Probabilistic and Combinatorial Methods - Part 2

Error-Correcting Codes: Theory and Practice Boot Camp

Jonathan Mosheiff
Ben-Gurion University

Let’s recall

Definition: A **random code ensemble** $C \subseteq \mathbb{F}_q^n$ is **k-locally-similar** to an **RLC of rate R** if

$$\Pr\left[\{v_1, \ldots, v_k\} \subseteq C\right] \lesssim 2^{-(1-R)\cdot n \cdot \dim\{v_1,\ldots,v_k\}}$$

for all $v_1, \ldots, v_k \in \mathbb{F}_q^n$.

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for all $v_1, \ldots, v_k \in \mathbb{F}_q^n$.

Theorem: If a **random code ensemble** C is k**-locally-similar** to an **RLC of rate** R then it **achieves the list-decoding GV-bound** and any other **monotone,** k**-local and symmetric property** of **RLC codes** with high probability.

Warm up: The generalized Wozencraft Ensemble

Definition: Let $n = 2m$. Let $\varphi : \mathbb{F}_{2^m} \to \mathbb{F}_2^m$ be the natural binary encoding.

Sample α uniformly at random from $\mathbb{F}_q$.

The **Wozencraft ensemble** is

$$C_\alpha = \left\{ (\varphi(x), \varphi(\alpha x)) \mid x \in \mathbb{F}_q) \right\} \subseteq \mathbb{F}_2^n$$

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Claim: C_α is 1**-locally-similar** to an **RLC of rate** $\frac{1}{2}$**.**

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Claim: C_α is **1-locally-similar** to an **RLC of rate $\frac{1}{2}$.**

Proof: Let $y \in \mathbb{F}_2^n \backslash \{0\}$.

There is a unique way to write $y = (\varphi(x), \varphi(\beta x))$ for some $\beta \in \mathbb{F}_2^n$.

So $y \in \mathbb{C}_\alpha$ only if $\alpha = \beta$, which happens with probability $2^{-m} = 2^{-\frac{n}{2}}$.

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Claim: C_α is 1**-locally-similar** to an **RLC of rate** $\frac{1}{2}$**.**

Corollary: The **Wozencraft ensemble** achieves the same 1**-local properties** as an **RLC of rate** $\frac{1}{2}$**.**

In particular, it **achieves the GV-bound for minimal distance**.

Warm up: The generalized Wozencraft Ensemble

Definition: Let $n = 2m$. Let $\varphi : \mathbb{F}_{2^m} \to \mathbb{F}_2^m$ be the natural binary encoding.

Sample a uniformly random polynomial p of degree $< k$ from $\mathbb{F}_q[x]$.

The k**-generalized Wozencraft ensemble** is

$$C_p = \left\{ (\varphi(x), \varphi(p(x))) \mid x \in \mathbb{F}_q \right\} \subseteq \mathbb{F}_2^n$$

Claim: C_p is k**-locally-similar** to an **RLC of rate** $\frac{1}{2}$**.**

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Corollary: The **Wozencraft ensemble** achieves the same k**-local properties** as an **RLC of rate $\frac{1}{2}$.**

In particular, it **achieves the list-decoding GV-bound** for list size up to k.

Randomly Punctured Low-Bias codes

Theorem [Guruswami-M]: Let D be a **low-bias code** and let C be a **random puncturing of D.** Then C is **k-locally similar** to an **RLC** of similar rate.

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Partially derandomized by **[Putterman-Pyne]**

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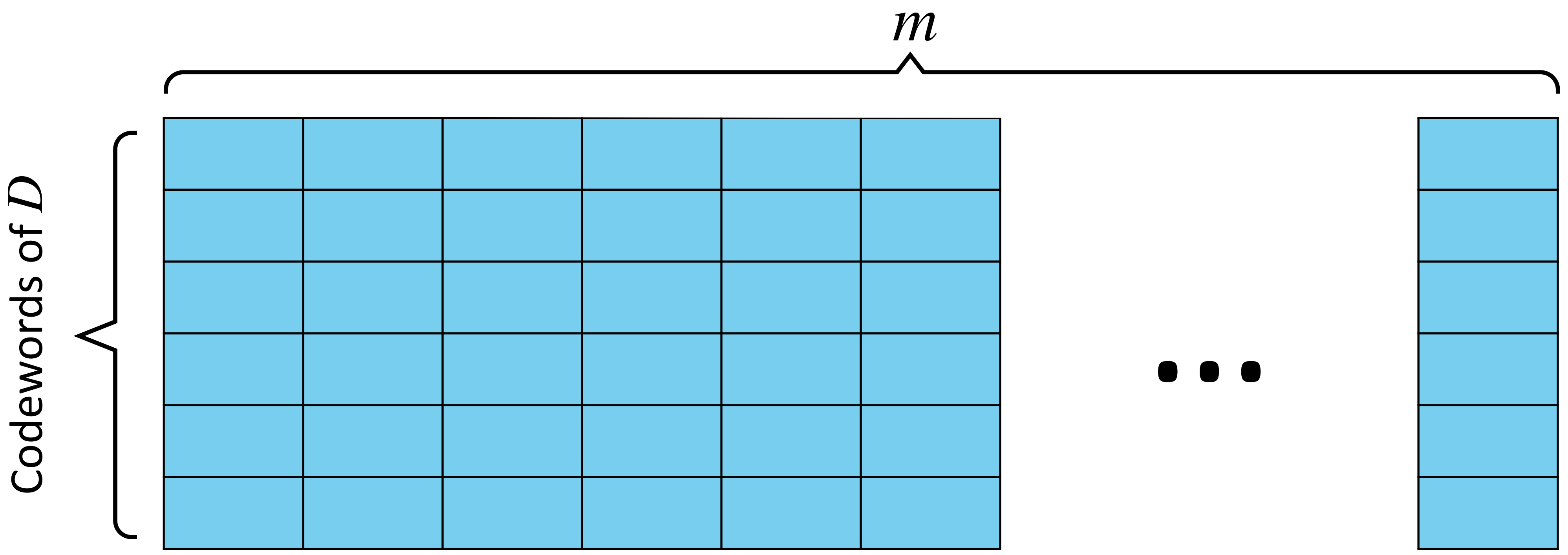
Puncturing of Codes



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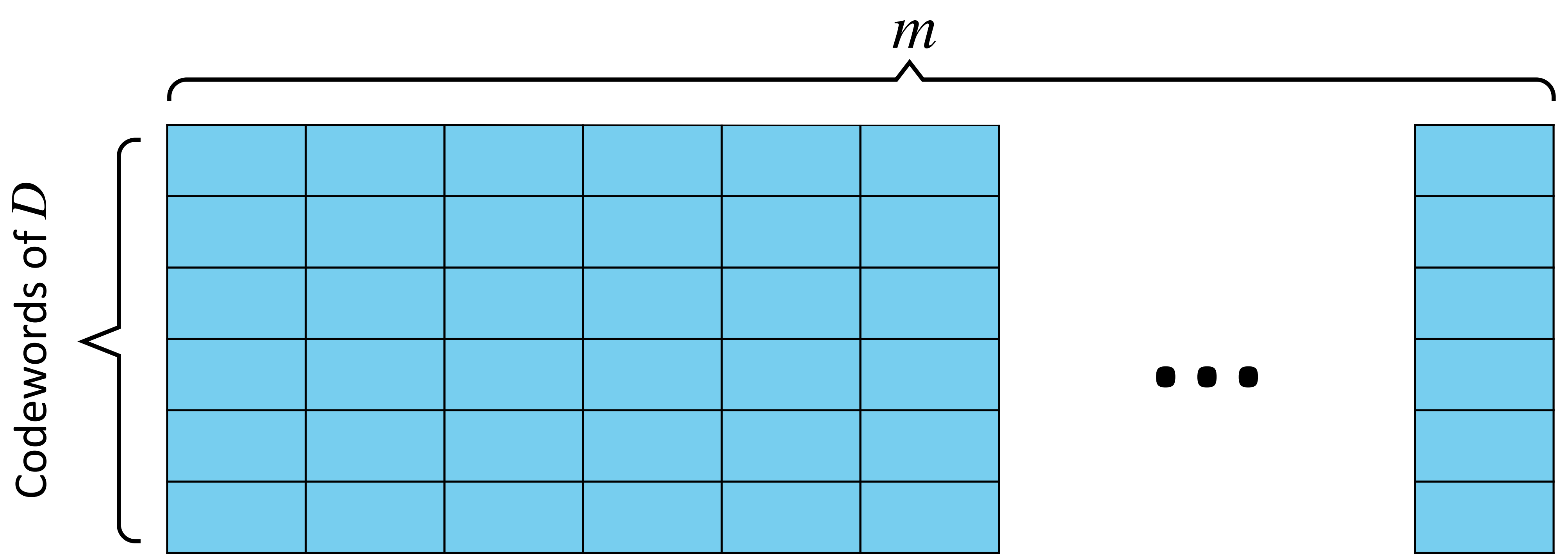
- From a code $D \subseteq \mathbb{F}_q^m$ create a new code $C \subseteq \mathbb{F}_q^n$. Usually $n \ll m$.



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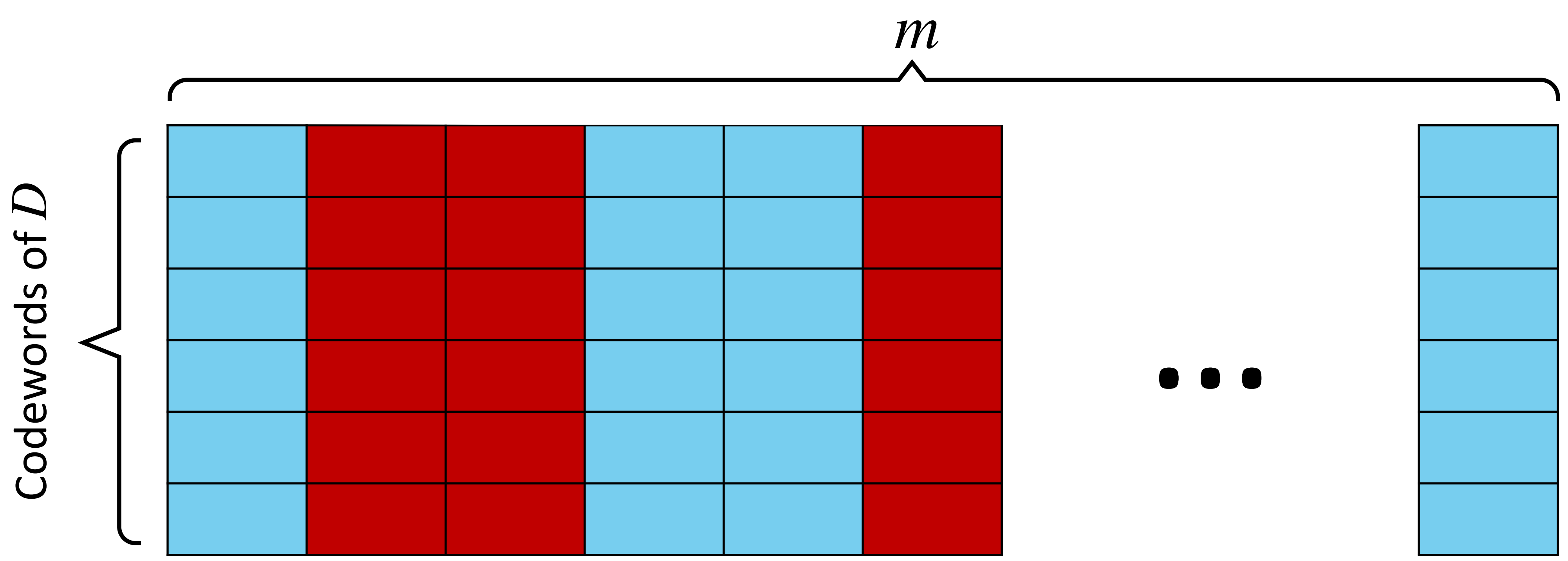

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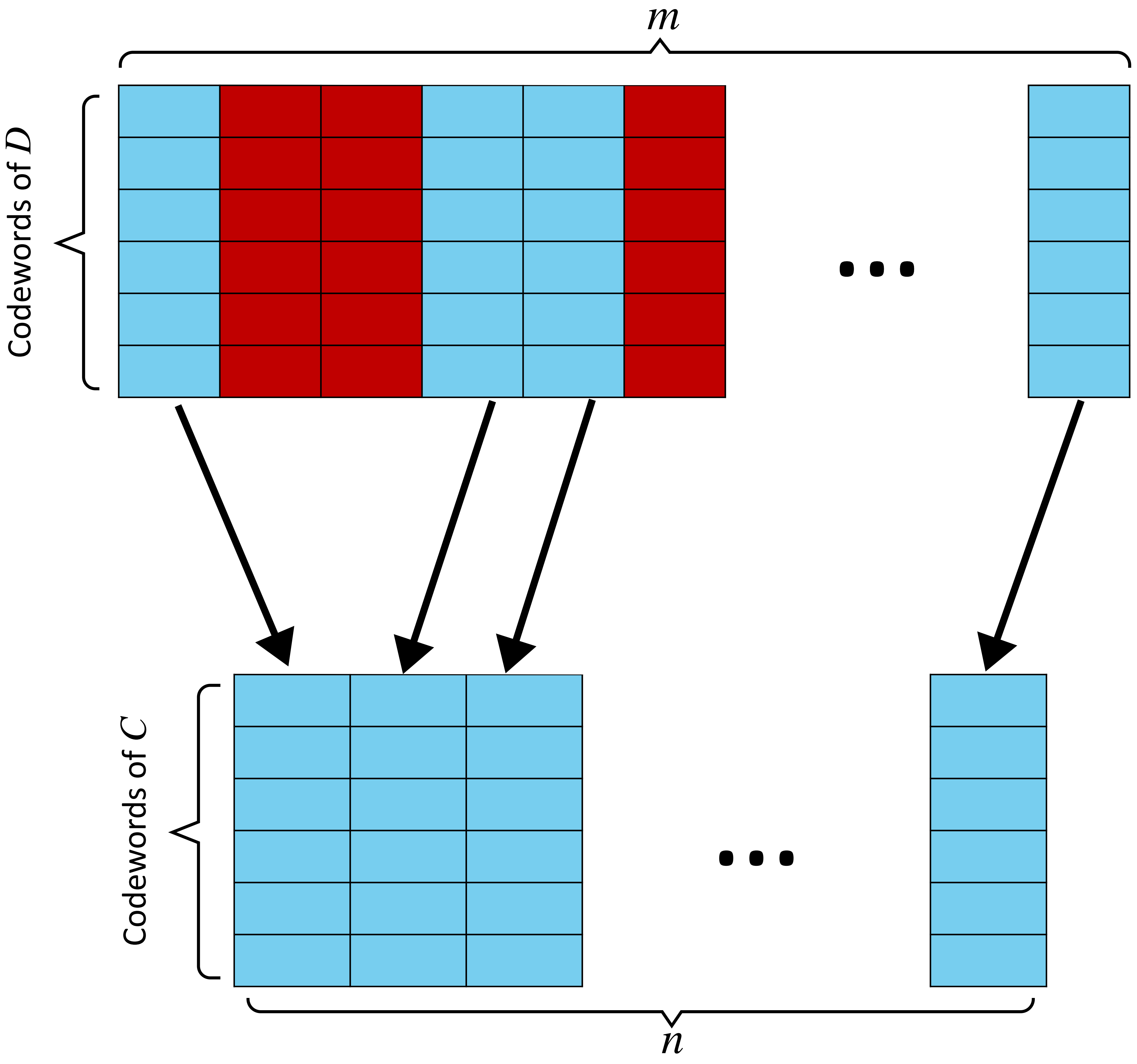

