Probabilistic and Combinatorial Methods

Error-Correcting Codes: Theory and Practice Boot Camp

Jonathan Mosheiff
Ben-Gurion University

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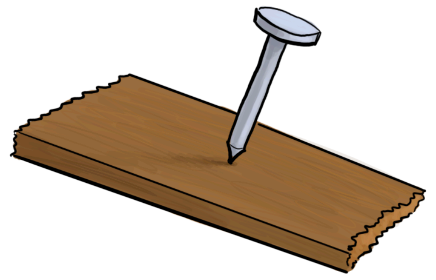

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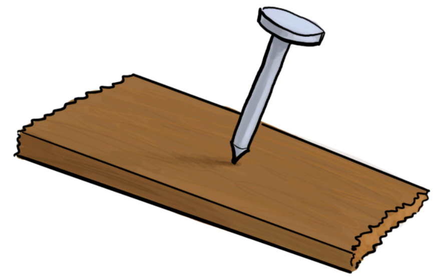

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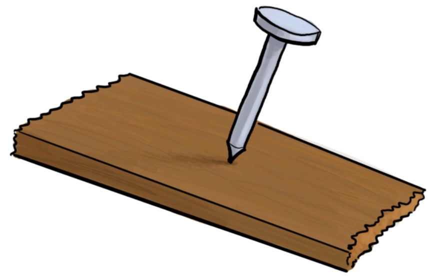

- **A star player:** The Random Linear Code (RLC)
- **Technique:** We **reduce** from RLC to more structured codes.

List-Decoding

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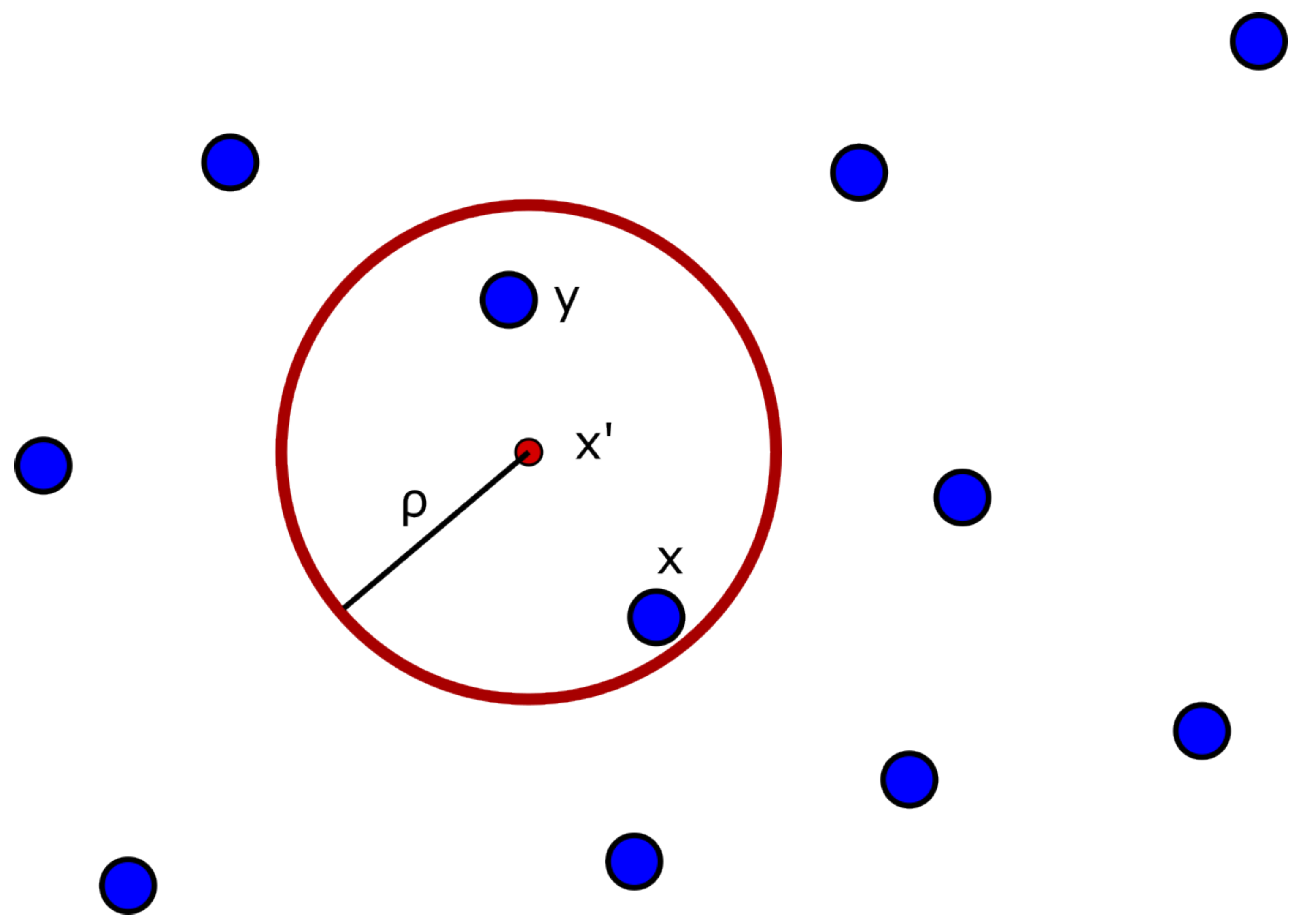

