floquet codes

Quantum error-correcting codes and quasiparticle excitations

We can write down Hamiltonians whose ground space is the code space of an error correcting code

$$\mathcal{S} = \langle S_j \rangle \qquad \rightarrow \qquad H = -\sum_j S_j$$



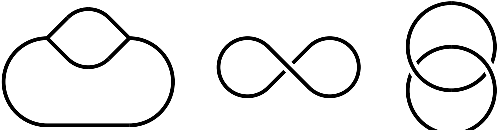
In this picture, violated stabilizers can be interpretted as excitations.

Topological quantum error-correcting codes

We write down a set of charge labels,
e.g., the toric code charge labels are:

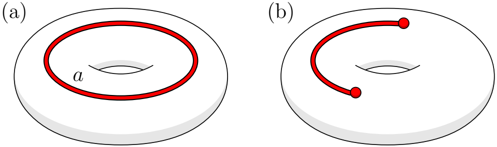
$$\mathcal{C} = \{1, \quad \mathtt{e}, \quad \mathtt{m}, \quad \mathtt{f} = \mathtt{e} \times \mathtt{m}\}\,.$$

Anyon models have exchange, fusion and braid statistics



(a) (b) (c)

Logical operators of topological codes can be viewed as hopping operators over some manifold (or lattice of qubits)



The vacuum, charge label 1

The vacuum, charge label 1

- The vacuum has trivial exchange (i.e., bosonic):

$$R_{1,1} = 1$$

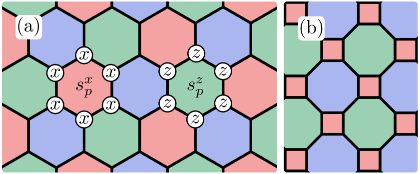
- The vacuum fuses trivially with all anyons $a \in \mathcal{C}$:

$$a \times 1 = a$$

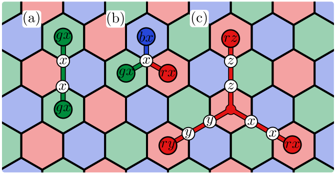
- The vacuum braids trivially with all anyons $a \in \mathcal{C}$:

$$S_{a,1} = 1$$

The color code



Stabilizer code $S_j \in \mathcal{S}$



Excitations

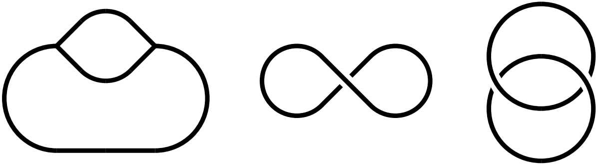
$$S_j|\psi\rangle = (+1)|\psi\rangle \qquad H = -\sum_j S_j$$

Color code bosons are labeled with:
a Pauli type $\mathtt{x}, \mathtt{y}, \mathtt{z}$
and a color $\mathtt{r}, \mathtt{g}, \mathtt{b}$
e.g.

$$\mathtt{rx}, \mathtt{gz}, \mathtt{by}, \mathtt{bz} \text{ etc. etc. etc.}$$

Bombin and Martin-Delgado, Phys. Rev. Lett. (2006)

Color code anyons



(a) (b) (c)

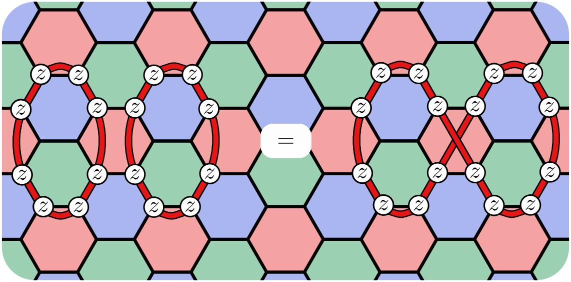
rx	gx	bx
ry	gy	by
rz	gz	bz

$\longrightarrow$

e1		1e
1m		m1

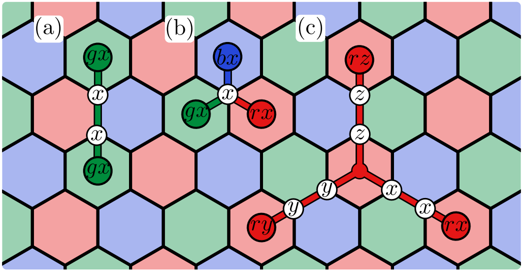
Kubica *et al.*, New J. Phys. (2015)