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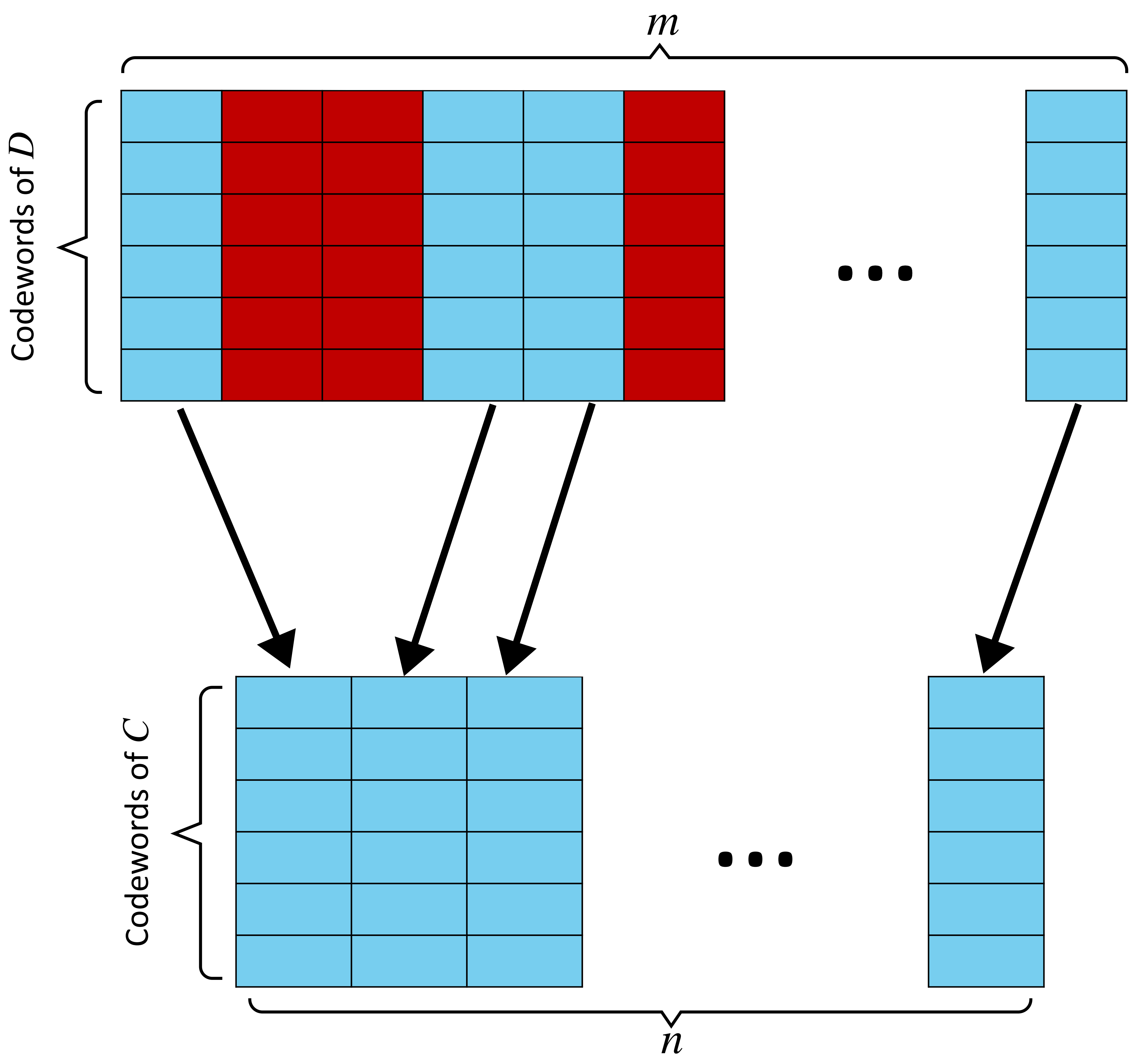

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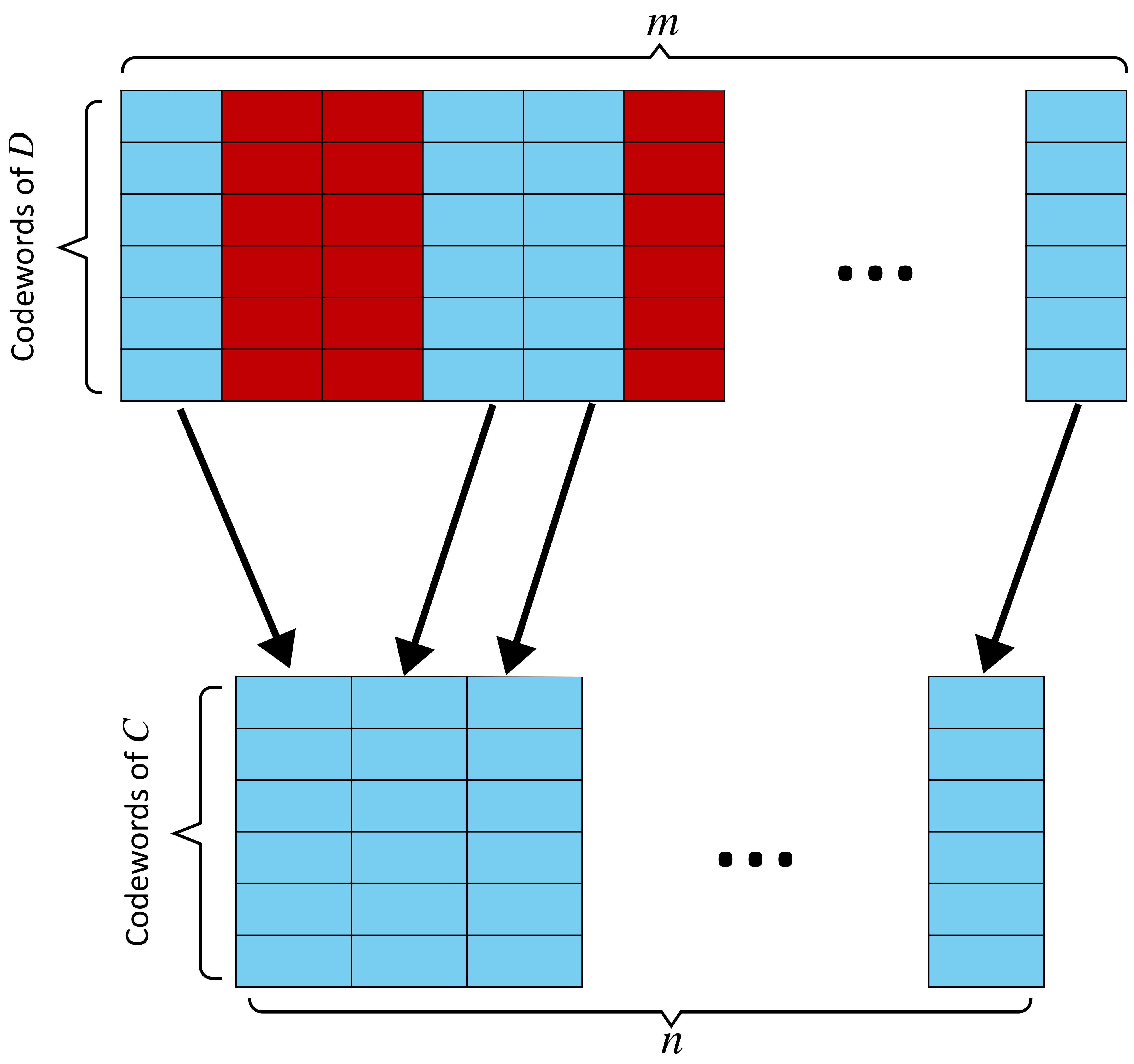
- **Example:** An **RLC** of rate R in $\mathbb{F}_q^n$ is a **random puncturing** of the **Hadamard code** $H \subseteq \mathbb{F}_q^{q^{Rn}}$.



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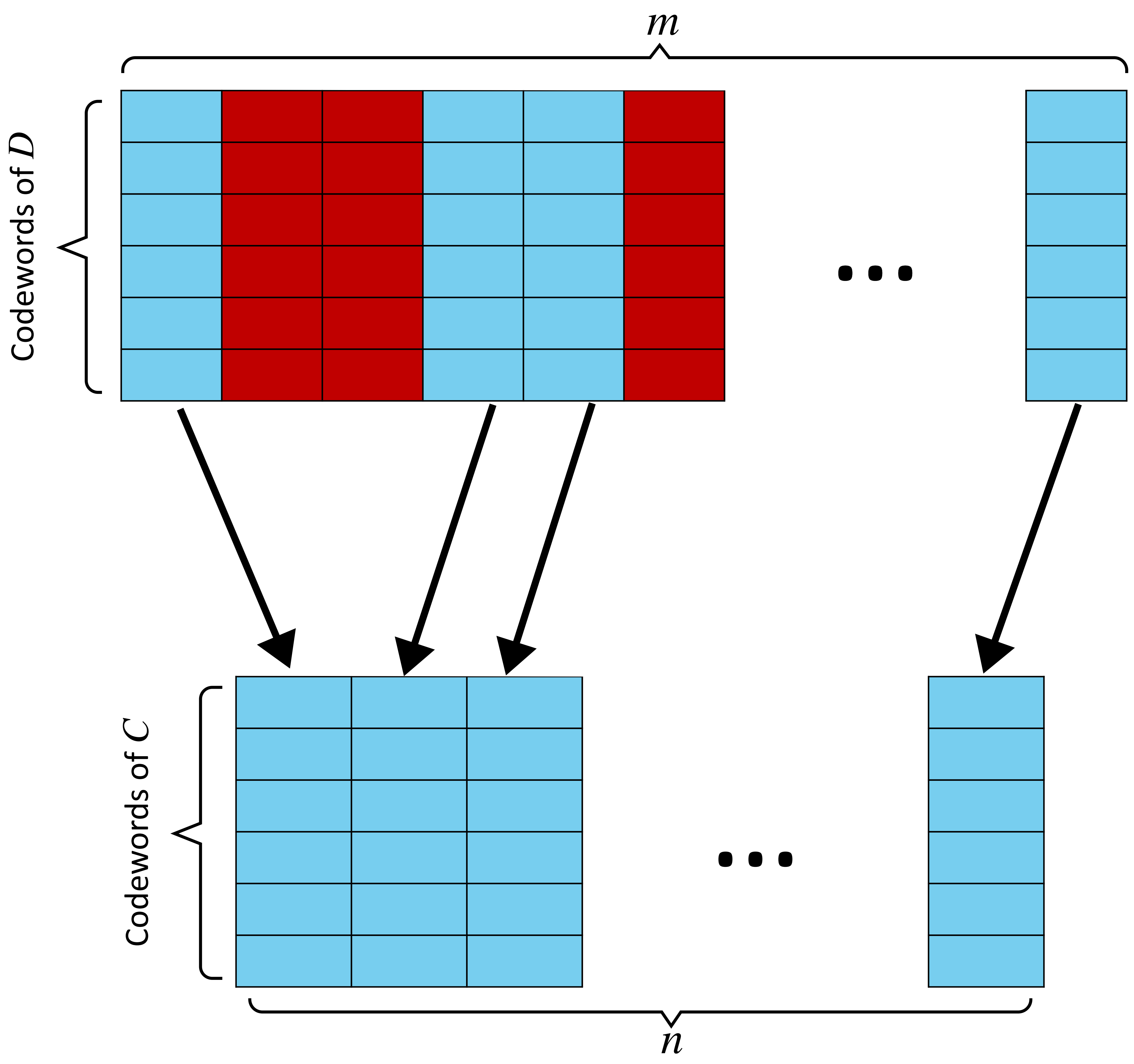
- A **Reed-Solomon code over a random evaluation set** is a **random puncturing** of the **full Reed-Solomon code.**



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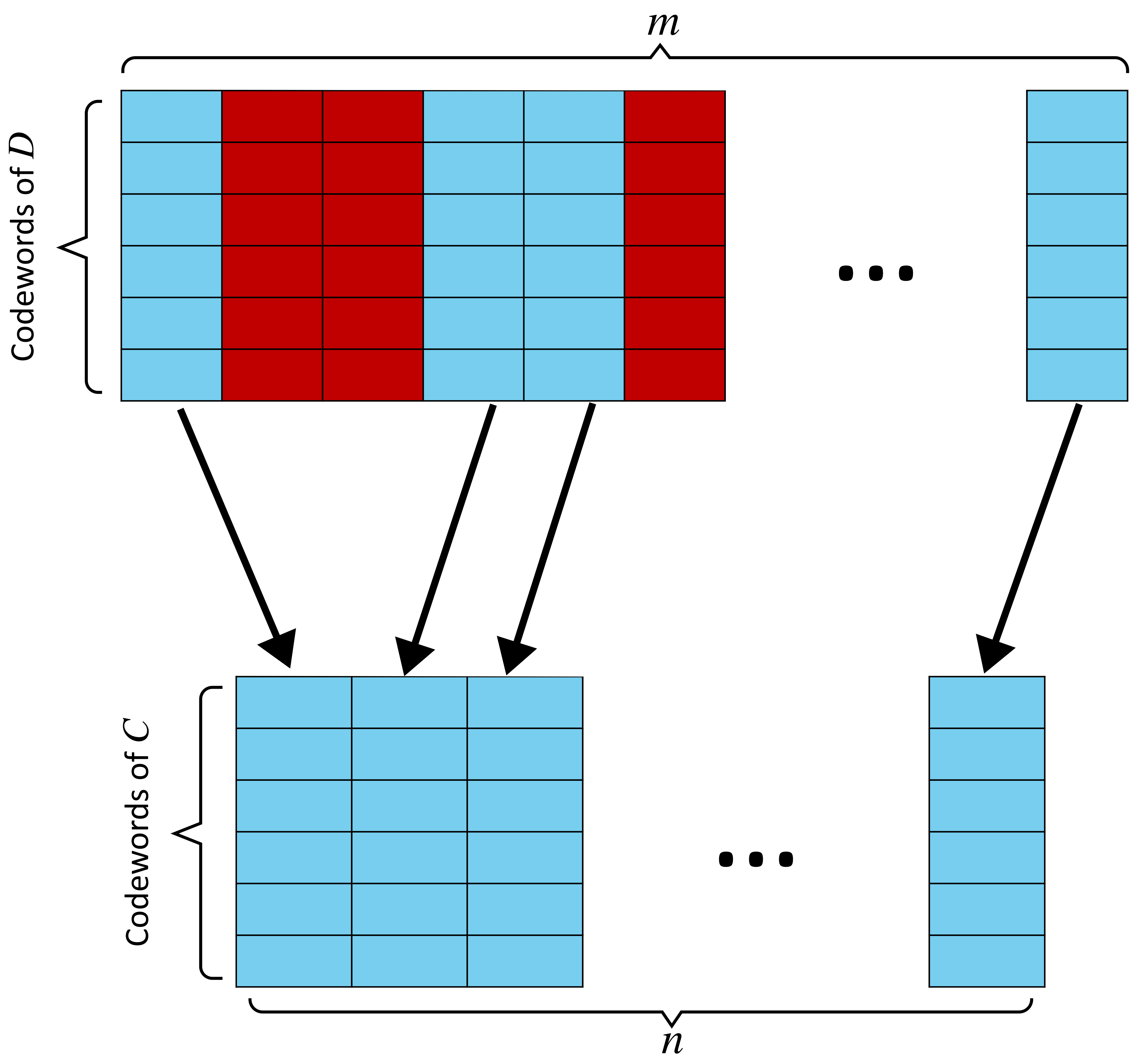

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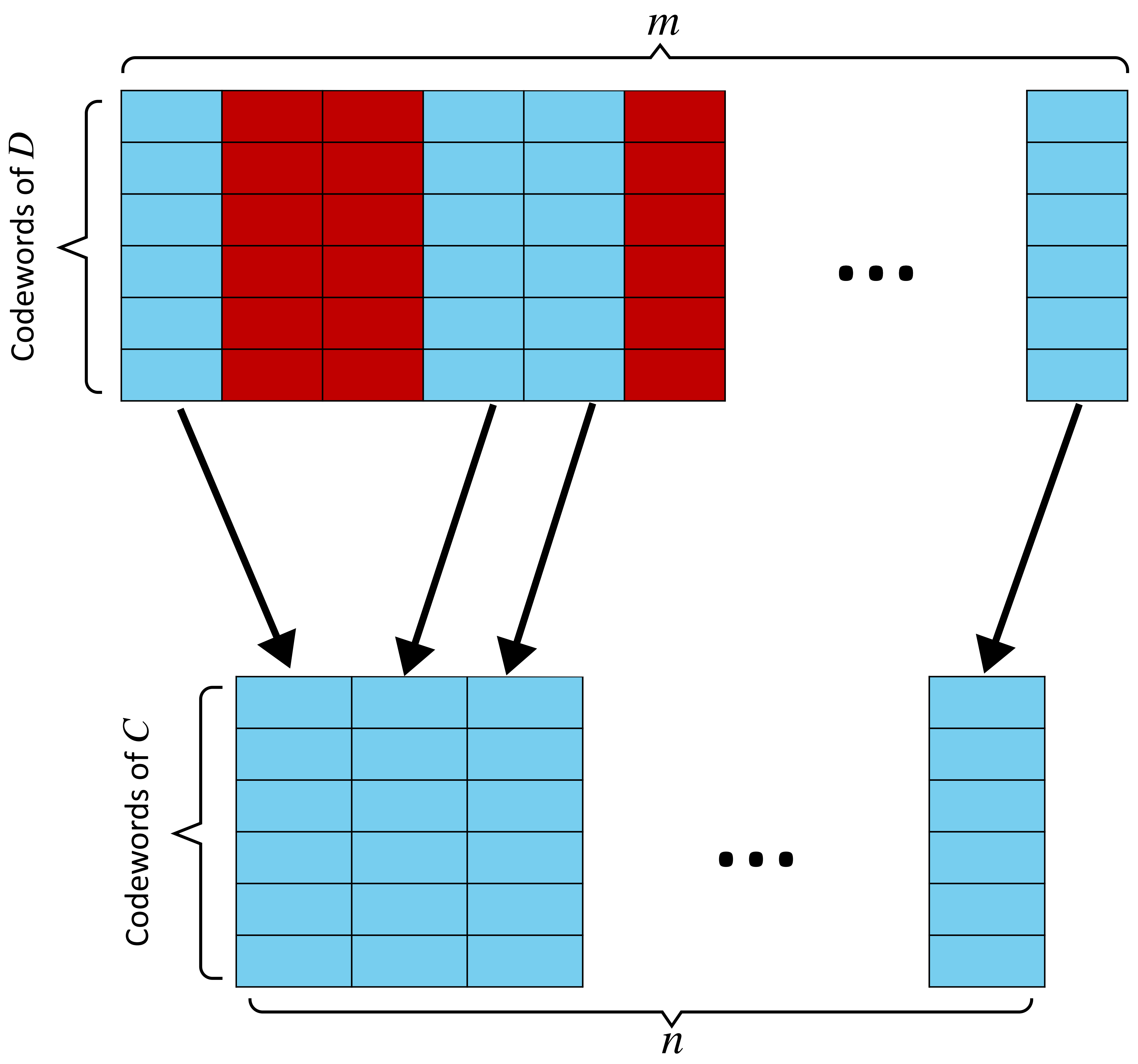