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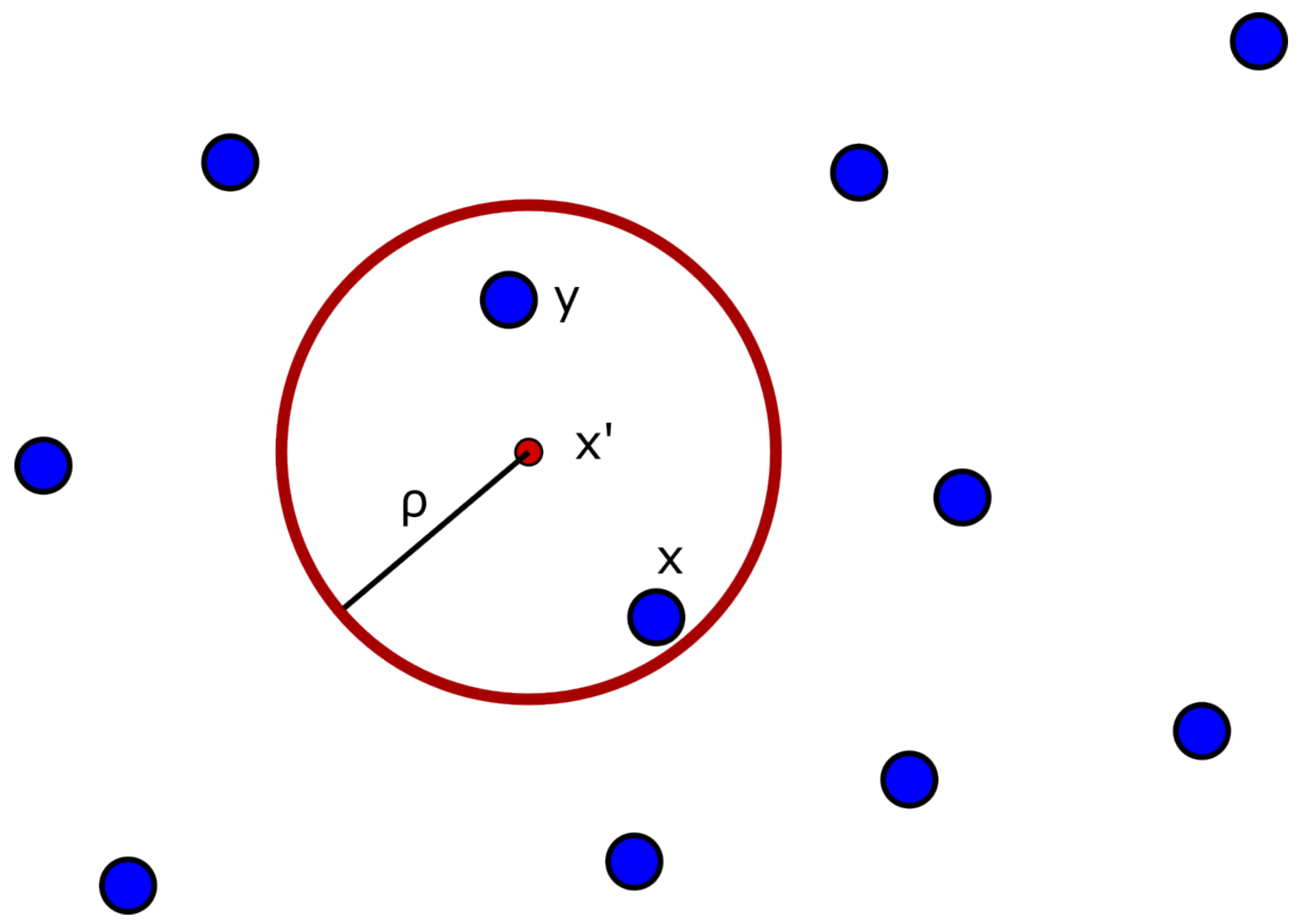

- C is **(ρ,L)-list-decodable** if the receiver can always recover a **list** of at most L codewords, such that the list contains x.

List-Recovery

List-Recovery

- In **List-Decoding** we want every Hamming ball to contain a small number of codewords.

- In **List-Recovery** we care about **combinatorial rectangles** instead of **balls.**

List-Recovery

We say that $C \subseteq \mathbb{F}_q^n$ is **(ℓ, L)-list-recoverable** if:

For every $S_1, \ldots, S_n \subseteq \mathbb{F}_q$ with $|S_i| \leq \ell$ we have

$$|C \cap (S_1 \times S_2 \times \ldots \times S_n)| \leq L.$$

$S_1 \times S_2 \times \ldots \times S_n$ is called a **combinatorial rectangle**

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- An **RLC** of **length** n and **rate** R over **alphabet** $\mathbb{F}_q$ is a uniformly-sampled Rn-dimensional linear subspace of $\mathbb{F}_q^n$.
- The go-to code for **existence proofs**!

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- **Achieves with high probability:**
  - The **Gilbert-Varshamov Bound** *
    $R \approx 1 - h_q(\delta)$
  - The **"List-decoding GV-bound":**
    $$R = 1 - h_q(\delta) - O\left(\frac{1}{L}\right)$$
  - **List-recovery** results as well.

* $H_q(\rho) = \rho \log_q(q-1) - \rho \log_q \rho - (1-\rho)\log_q(1-\rho)$

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- **However:**
  - Decoding is **probably hard**
  - Certifying is **probably hard**
  - Construction requires $\Theta\left(n^2\right)$ **random bits**.

* $H_q(\rho) = \rho \log_q(q-1) - \rho \log_q \rho - (1-\rho)\log_q(1-\rho)$

The only thing you need to know about RLCs

Let C be an **RLC** of rate R. Fix $v_1, \ldots, v_k \in \mathbb{F}_2^n$.

Then:

$$\Pr\left[\{v_1, \ldots, v_k\} \subseteq C\right] = 2^{-(1-R)\cdot n \cdot \dim\{v_1,\ldots,v_k\}}$$

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- **Motivation:** Show that a binary **RLC** achieves the **list-decoding GV-bound**.