Color code anyons (bosonic self statistics)



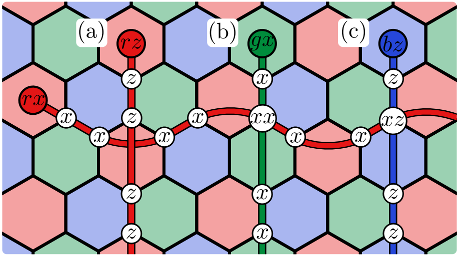
rx	gx	bx
ry	gy	by
rz	gz	bz

Color code anyons (fusion rules)



rx	gx	bx
ry	gy	by
rz	gz	bz

Color code anyons (mutual braid statistics)



rx	gx	bx
ry	gy	by
rz	gz	bz

Anyon condensation

We can condense a single boson.

●	gx	bx
ry	gy	by
rz	gz	bz

This means we identify a boson, e.g. rx with the vacuum. We write

$$\mathtt{rx} \simeq 1.$$

In the boson table

● = 'condensed'

What does this mean with respect to the anyon model?

Remember, now charge label $1 \simeq \texttt{rx}$

- The vacuum has trivial exchange (i.e., bosonic):

$$R_{1,1} = 1$$

- The vacuum braids trivially with all anyons $\mathrm{a} \in \mathcal{C}$:

$$S_{\mathrm{a},1} = 1$$

- The vacuum fuses trivially with all anyons $\mathrm{a} \in \mathcal{C}$:

$$\mathrm{a} \times 1 = \mathrm{a}$$

What does this mean with respect to the anyon model?

Remember, now charge label $1 \simeq \texttt{rx}$

- The vacuum has trivial exchange (i.e., bosonic):

$$R_{\texttt{rx},\texttt{rx}} = 1 \qquad \checkmark$$

- The vacuum braids trivially with all anyons $\mathrm{a} \in \mathcal{C}$:

$$S_{\mathrm{a},1} = 1$$

- The vacuum fuses trivially with all anyons $\mathrm{a} \in \mathcal{C}$:

$$\mathrm{a} \times 1 = \mathrm{a}$$

What does this mean with respect to the anyon model?

Remember, now charge label $1 \simeq \texttt{rx}$

- The vacuum braids trivially with all anyons $\mathsf{a} \in \mathcal{C}$:

$$S_{\mathsf{a},\texttt{rx}} = 1$$

As $S_{\times,\texttt{rx}} \neq 1$, we have $\times$ anyons are now forbidden. They are 'confined'.

●	gx	bx
ry	×	×
rz	×	×

● = 'condensed'
× = 'confined'

What does this mean with respect to the anyon model?

Remember, now charge label $1 \simeq \mathtt{rx}$

- The vacuum braids trivially with all anyons $\mathrm{a} \in \mathcal{C}$:

$$S_{\mathrm{a},\mathtt{rx}} = 1$$

for

$$\mathrm{a} = \mathtt{ry}, \mathtt{rz}, \mathtt{gx}, \mathtt{bx}.$$

$$\checkmark$$

Now, $\mathtt{ry}$, $\mathtt{rz}$, $\mathtt{gx}$, $\mathtt{bx}$, remain ‘deconfined’.

●	gx	bx
ry	×	×
rz	×	×

What does this mean with respect to the anyon model?