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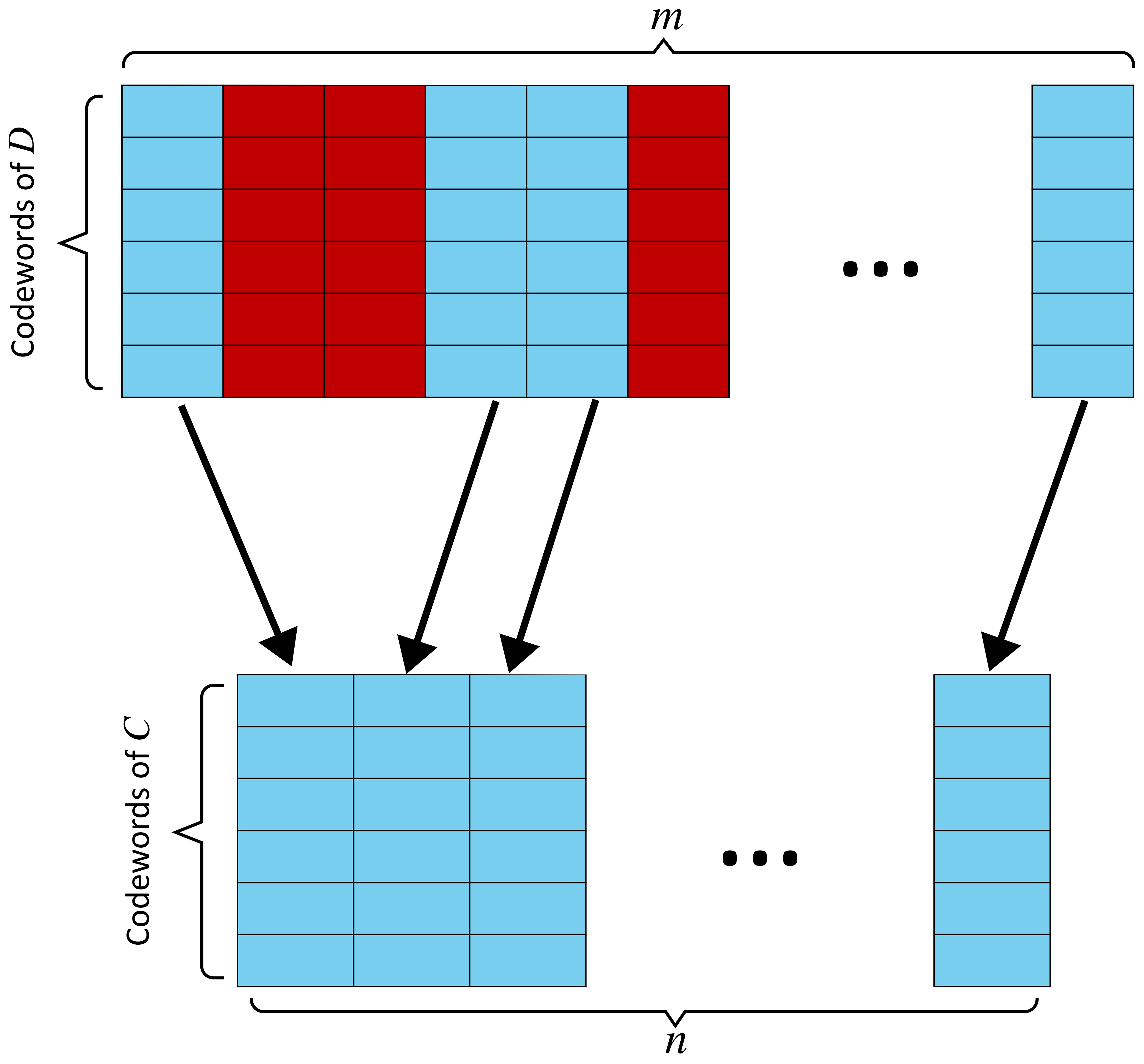

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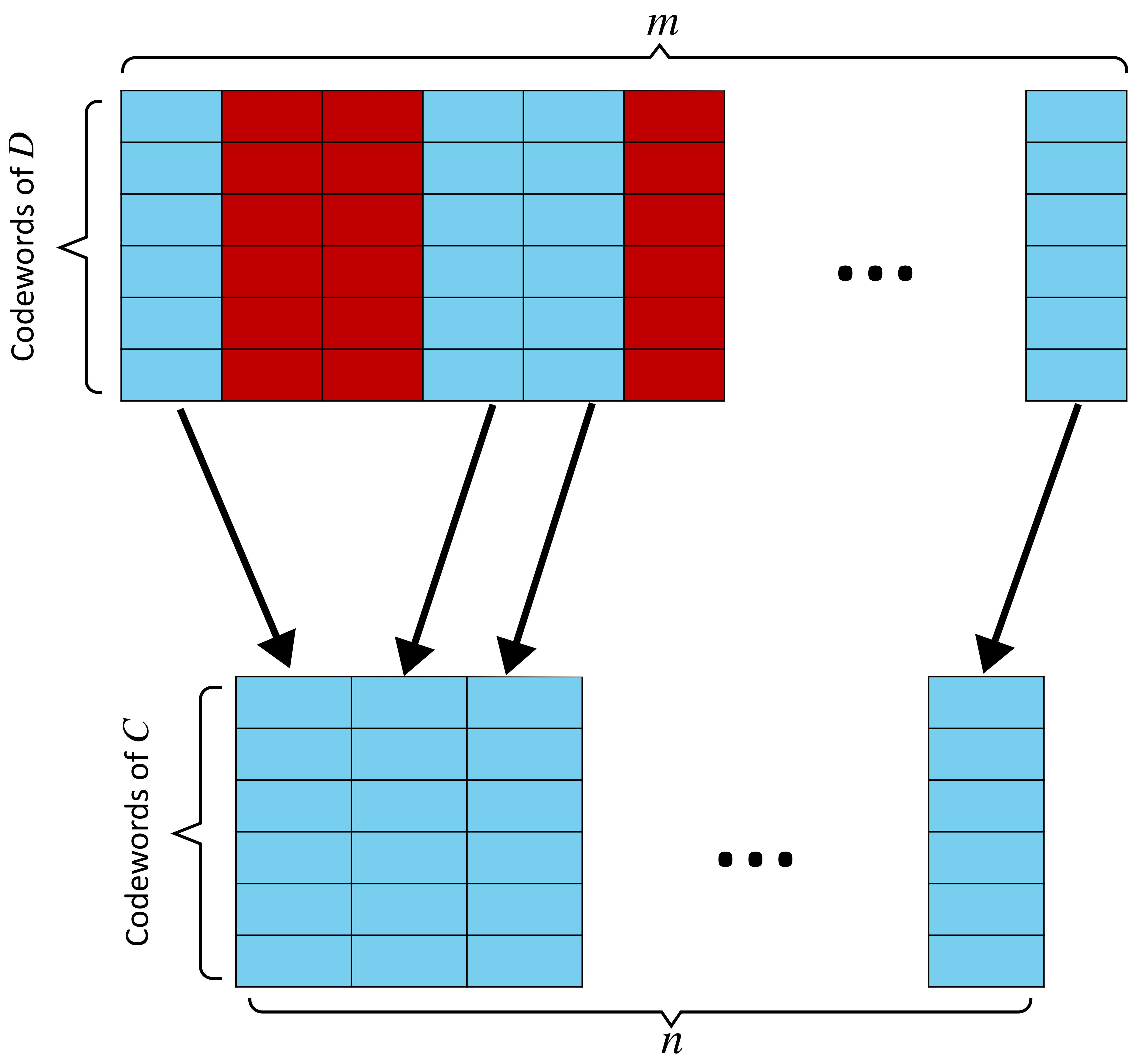

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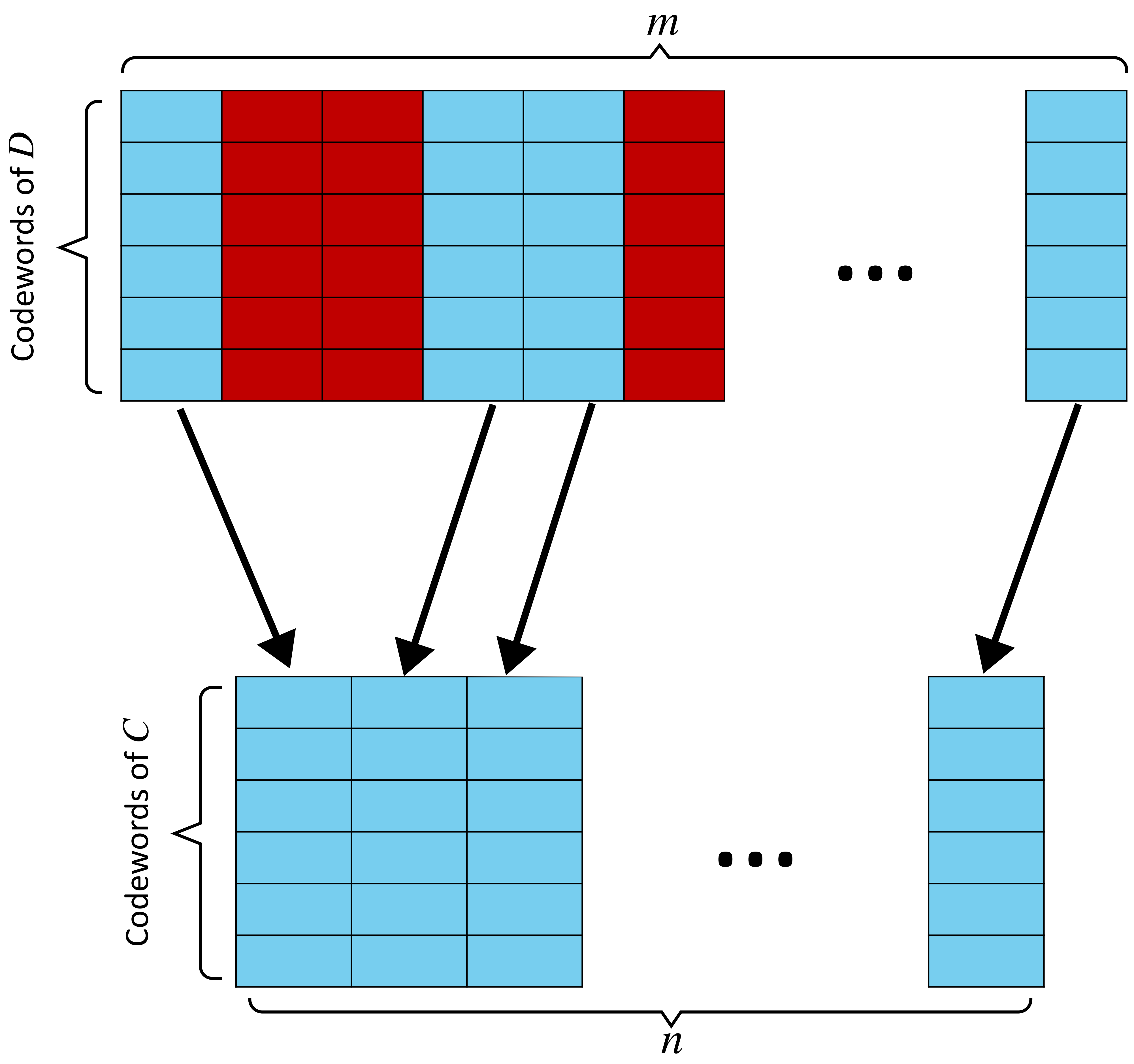

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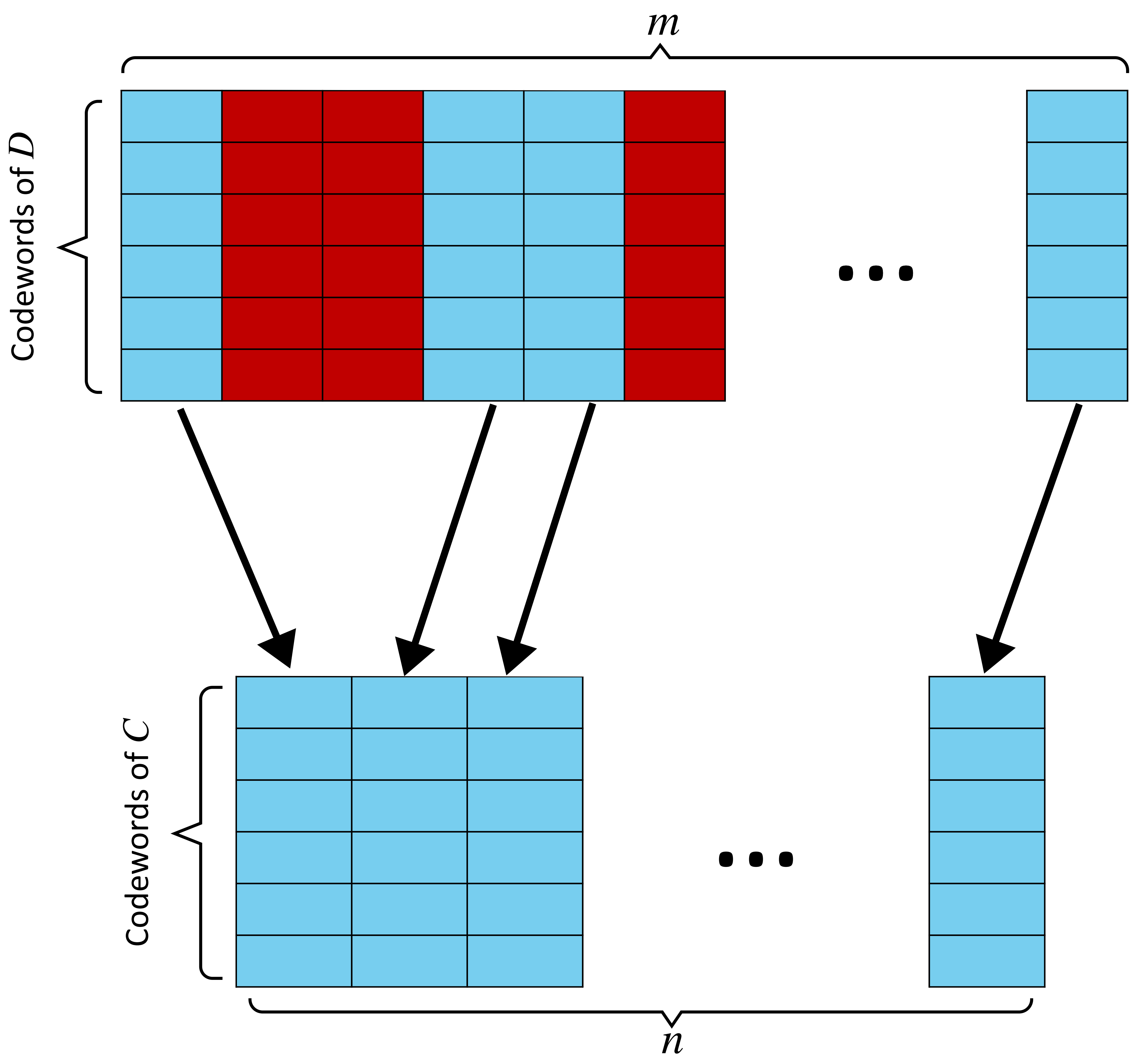
- **Theorem:** C is **locally-similar** to an **RLC**.
- **Corollary:** C is as **list-decodable** and **list-recoverable** as an **RLC**.