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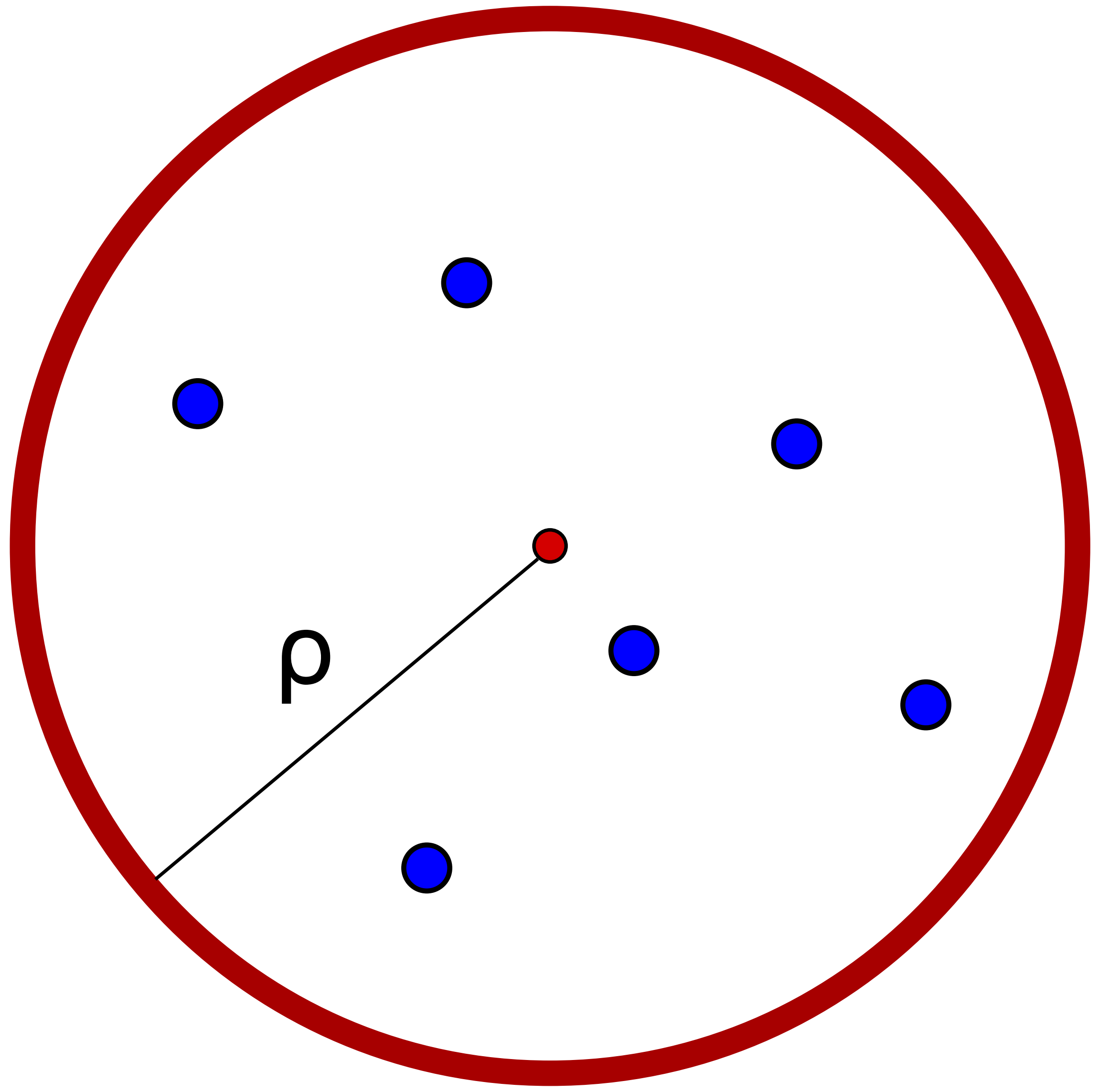
- **More precisely**: Show that an **RLC** with $R = 1 - h(\rho) - \epsilon$ is $(\rho, O(1/\epsilon))$**-list-decodable** with high probability.

List-Decodability of an RLC

- Say that the vectors $x_1, \ldots, x_{L+1}$ are **ρ-clustered** if they are **distinct** and **contained in some radius ρ ball**.

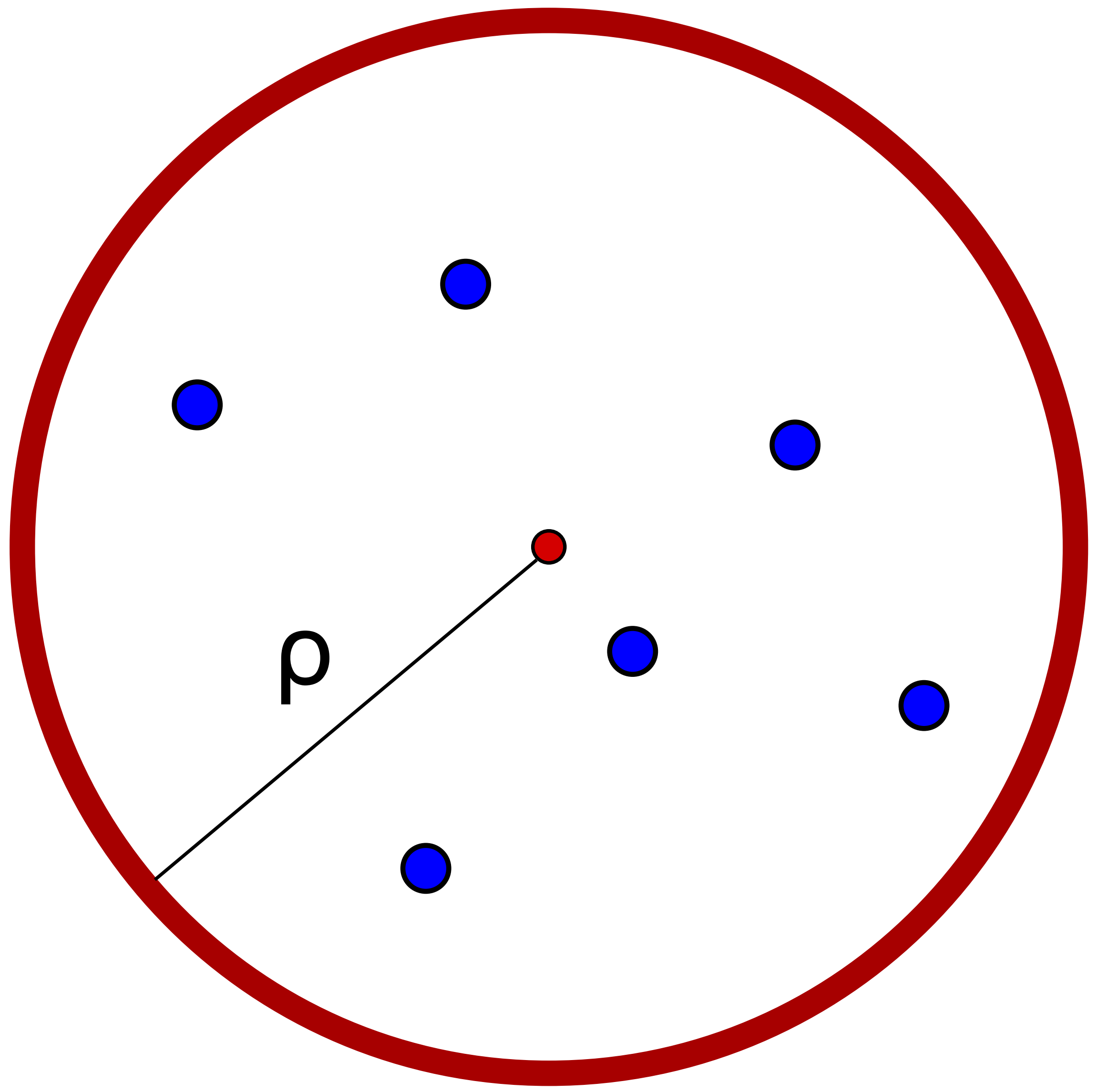
- The tuple $(x_1, \ldots, x_{L+1})$ is a **witness** to C **not being (ρ, L)-list-decodable.**