Remember, now charge label $1 \simeq \texttt{rx}$

- The vacuum fuses trivially with all anyons $\texttt{a} \in \mathcal{C}$:

$$\texttt{a} \times 1 = \texttt{a}$$

What does this mean with respect to the anyon model?

Remember, now charge label $1 \simeq \texttt{rx}$

- The vacuum fuses trivially with all anyons $\texttt{a} \in \mathcal{C}$:

$$\texttt{a} \times \texttt{rx} = \texttt{a},$$

In the parent model we have:

$$\texttt{bx} \times \texttt{rx} = \texttt{gx}$$

But, for condensed $\texttt{rx}$ we need

$$\texttt{bx} \times \texttt{rx} \doteq \texttt{bx}.$$

For the above two relationships to hold we find the identification:

$$\texttt{gx} \simeq \texttt{bx},$$

Likewise

$$\texttt{ry} \simeq \texttt{rz}$$

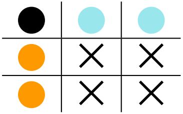
by the same argument.

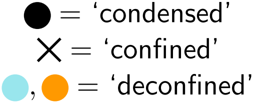
What does this mean with respect to the anyon model?

Remember, now charge label $1 \simeq \texttt{rx}$

rx	gx	bx
ry	gy	by
rz	gz	bz

$\longrightarrow$





where now the following identifications are made

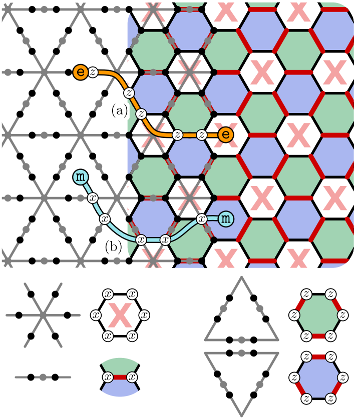
$\texttt{ry} \simeq \texttt{rz} \simeq$ 

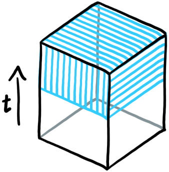
and

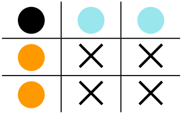
$\texttt{gx} \simeq \texttt{bx} \simeq$

due to condensation. ✓

Condensing a single boson creates a toric code phase







● = 'condensed'

× = 'confined'

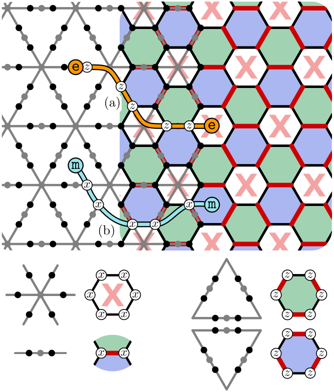
●, ● = 'deconfined'

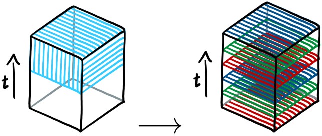
Physically, in the code picture we condense the rx charges by measuring the red XX edge terms.

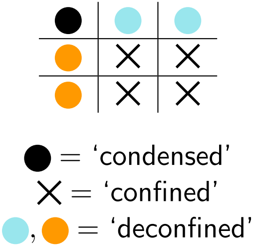
In the Hamiltonian picture we might add hopping terms V for rx charges $H = H_0 + V$

Floquet codes

Hastings and Haah recently proposed Floquet codes, such as the honeycomb code, where syndromes are read out using a sequence of two-body measurements.