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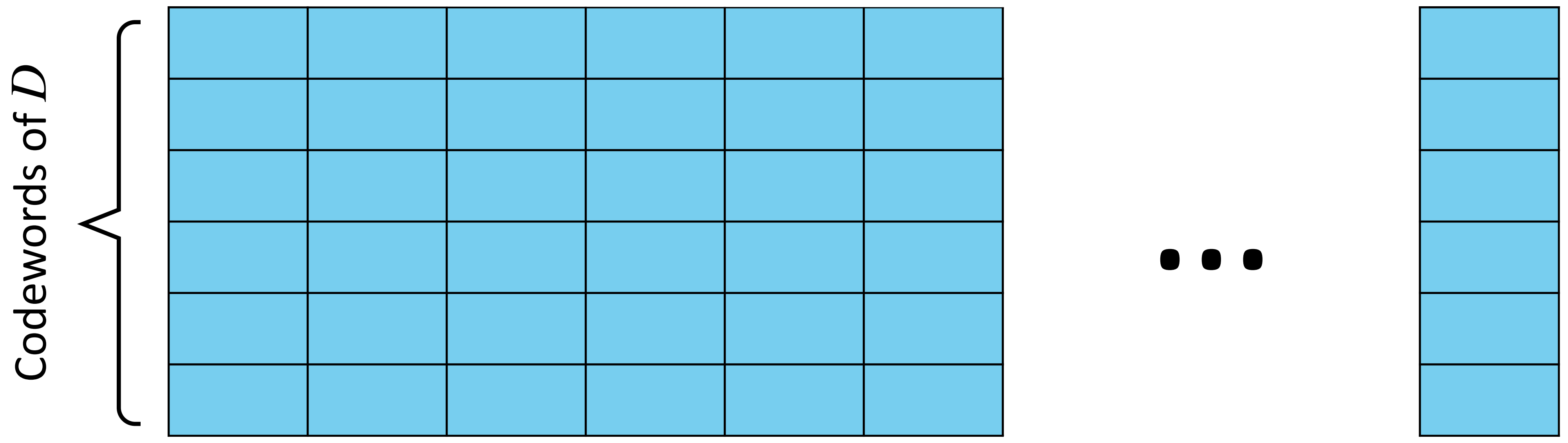
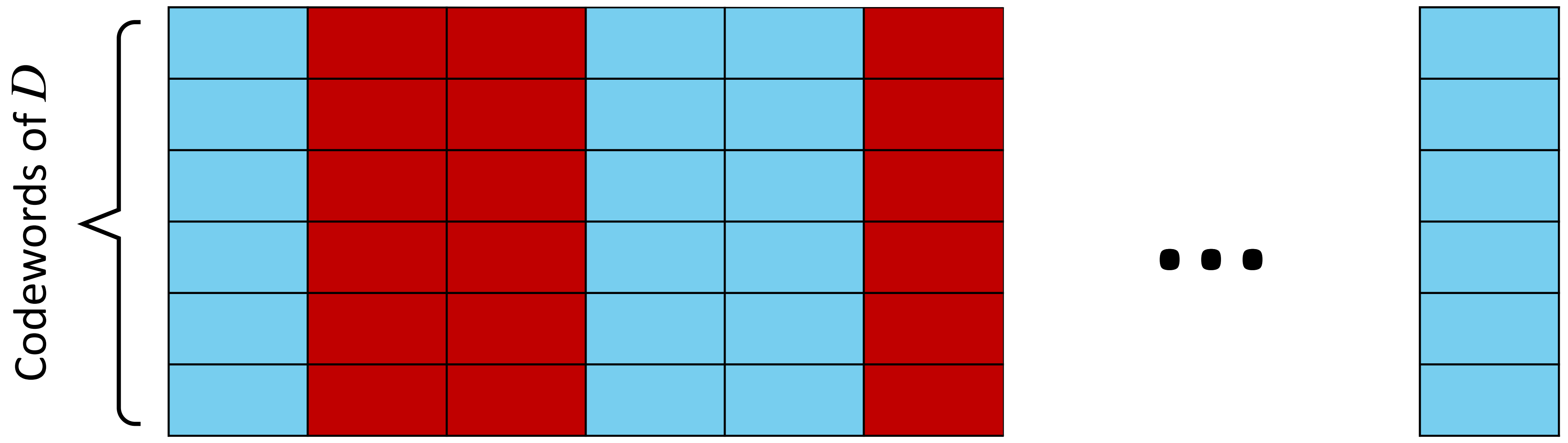


Proof sketch: C is locally-similar to an RLC.



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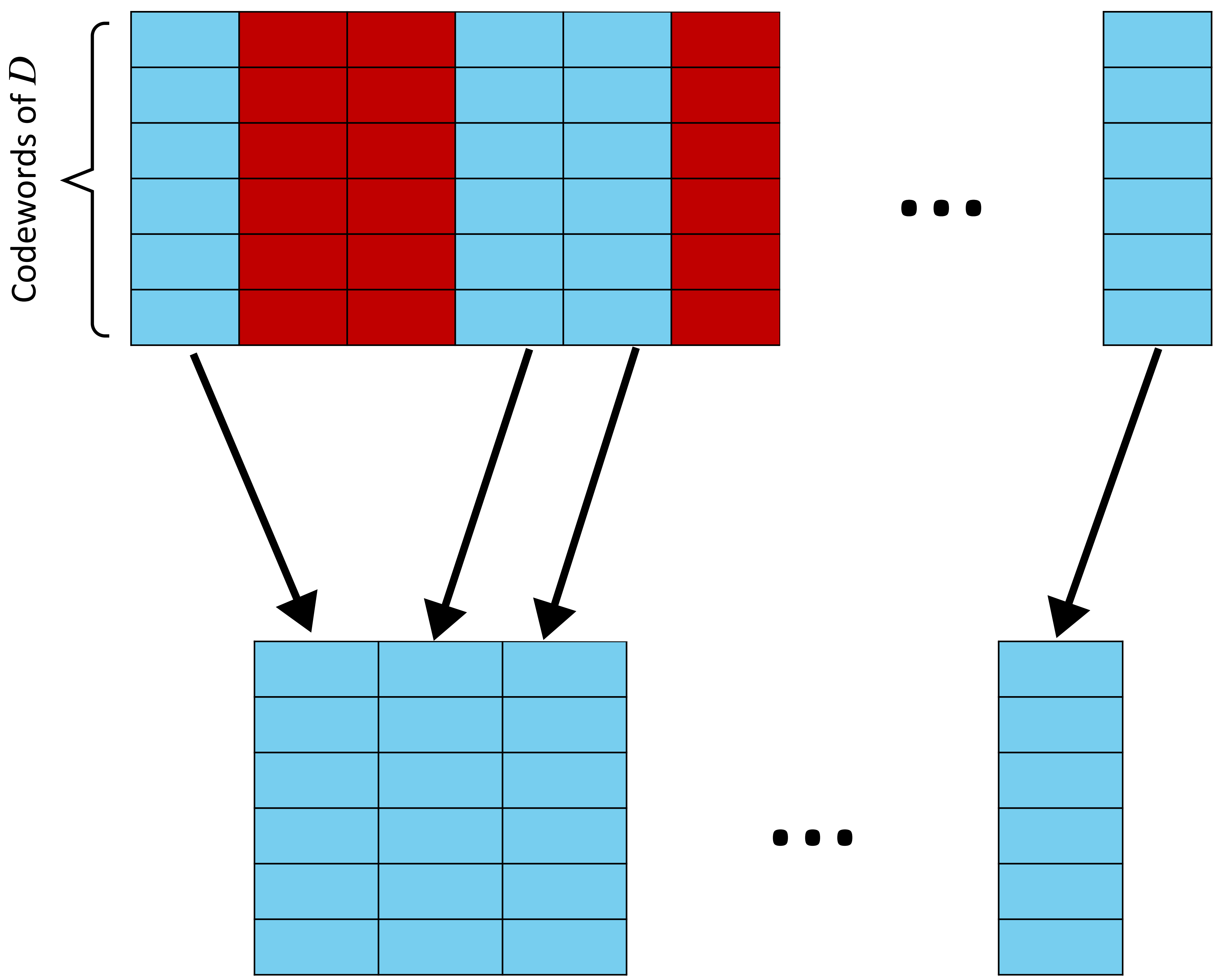

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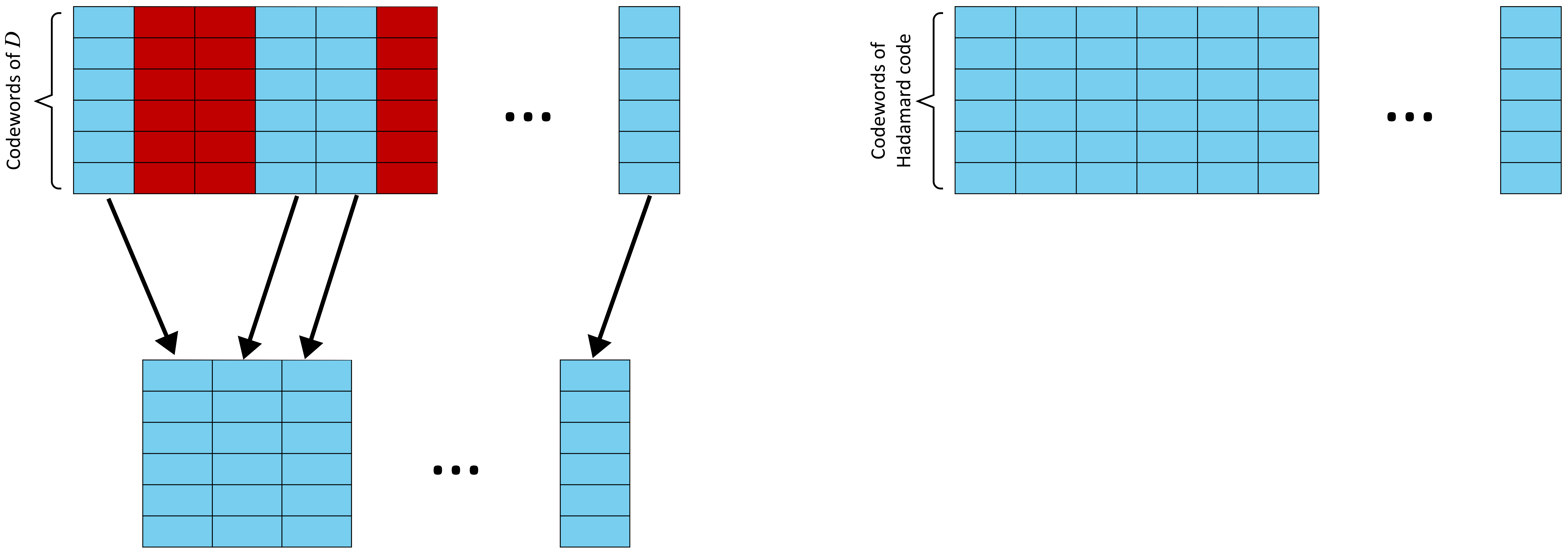

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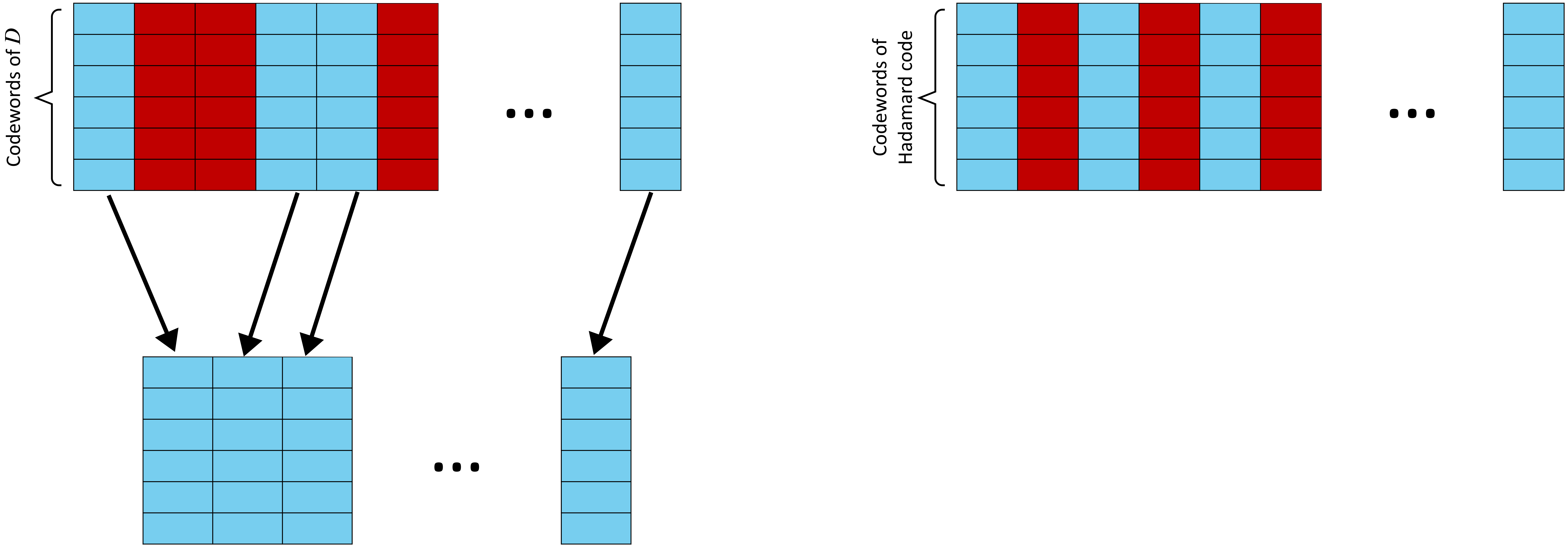

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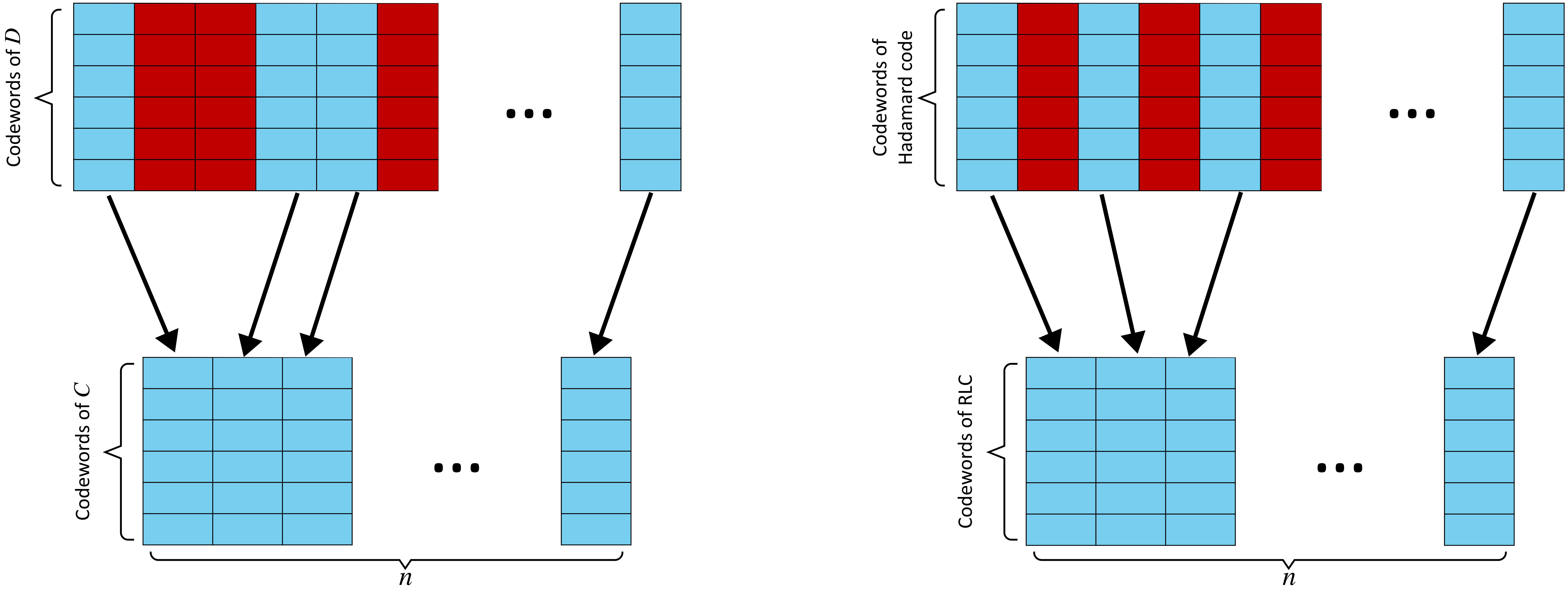