List-Decodability of an RLC

Let's try an expectation approach:

Try to Prove that the expected number of **clustered tuples** in an **RLC** is $o(1)$.



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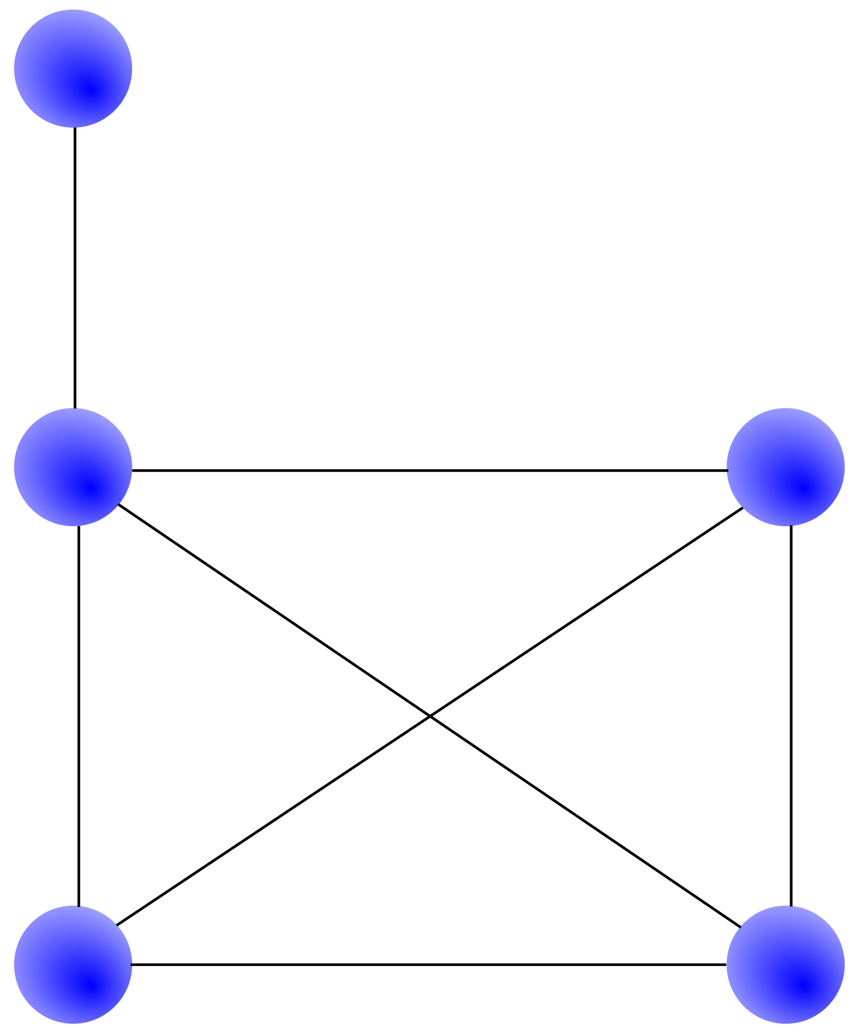

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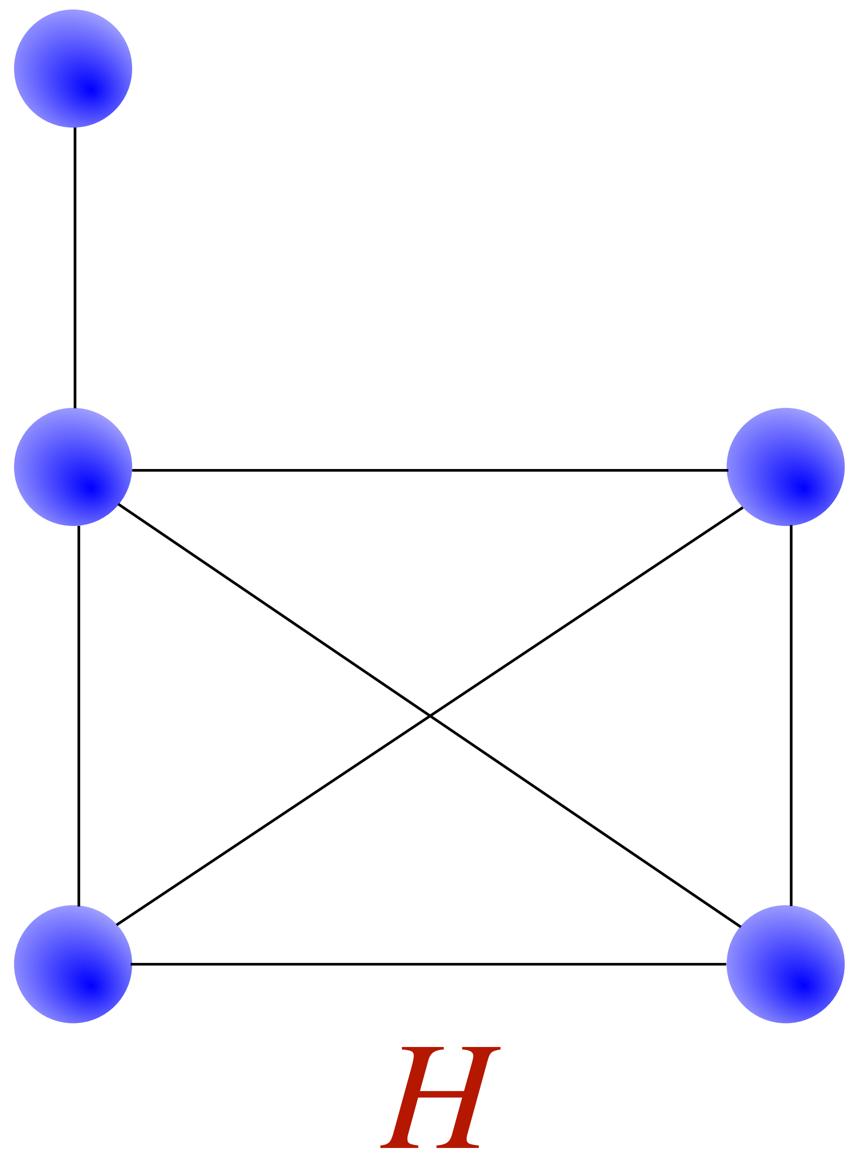
  - $\mathbb{E}(\#H \text{ in } G) \approx n^5 \cdot p^7$