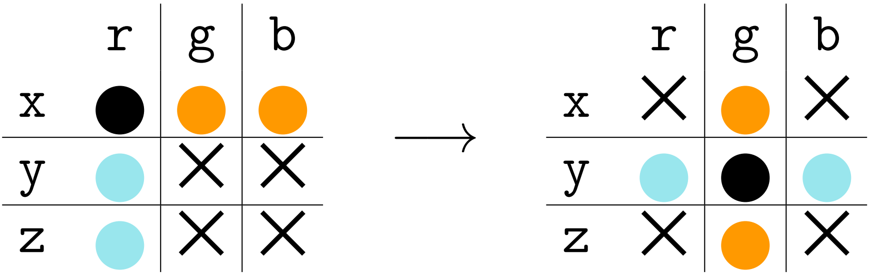




Hastings and Haah, Quantum (2021)

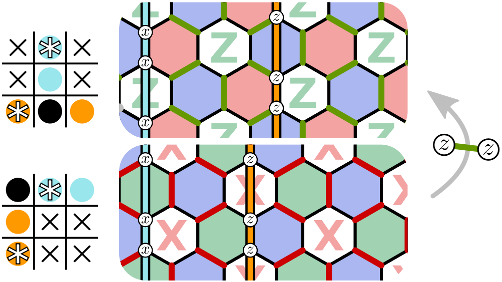
Floquet code transformations

Floquet code transformations can be viewed as a condensation operation on a confined boson.



Floquet code transformations

A single transformation from the `rx` condensed code to the gz condensed code

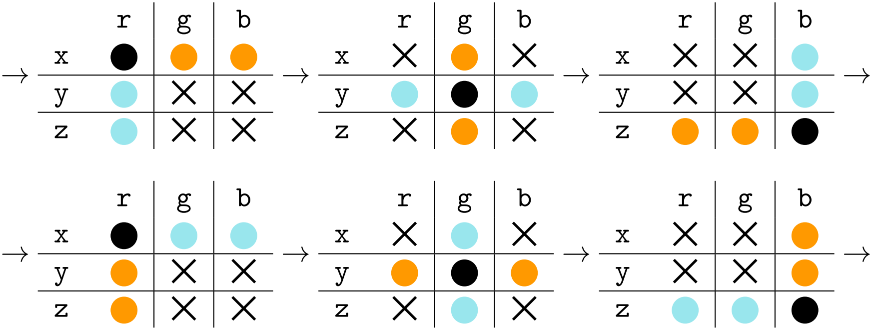


Over the transformation:

- Green Pauli-X stabilizers are turned off
- Red Pauli-Z stabilizers are initialised
- Blue Pauli-Z stabilizers are measured
- A pair of deconfined charges are preserved over the transformation
- Up to a color and Pauli label, the code is preserved

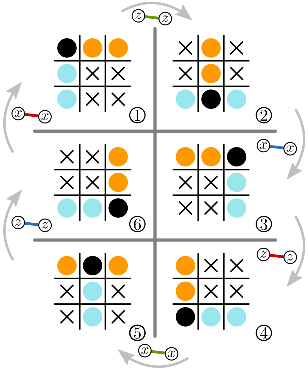
The honeycomb code

The honeycomb code cyles between a $\to$ rx $\to$ gy $\to$ bz $\to$ condensate



The Floquet color code

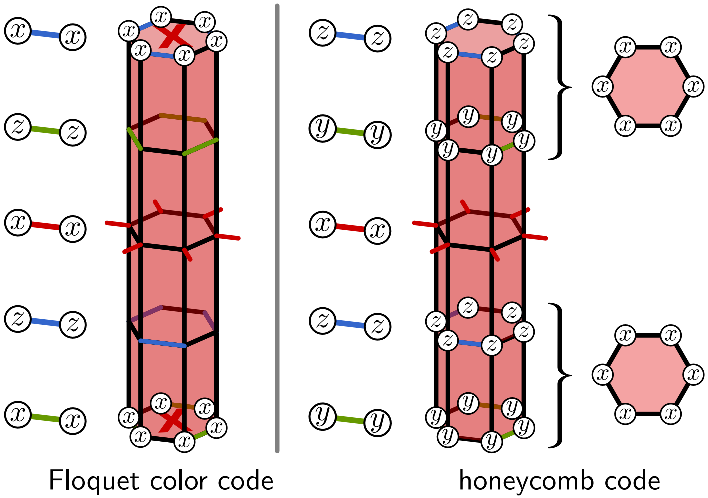
We propose a CSS Floquet code, that we call the Floquet color code, by changing the condensation sequence.