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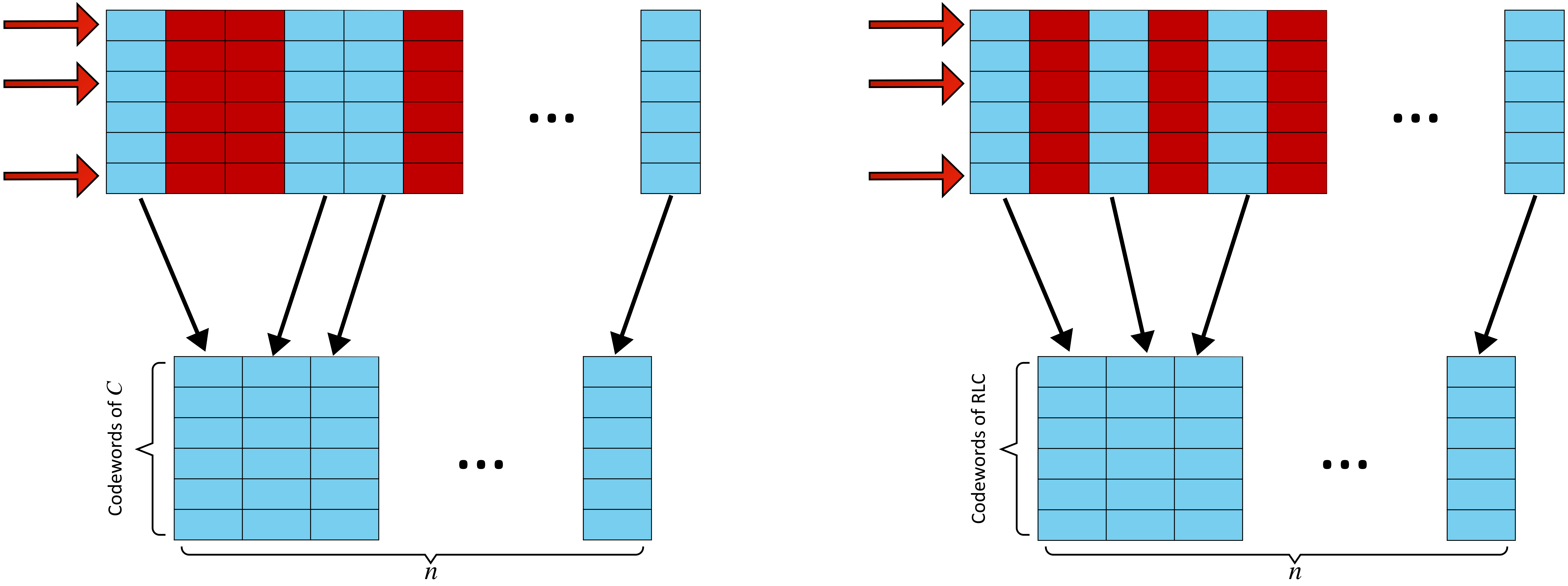

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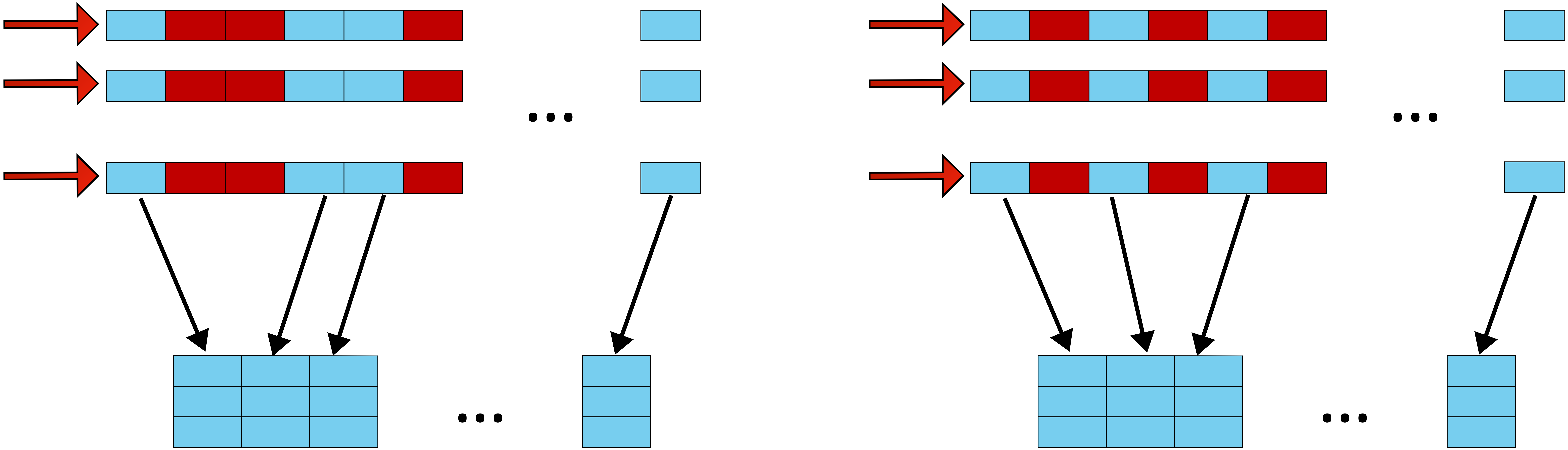

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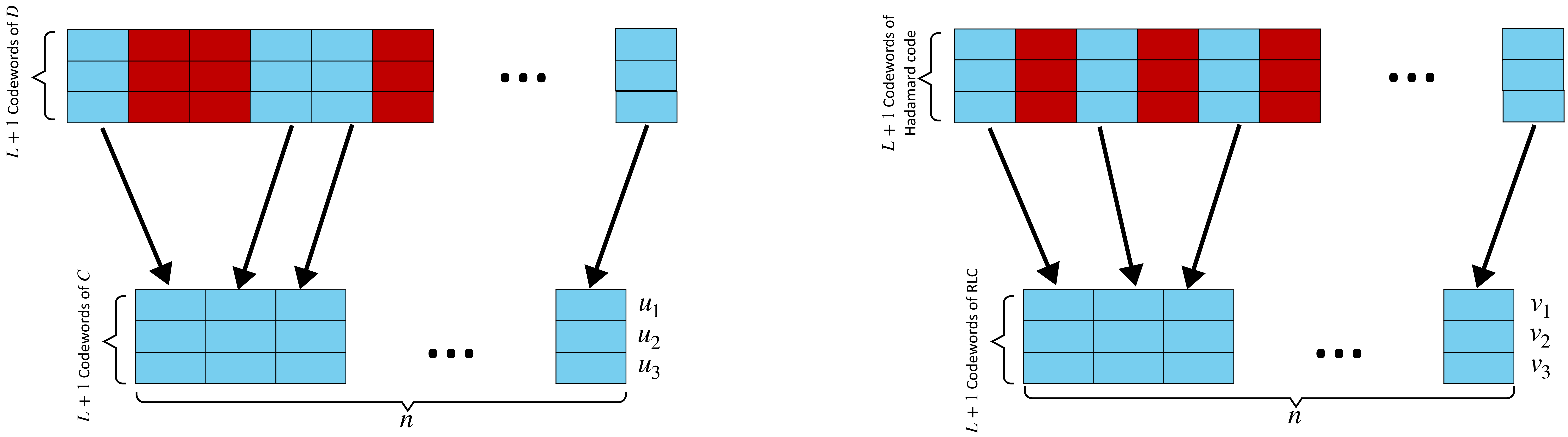

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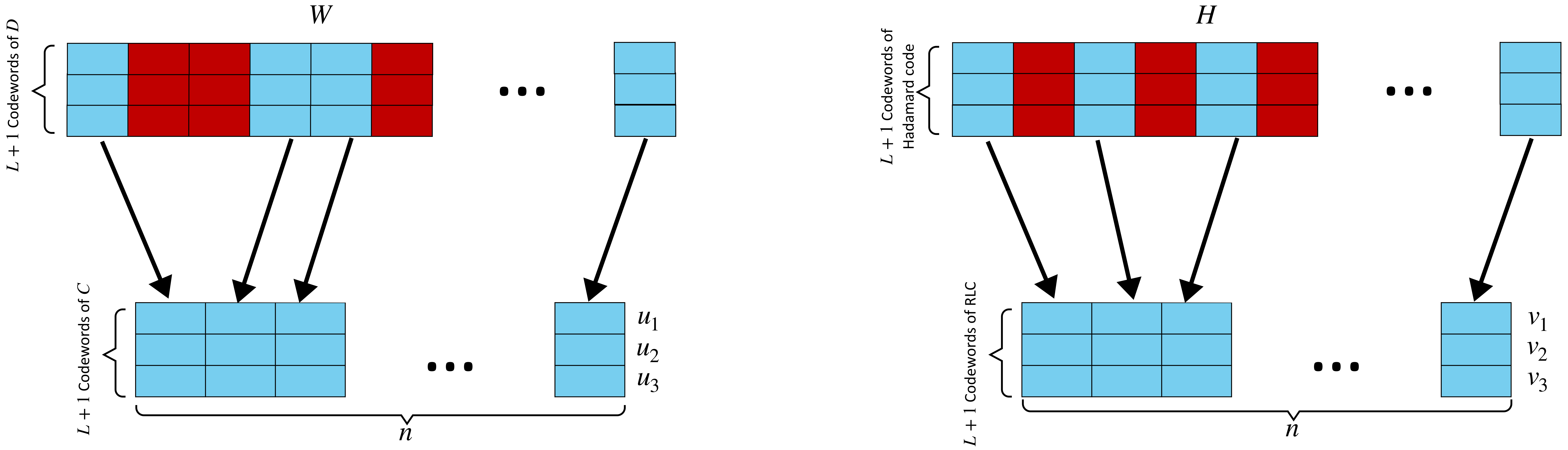

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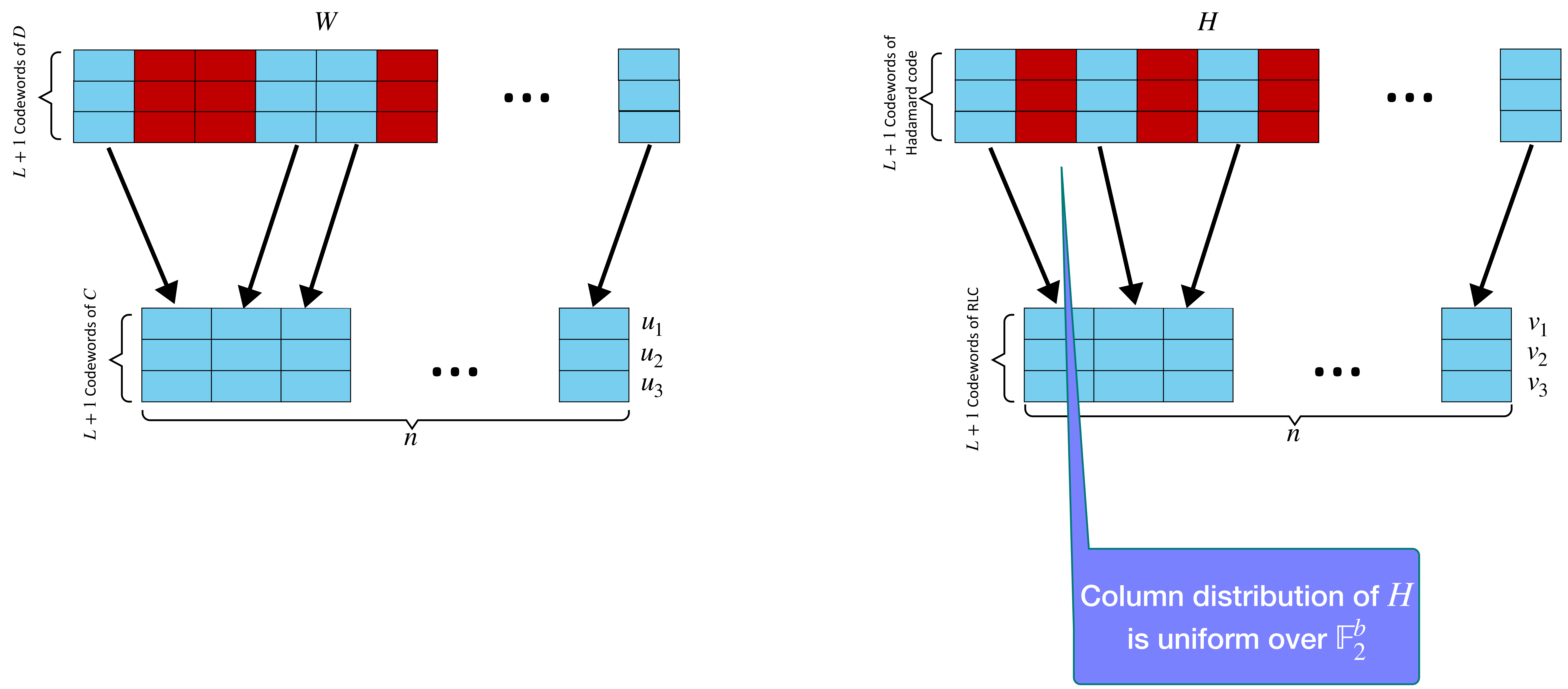

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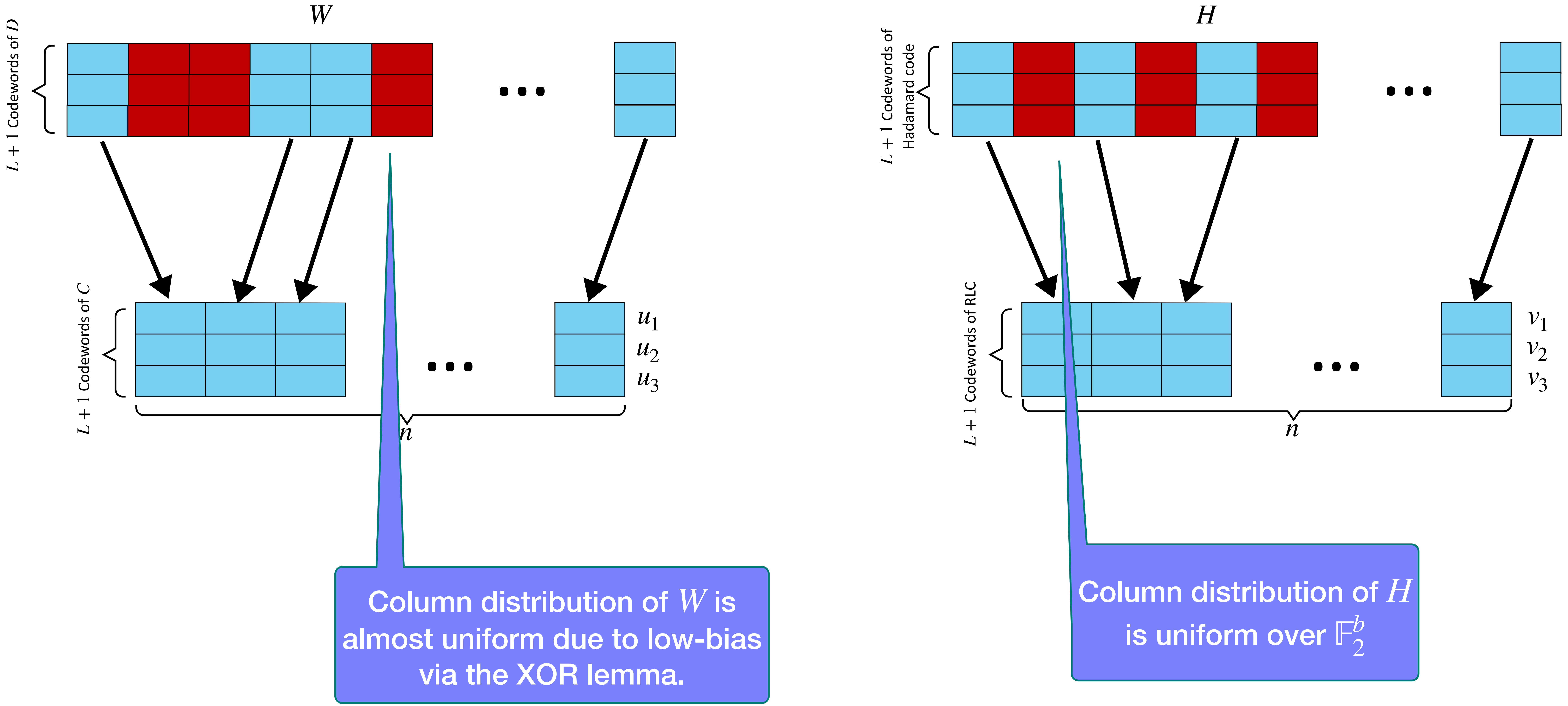

Drawbacks of the method

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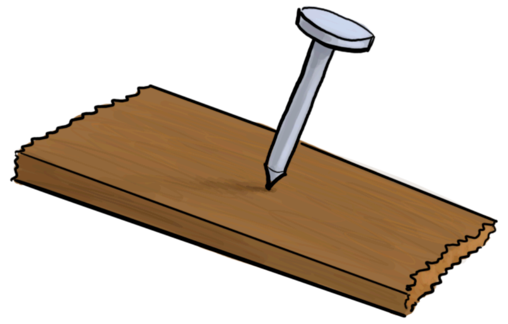
Open problem:

Let $C \subseteq \mathbb{F}_q^n$ be an **RLC** and fix $\epsilon > 0$.

Prove that C is $\left(\frac{q}{2},\ q^{Rn} \cdot 2^{-n} \cdot (1+\epsilon)\right)$**-list-recoverable** with high probability.



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Can only deal with “Σ_1” properties.