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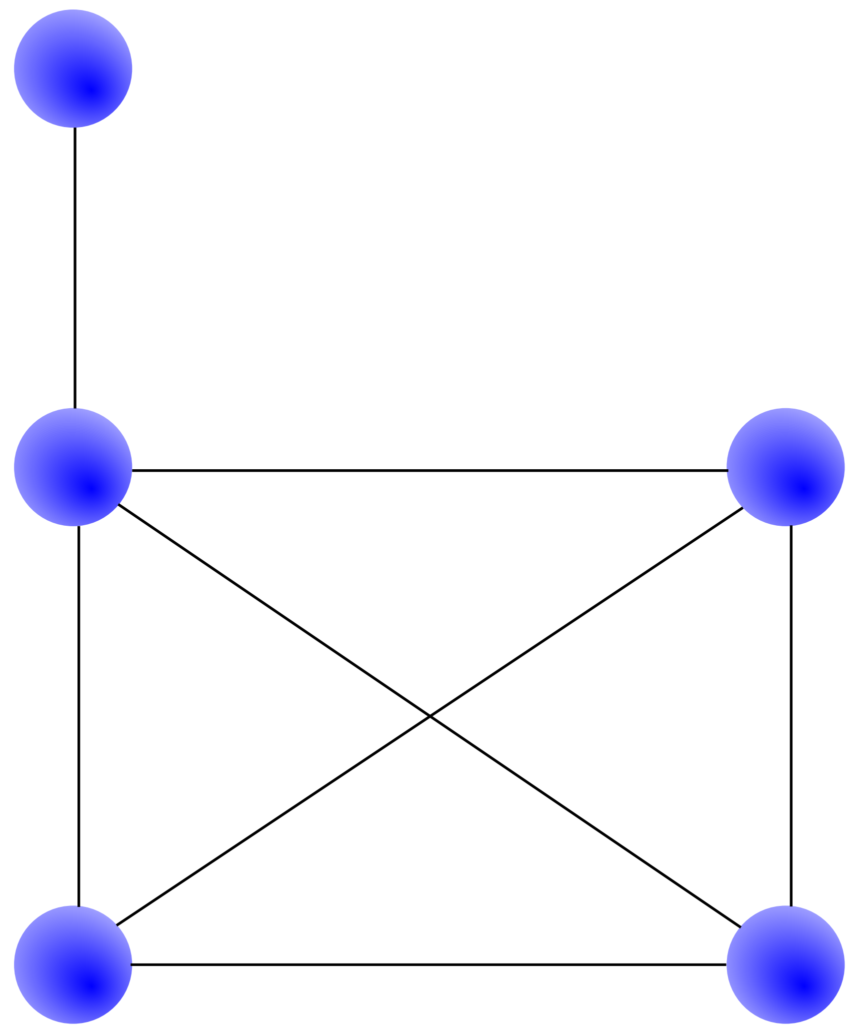

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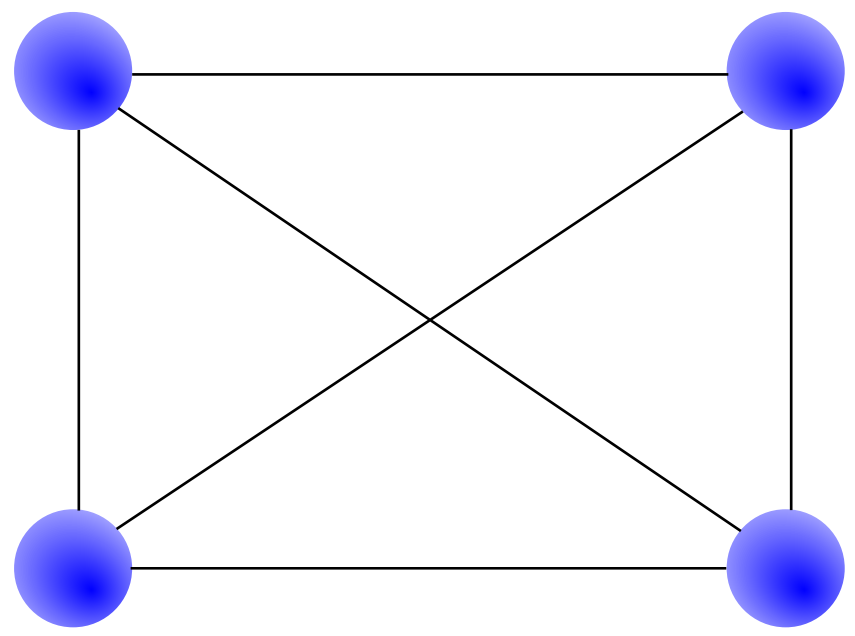