Davydova *et al.* PRX Quantum (2023)
Kesselring *et al.*, 2212.00042
Bombin *et al.*, arXiv:2303.08829

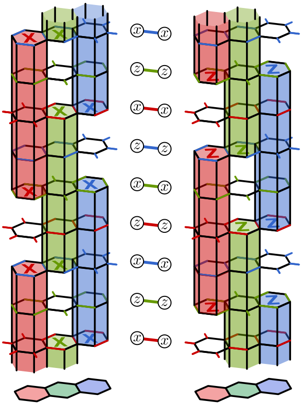
Detection cells - stabilizers in spacetime

Measurements on each hexagon are compared over time to detect errors.



Detection cells in spacetime

In the Floquet color code, we measure Pauli-X and Pauli-Z detection cells on all hexagons.

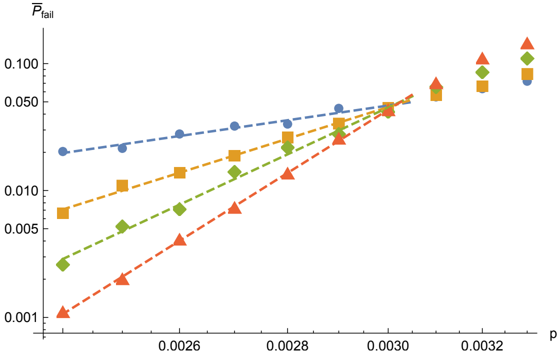


However, we also TURN OFF stabilizers.

Expressed as a subsystem code, the Floquet color code has no local stabilizers

A numerical threshold for circuit-level noise

We simulated the Floquet color code for standard circuit level noise.

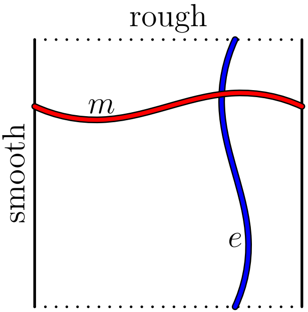


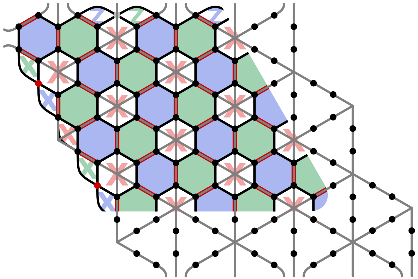
The threshold is competitive with the honeycomb code.

Gidney, Newman, McEwen, arXiv:2202.11845 (2022)
See also numerics in Ref.
Paetznick *et al.* arXiv:2202.11829 (2022)

Boundaries in the condensation picture

The condensation picture shows us a constructive way to write down boundaries

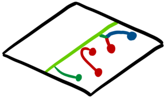




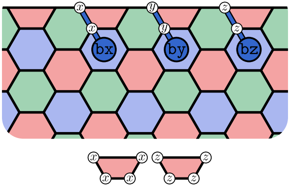
A planar implementation of a Floquet code requires rough and smooth boundaries.