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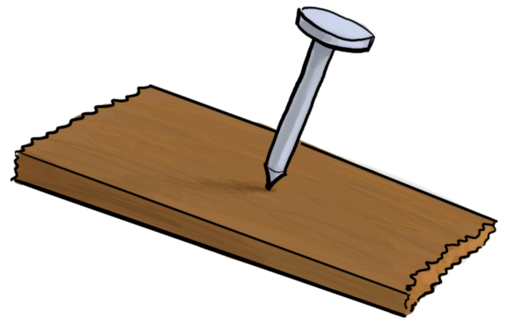
Open problem:

Say that a code C is (ρ, L)-**covering** if every $x \in \mathbb{F}_2^n$ is ρ-**close** to at least L codewords of C.

Find the rate threshold for (ρ, L)-**covering** with regard to **RLCs**.



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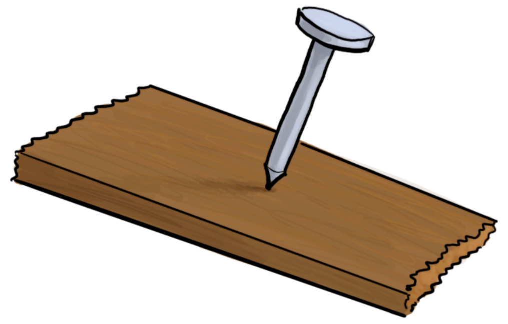
Open problem:

Construct a code **achieving the GV bound** with $o(n)$ random bits.



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- For general q this is n^{q^L}.
- Suppose $q = n$, then **there are n^{n^L} types!** 😱

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- Is there any hope for reasoning about **Reed-Solomon** codes with this method?

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- Note that $n \leq q$.

List-Decodability of RS codes

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Problem:

Are there **RS codes** that **achieve the list-decoding GV-bound**?

How large does q need to be in terms of n?

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- Many works about **list-decodability** of $\mathrm{RS}[S,k]$ where $S \subseteq \mathbb{F}_q$ is random ("**random RS code**")
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- Most recently:
  - **[BGM]** - **List-decoding GV-bound** with $q = 2^{O(n)}$
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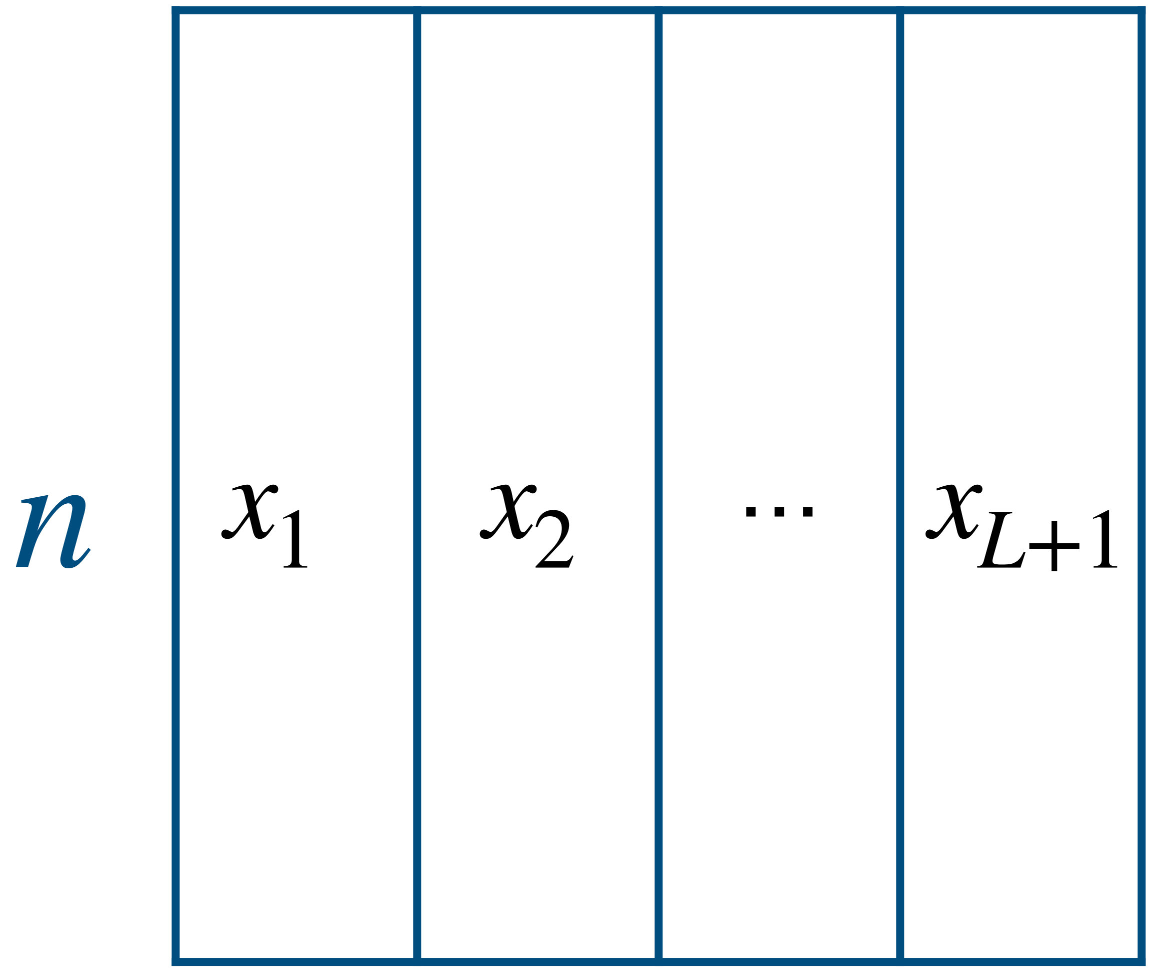

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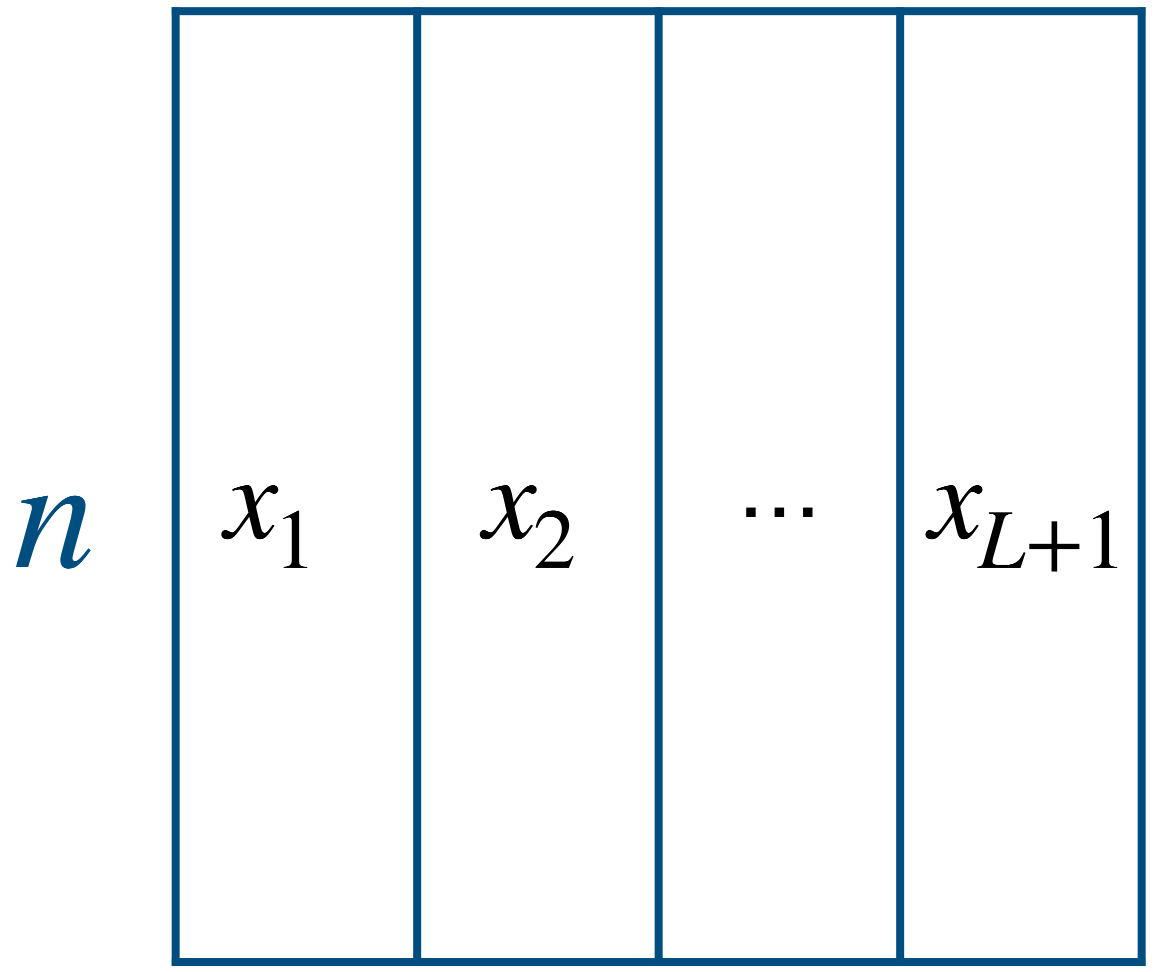

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- The row distribution of A contains **too much information**.
- For a given row, we only care about the **identity relation**.

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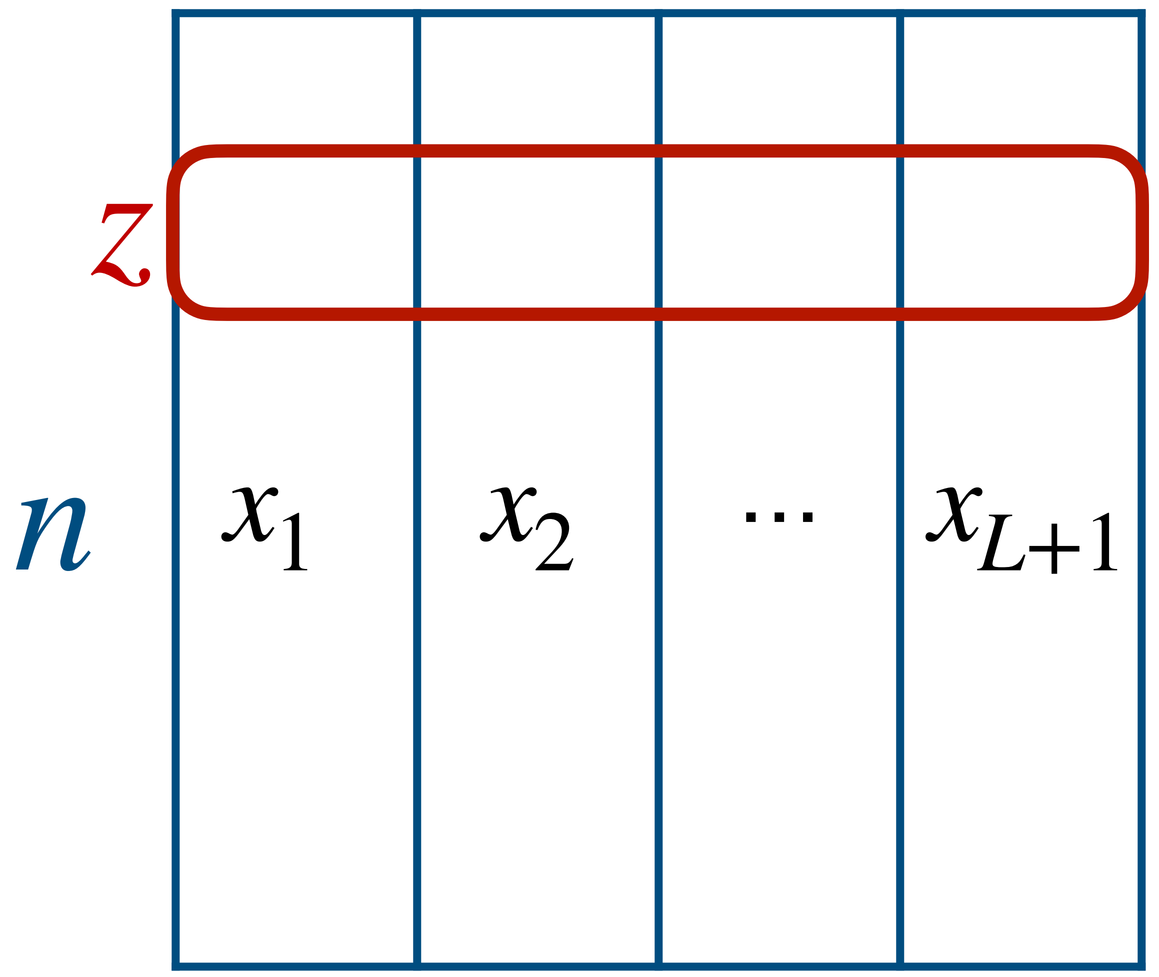
Given $z \in \mathbb{F}_q^{L+1}$ let P_z denote the partition of $\{1,\ldots,L+1\}$ where

$$i \sim_{P_z} j \quad \Longleftrightarrow \quad z_i = z_j$$



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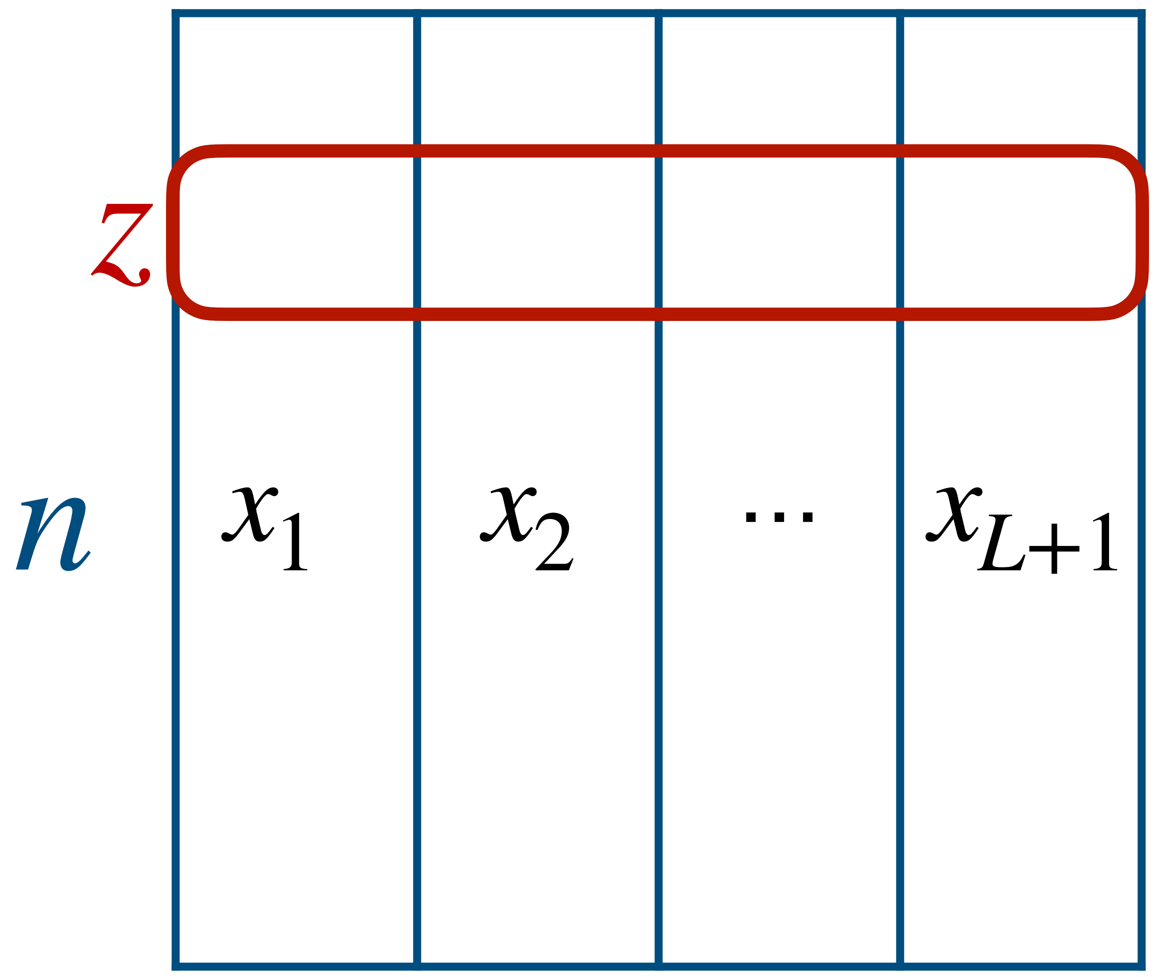

The **type** of a matrix $A \in \mathbb{F}_q^{n\times(L+1)}$ is a pair consisting of:

1. A list of partitions $\left(P_{A_i}\right)_{i=1}^{n}$

2. The row-span of A.

Types for large alphabet

Types for large alphabet

Observation:

If a matrix A is ρ**-clustered** then so are all matrices of the **same type**.