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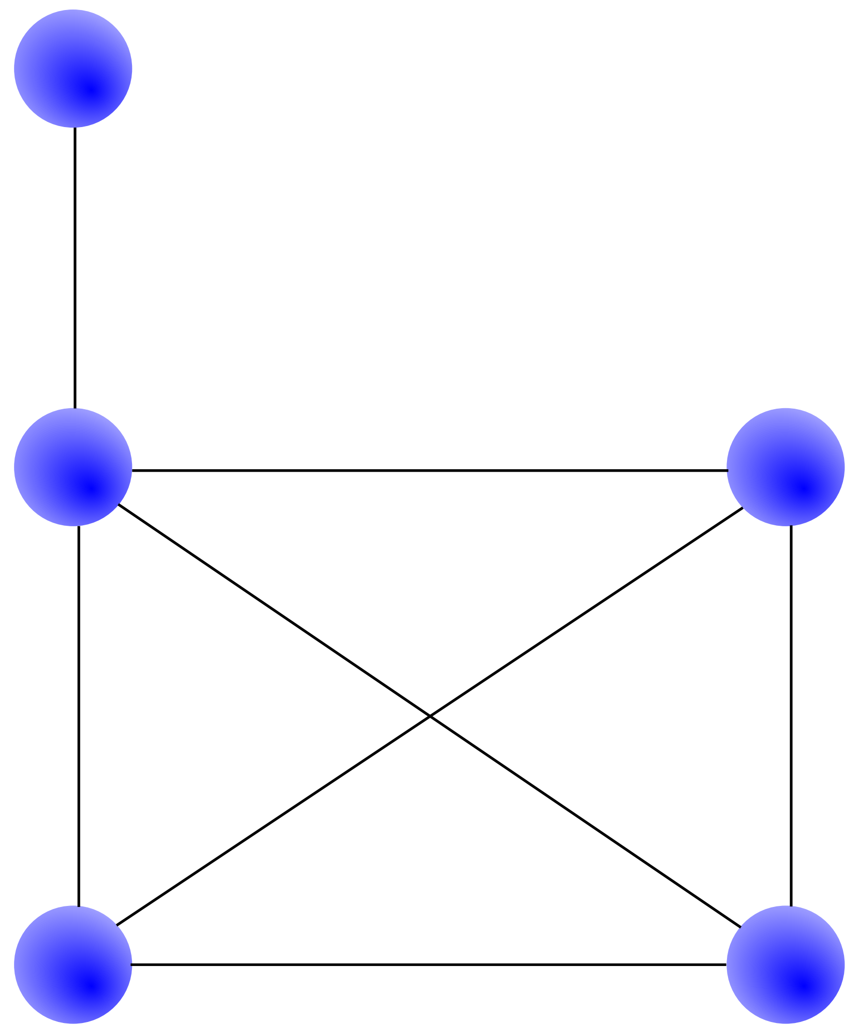
- Let $p = n^\alpha$, with $-5/7 < \alpha < -2/3$.
  - Then $\mathbb{E}(\#H \text{ in } G) \to \infty$ but $\mathbb{E}(\#S \text{ in } G) \to 0$.



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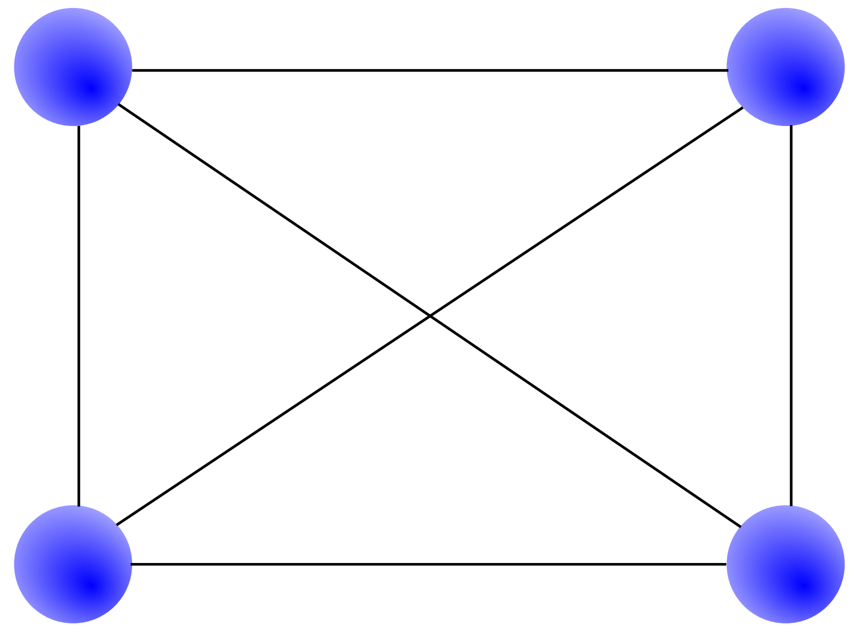