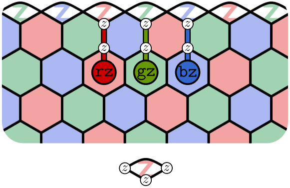
Dennis *et al.* J. Math. Phys. (2001)
Haah and Hastings, Quantum (2022)

Lagrangian subgroups and color code boundaries (maximal condensation)



rx	gx	bx
ry	gy	by
rz	gz	bz





×	×	●
×	×	●
×	×	●

×	×	×
×	×	×
●	●	●

Levin, Phys. Rev. X (2013)

Kesselring et al., Quantum **2**, 101 (2018)

Boundaries from the parent model

We condense the `rx` boson of the color code

rx	gx	bx
ry	gy	by
rz	gz	bz

$\longrightarrow$

●	●	●
●	×	×
●	×	×

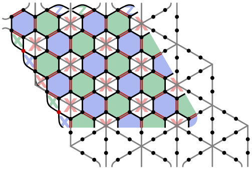
The parent rough boundary should condense red charges and the parent smooth boundary should condense Pauli-X charges.

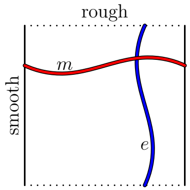
●	×	×
●	×	×
●	×	×

red parent boundary

●	●	●
×	×	×
×	×	×

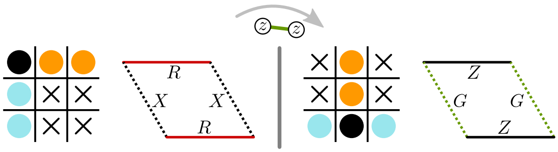
Pauli-X parent boundary





Boundary transformations

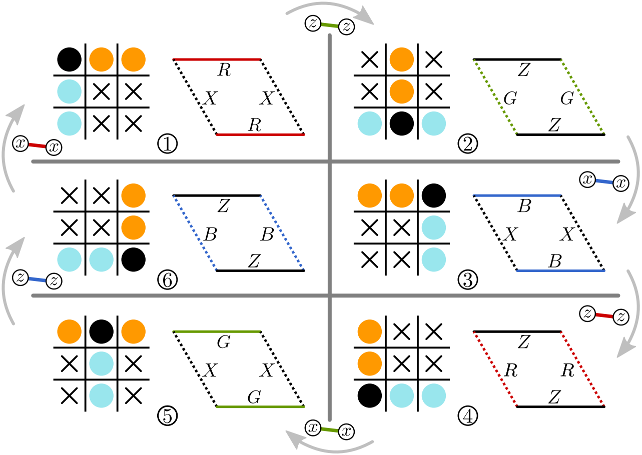
The condensation picture dictates how the boundaries must transform.



And the new boundaries can also be obtained from the parent color code theory.

Boundary transformations for the Floquet color code

The transformations can be extended to a full period of the Floquet color code



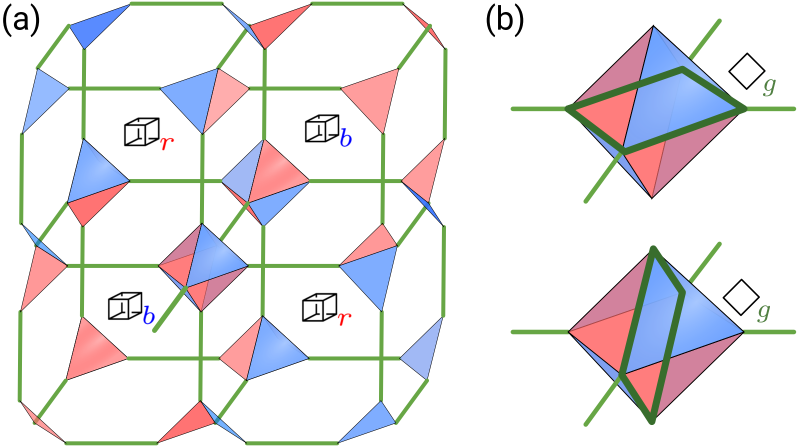
Outlook

- Floquet codes give us a practical route to realise quantum error correcting codes
- Anyon condensation is a helpful picture to obtain and generalise Floquet codes from a parent theory.

- Can we generalise Floquet codes for e.g. constant rate codes, or for more general logic gates?