So the **witnesses for non-list-decodability** are a union of **type classes**.

List-recoverability can also be expressed this way. A property expressible by **type classes** is called a **local identity property.**

$$n \quad \begin{array}{|c|c|c|c|} \hline x_1 & x_2 & \cdots & x_{L+1} \\ \hline \end{array}$$

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    - We can union bound over the **ρ-clustered types**.

Large alphabet types in RLCs - Intuition

	3
	{1,2},{3}
	{1,2,3}
	{1},{2},{3}
n	{1,2,3}
	...
	{1,3},{2}

Large alphabet types in RLCs - Intuition

- Consider the **type** $T = \left(P = \left(P_i\right)_{i=1}^{n}, \mathbb{F}_3^n\right)$

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- Let $\deg(P, \mathbb{F}_q^3) = 3Rn - \sum_i (3 - |P_i|)$.
- If $\deg(T) < 0$ then there is **probably no type T matrix in C**.

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	{1,2,3}
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Large alphabet types in RLCs - Intuition

$$\begin{array}{cc} & 3 \\ n & \begin{array}{|c|}\hline \{1,2\},\{3\} \\ \hline \{1,2,3\} \\ \hline \{1\},\{2\},\{3\} \\ \hline \{1,2,3\} \\ \hline \cdots \\ \hline \{1,3\},\{2\} \\ \hline \end{array}\end{array}$$

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- What if $\deg(P, \mathbb{F}_q^3) > 0$?

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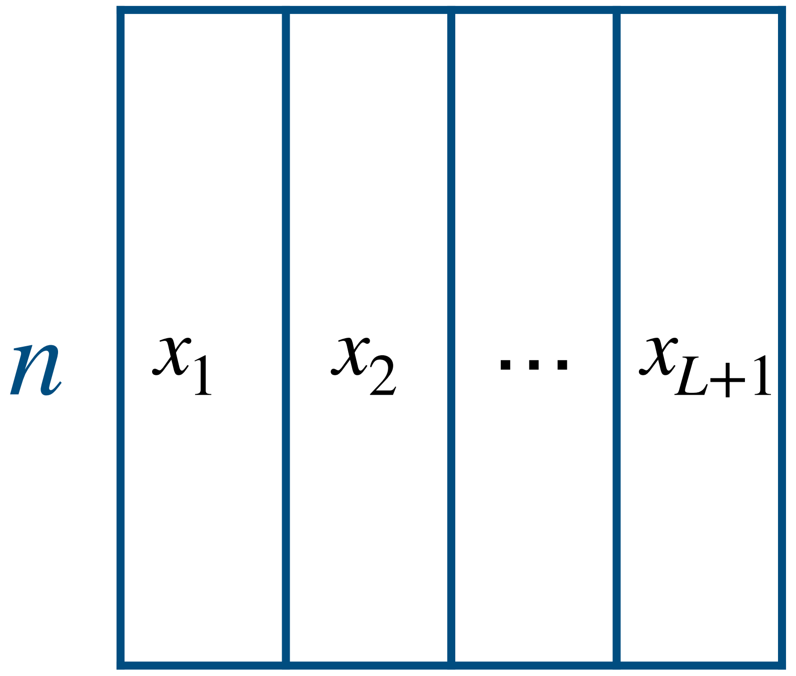

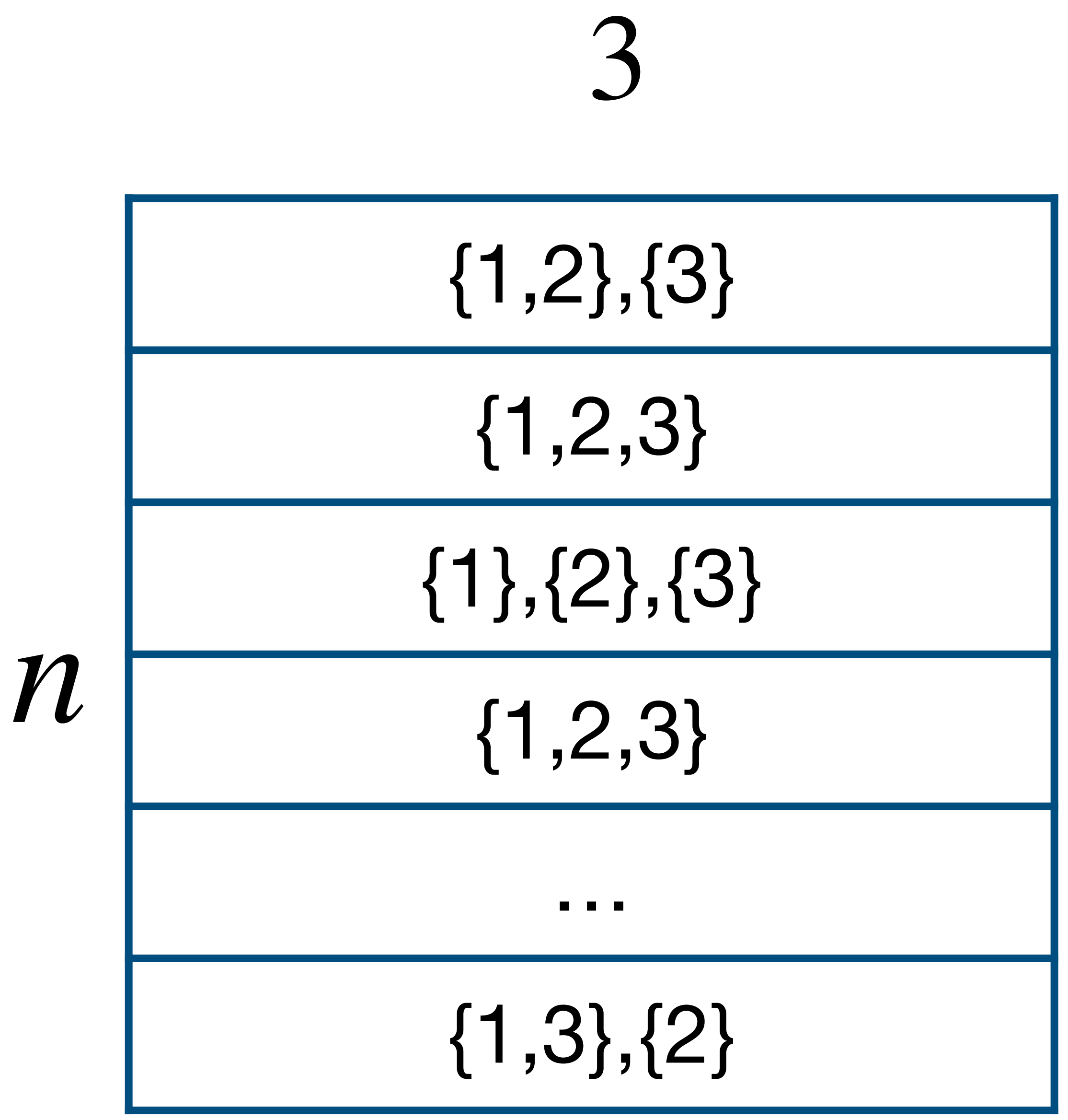

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- What if $\deg(P, \mathbb{F}_q^3) > 0$?
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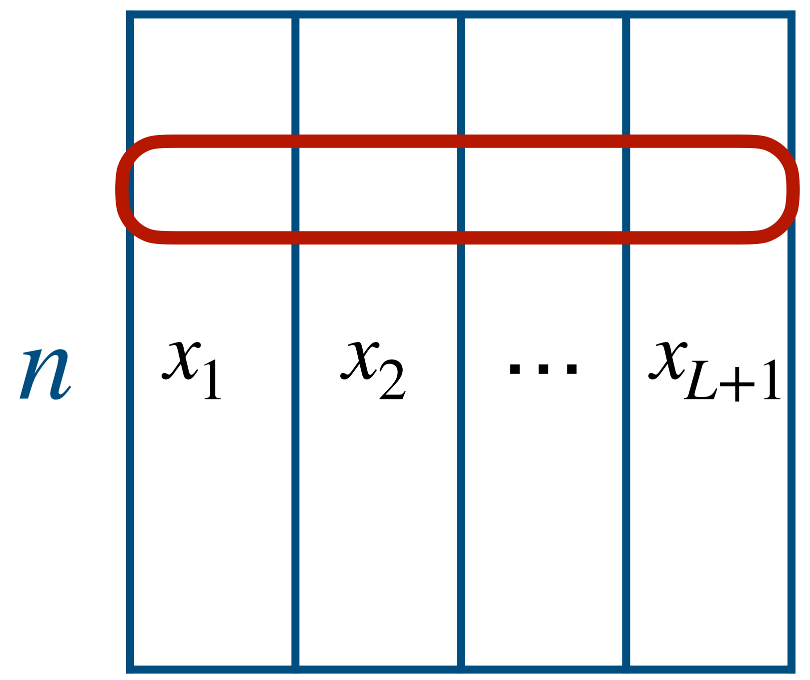

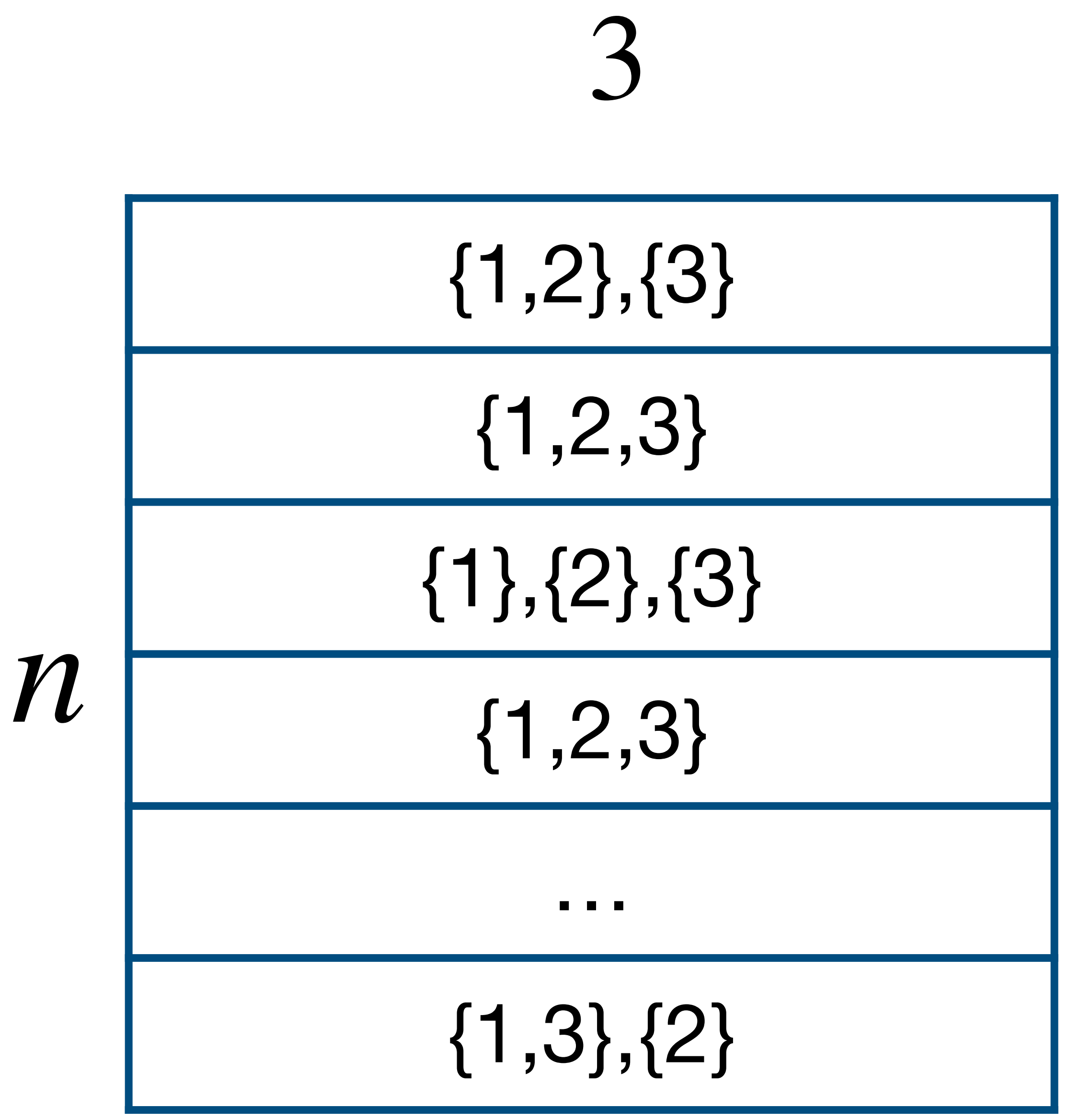

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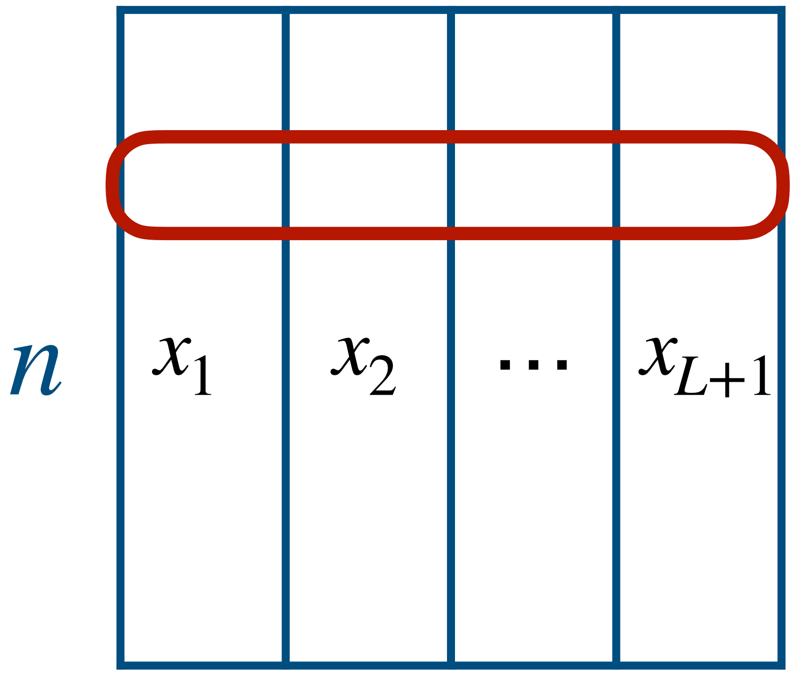