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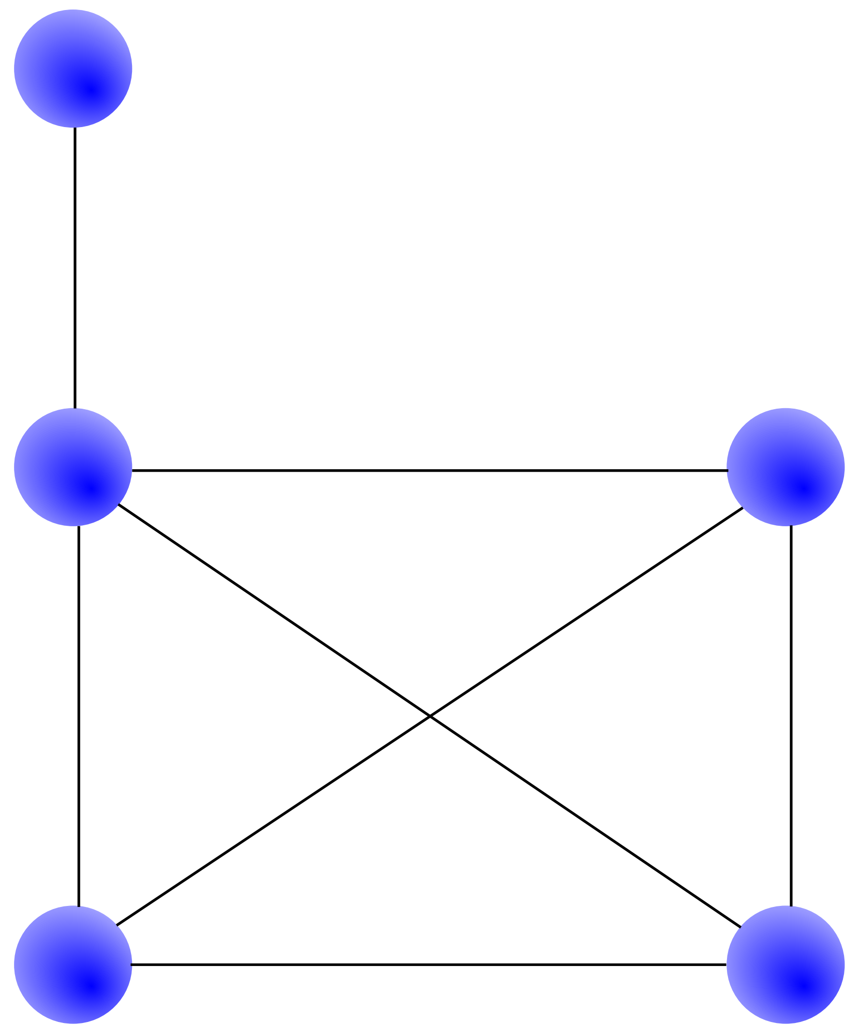
- Let $p = n^\alpha$, with $-5/7 < \alpha < -2/3$.
  - Then $\mathbb{E}(\#H \text{ in } G) \to \infty$ but $\mathbb{E}(\#S \text{ in } G) \to 0$.
  - So almost surely **not a single H can be found in G** even though many such subgraphs appear **in expectation**.



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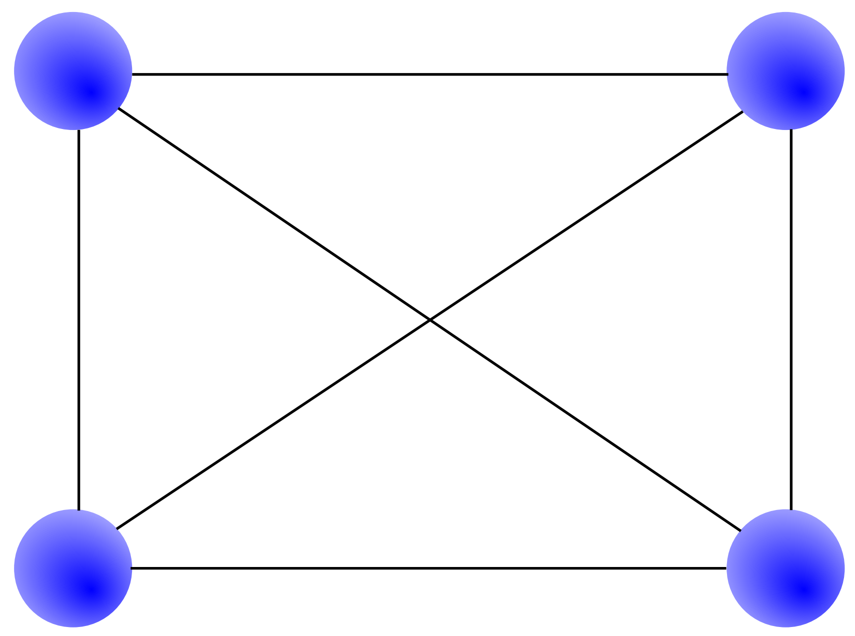

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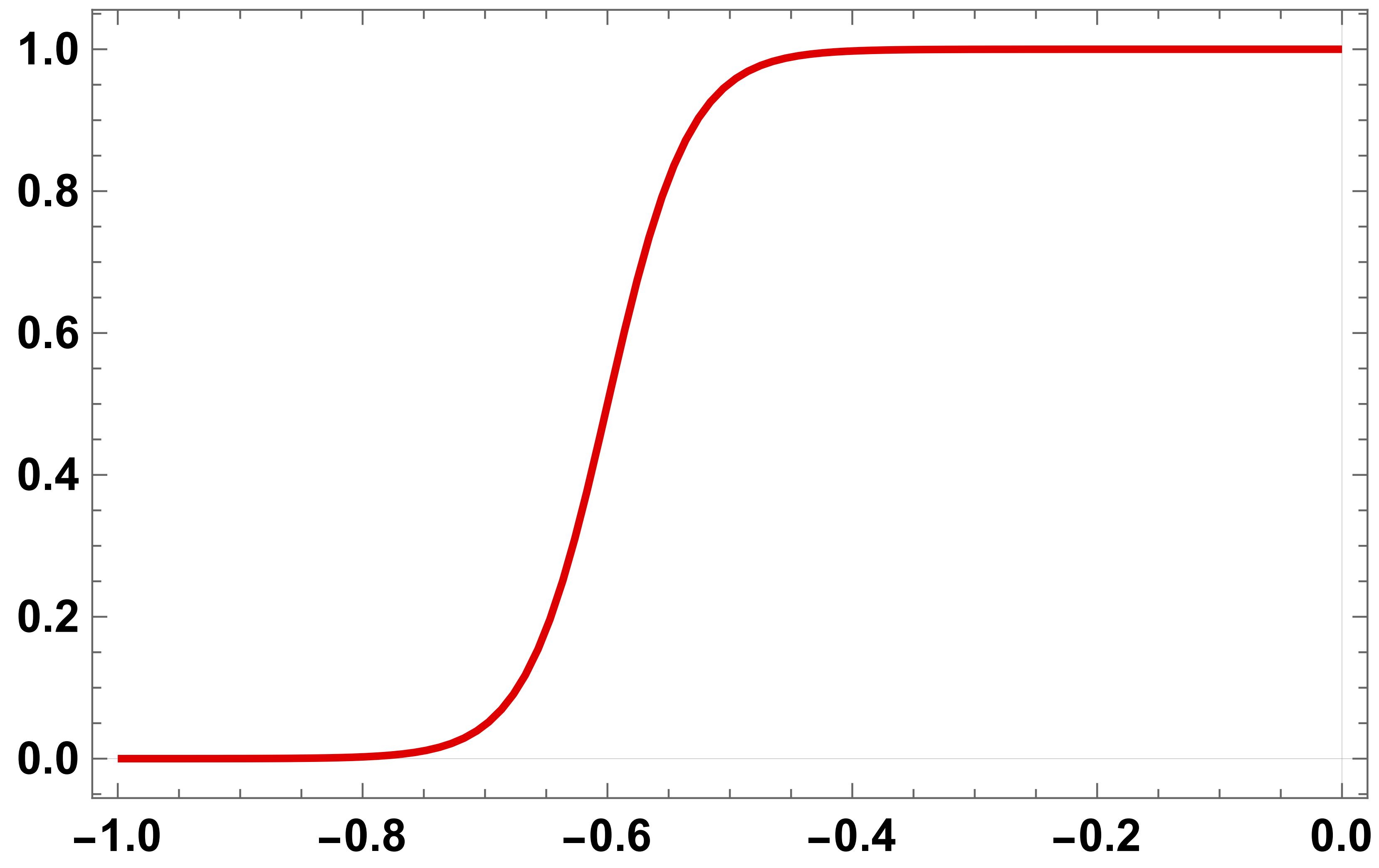
Threshold for random graphs