Other Floquet codes from condensation

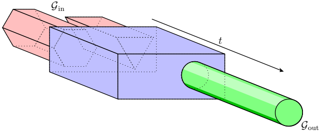
3D Floquet codes and Floquet fracton codes have been produced by condensation



Davydova *et al.*, PRX Quantum 4, 020341 (2023)
Dua *et al.*, arXiv:2037.13668

How can we think about Floquet codes?

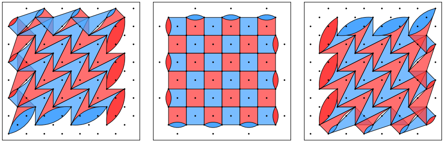
Floquet code transformations are just instances of code deformations/gauge fixing on subsystem codes



Vuillot *et al.* New J. Phys. 21, 033028

Brown and Roberts, Phys. Rev. Research 2, 033305 (2020)

Maybe we can think of these as generalised syndrome readout circuits



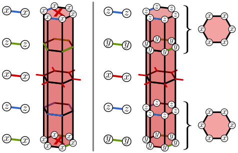
McEwen *et al.*, Quantum 7, 1172 (2023)

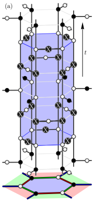
Bacon *et al.*, IEEE Trans. Info. Theor. 63, 2464 (2017)

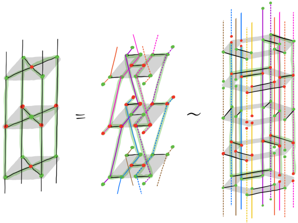
Delfosse and Paetznick, arXiv:2304.05943

A space-time picture for Floquet codes

Floquet codes look much more ‘ordinary’ in three-dimensional pictures





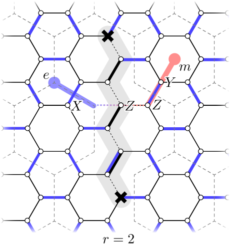


Foliated MBQC picture:
Brown and Roberts, Phys. Rev. Research **2**, 033305 (2020)
Paesani and Brown, Phys. Rev. Lett. **131**, 120603 (2023)

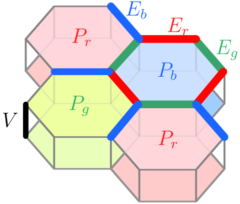
Tensor network picture:
Bombin *et al.*, arXiv:2303.08829
Bauer, arXiv:2303.16405
Townsend-Teague *et al.*, EPTCS 384, 265 (2023)

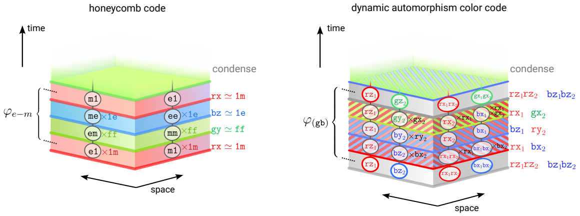
Logic gates

Can we find logic gates for Floquet codes?



Twist braiding: Ellison *et al.* arXiv:2306.08027

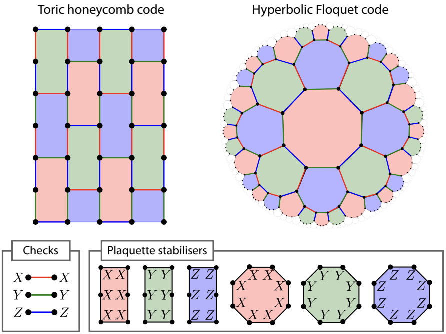




Transversal color code gates by measurement:
Davydova *et al.* arXiv:2307.10353

Good Floquet codes

Can we make Floquet versions of good LDPC codes?



Recent work on hyperbolic Floquet codes.
Higgott and Breuckmann, arXiv:2308.03750
Fahimniya *et al.* arXiv:2309.10033