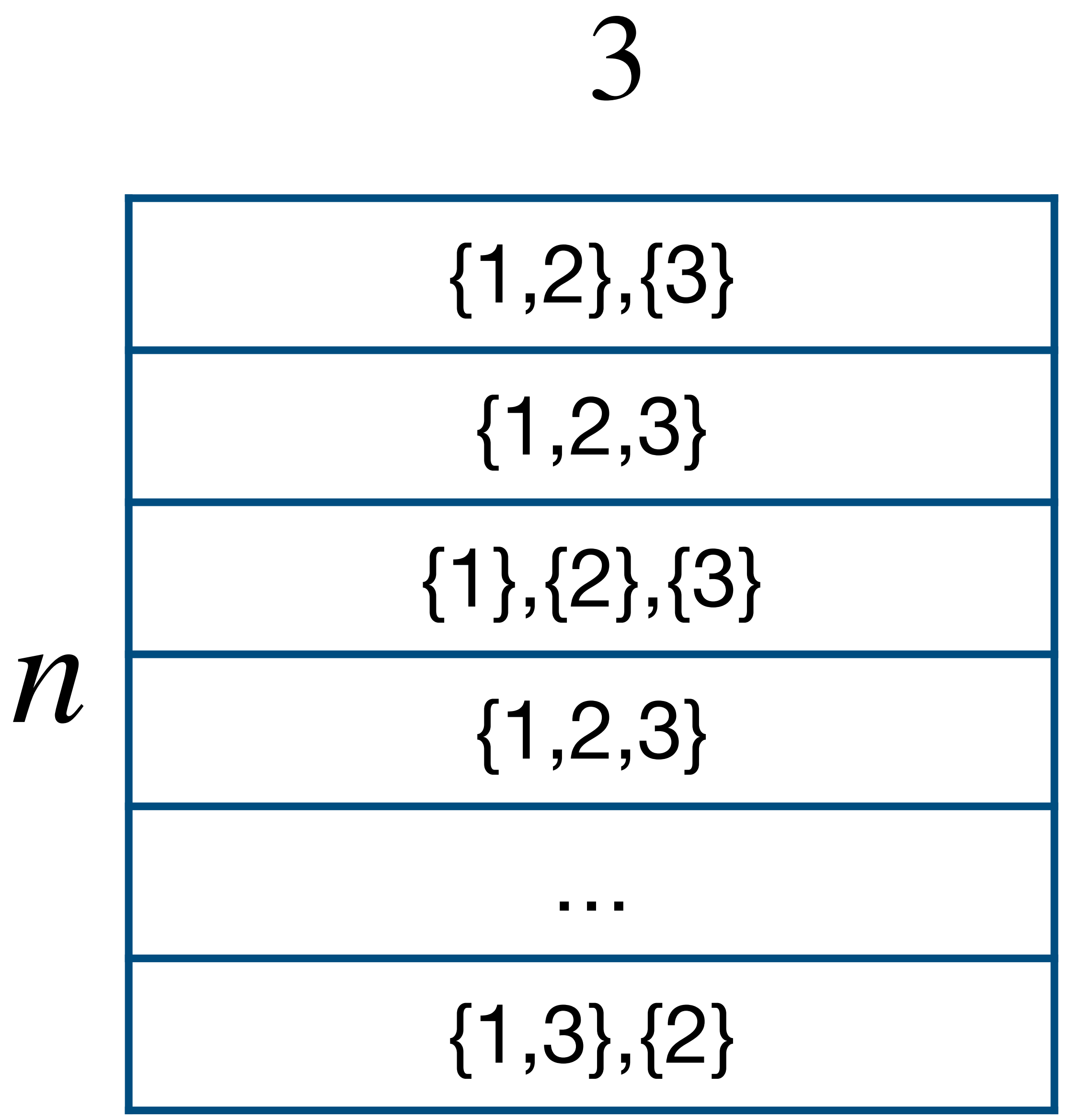

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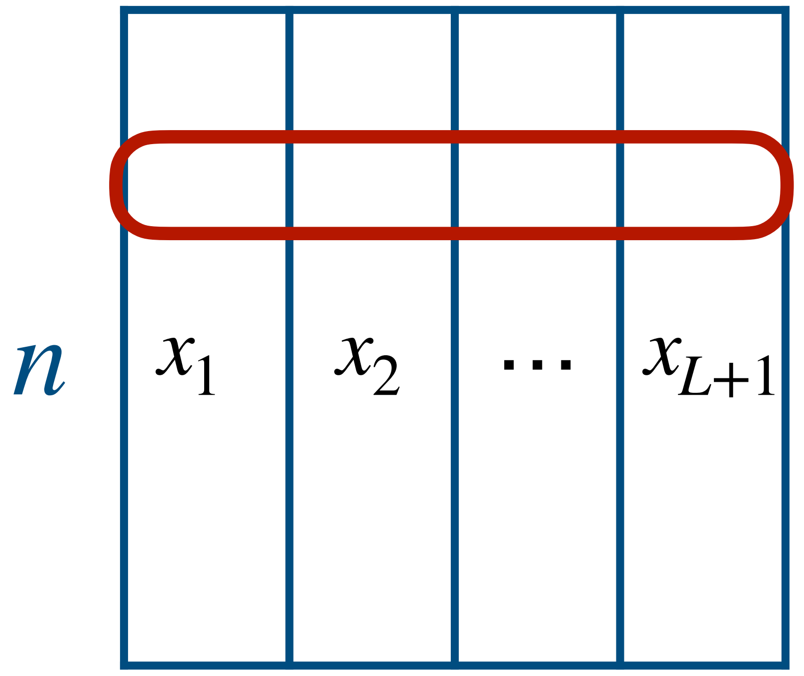
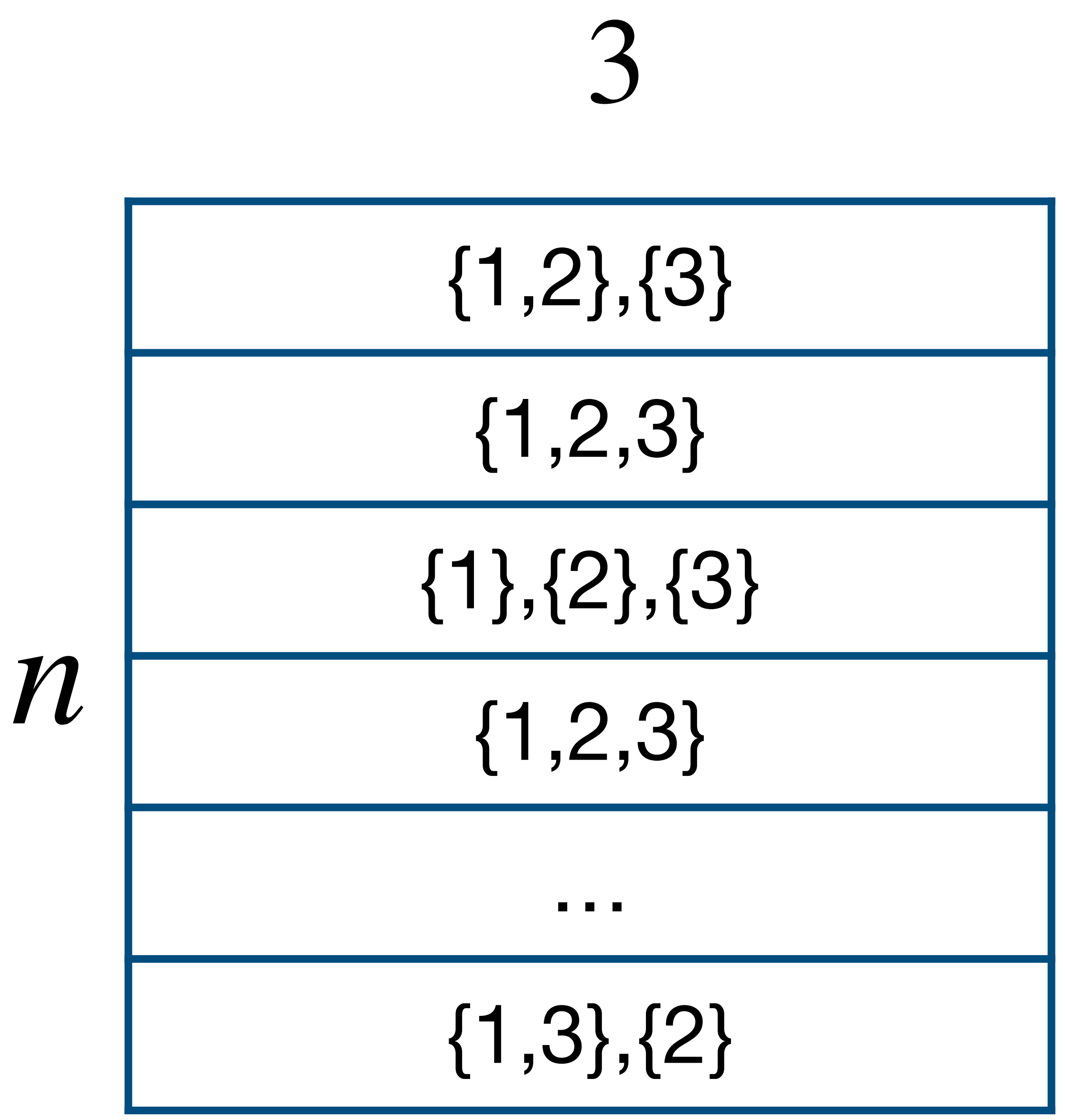
- **Maybe not!**
- It's possible that these x_1, x_2, x_3 are **not even distinct!**





Large alphabet types in RLCs - Intuition

3

n

{1,2},{3}
{1},{2},{3}
{1,2},{3}
{1,2},{3}
...
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- In this example we have $\deg(P, \mathbb{F}_q^3) > 0$.
- However, it's likely that all solutions will have $x_1 = x_2$!

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- z_3 is determined by z_1, z_2 so we only have $2Rn$ **degrees of freedom**.
- On the other hand, $z_1 = z_2 \Rightarrow z_1 = z_3$
    - So $\{1,2,3\}$ is just **1 constraint instead of 2**.

3

{1,2},{3}
{1,2,3}
{1},{2},{3}
{1,2,3}
...
{1,3},{2}

n

	3
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Where

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An **RLC** is **likely to contain a type (P, V) matrix** if and only if

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In particular
$\deg(P, V) > \deg(P, \{0\}) = 0$

Theorem [List-Decodability of RLC]
(previously proven by **[AGL]**):

For $q \geq 2^{\Omega(L)}$, an **RLC** in $\mathbb{F}_q^n$ **achieves the list-decoding GV bound**.

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Let $\mathscr{P}$ be a **local identity property achieved with high probability** by an **RLC**.

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Corollary:

A **random RS code** **achieves the list-decoding GV-bound**.
(Already proven by **[AGL]** using the **GM-MDS theorem**)

A **random RS code** is at **least as list-recoverable** as an **RLC**.

Proof sketch: Reduction from random RS to RLC

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Suppose that $\deg(P, \mathbb{F}_2^{L+1}) \leq -\epsilon n$.

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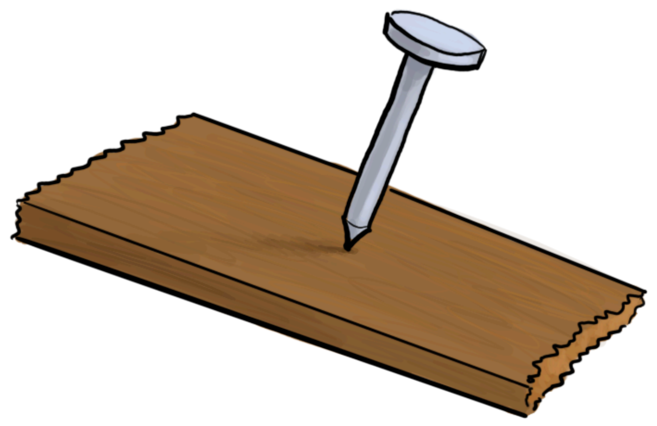
On the Board

Open Problems



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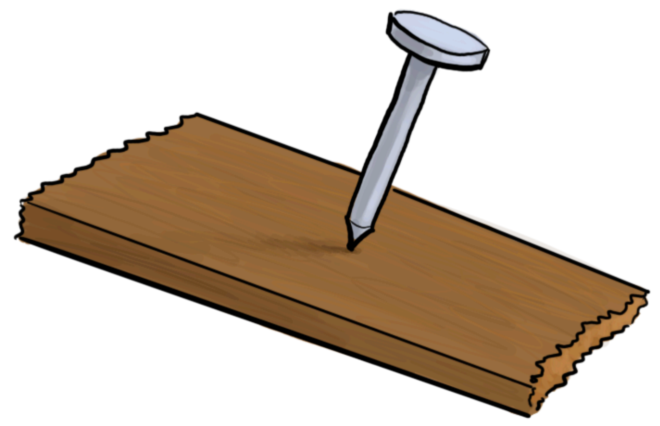

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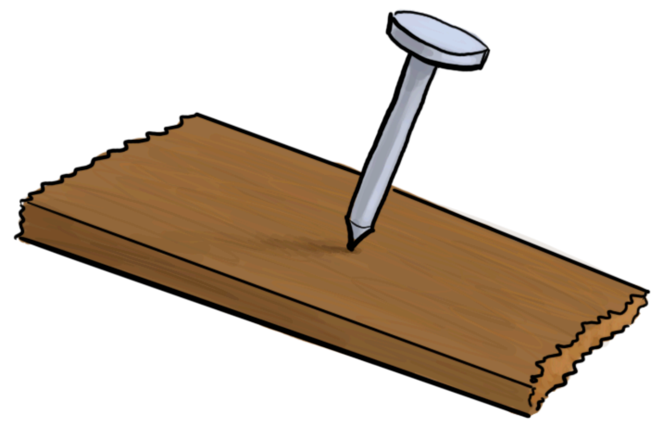

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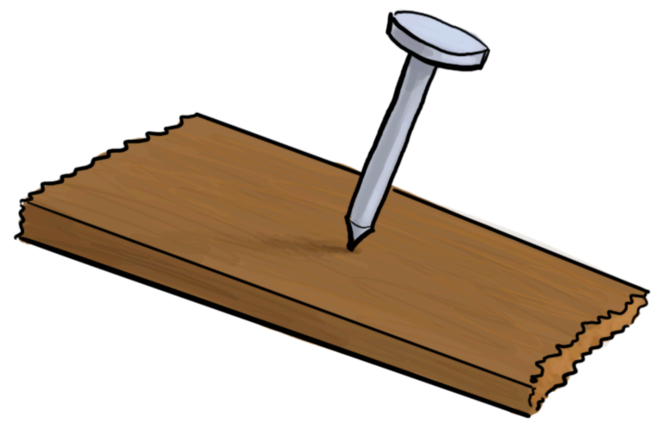

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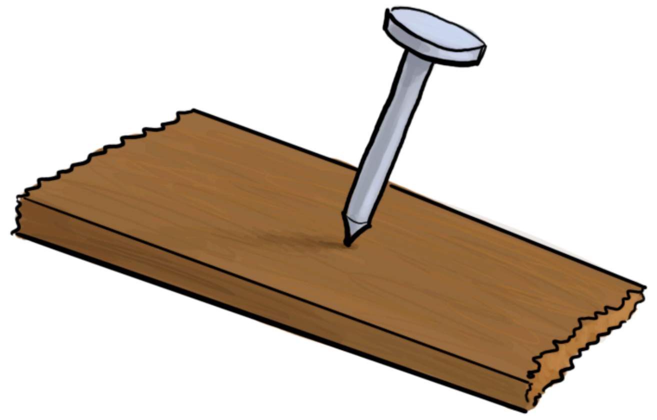

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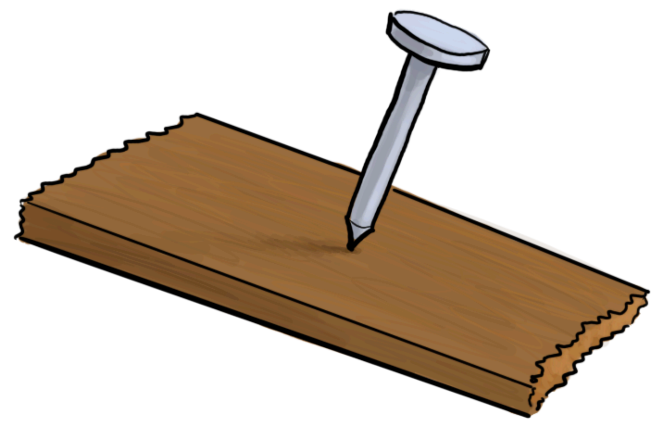

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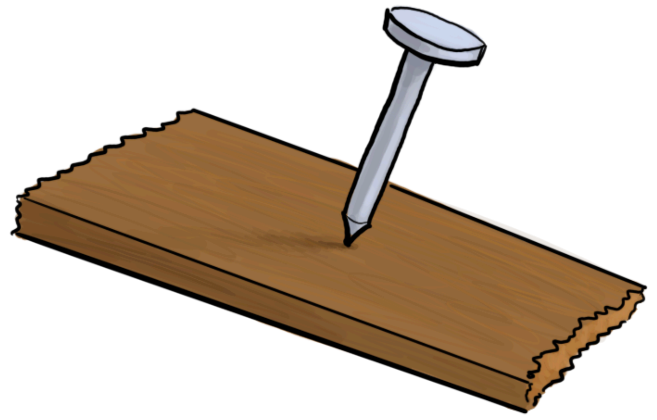

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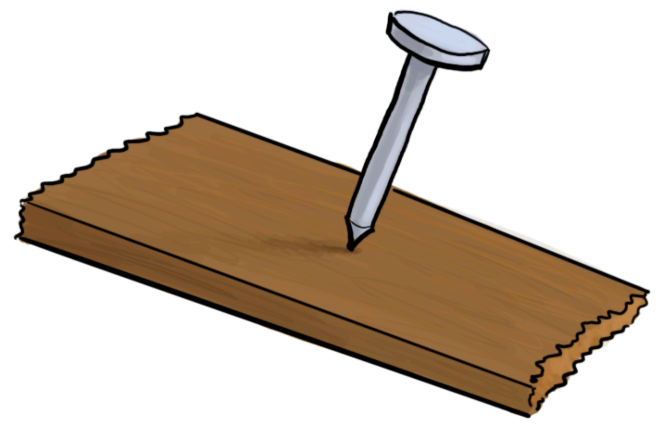

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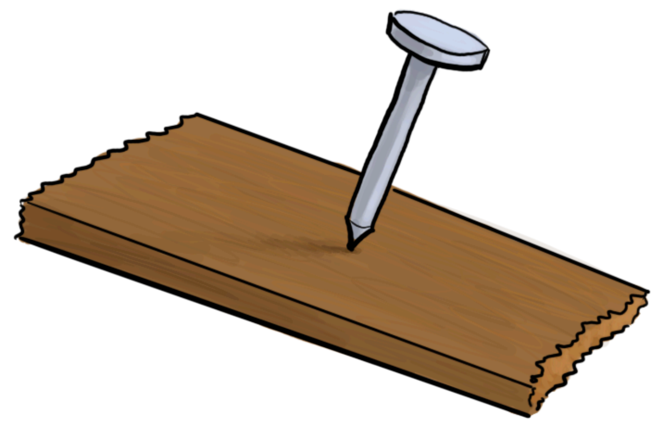

Thank you!