- **Theorem (Bollobás 1981):** A subgraph H is likely found in G if and only if $\mathbb{E}(\#S \text{ in } G) \to \infty$ for all $S \subseteq H$.

$\Pr(G(n, p^{\alpha})$ **contains** $H)$



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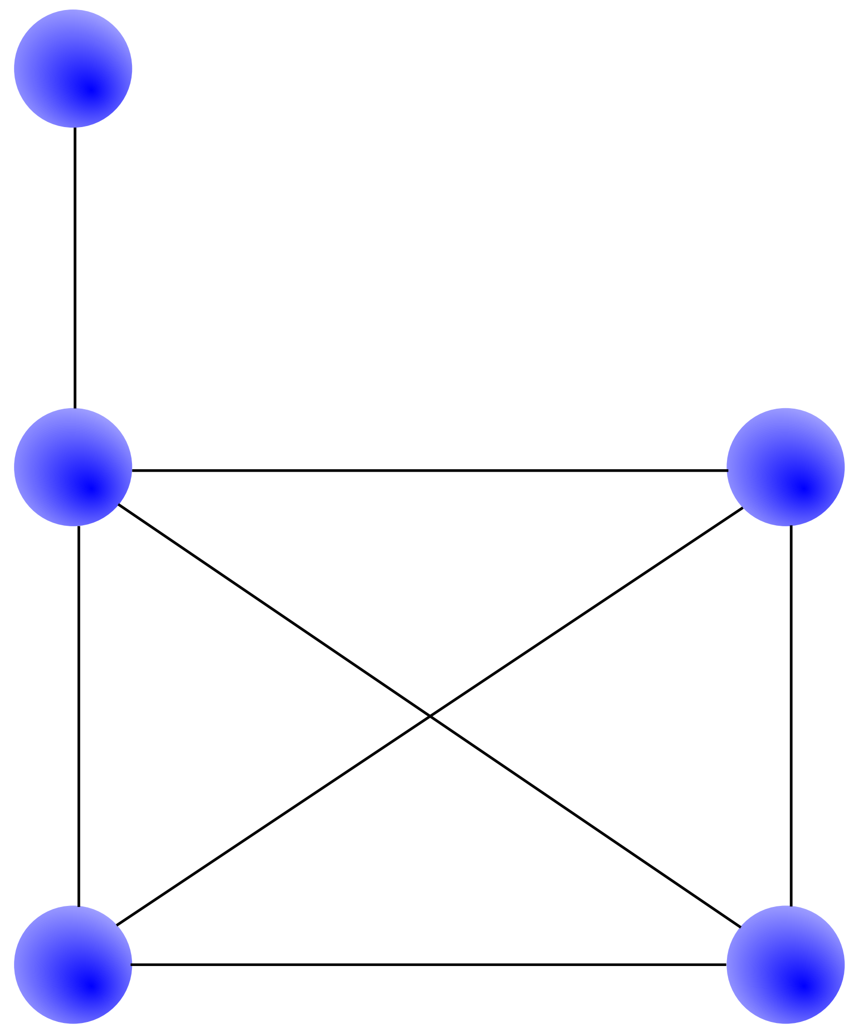

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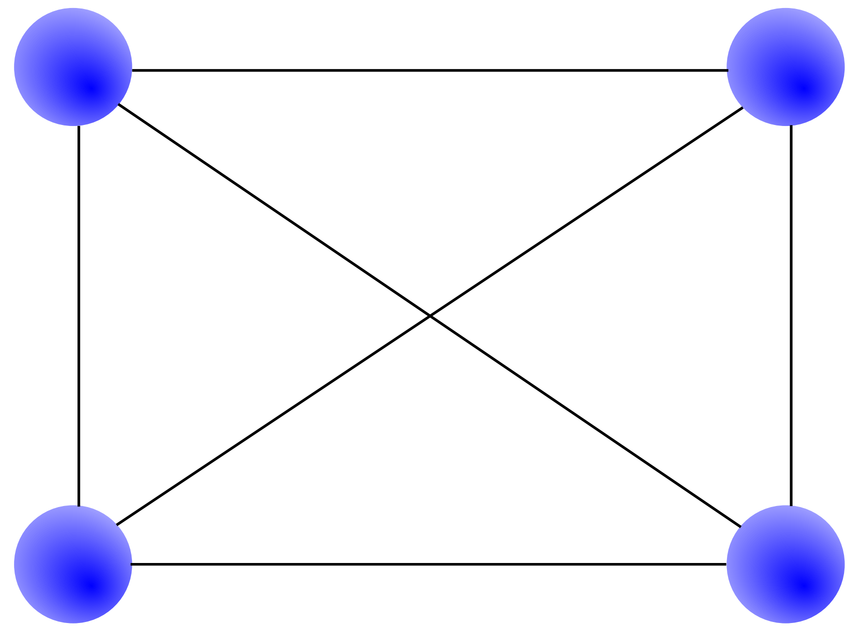

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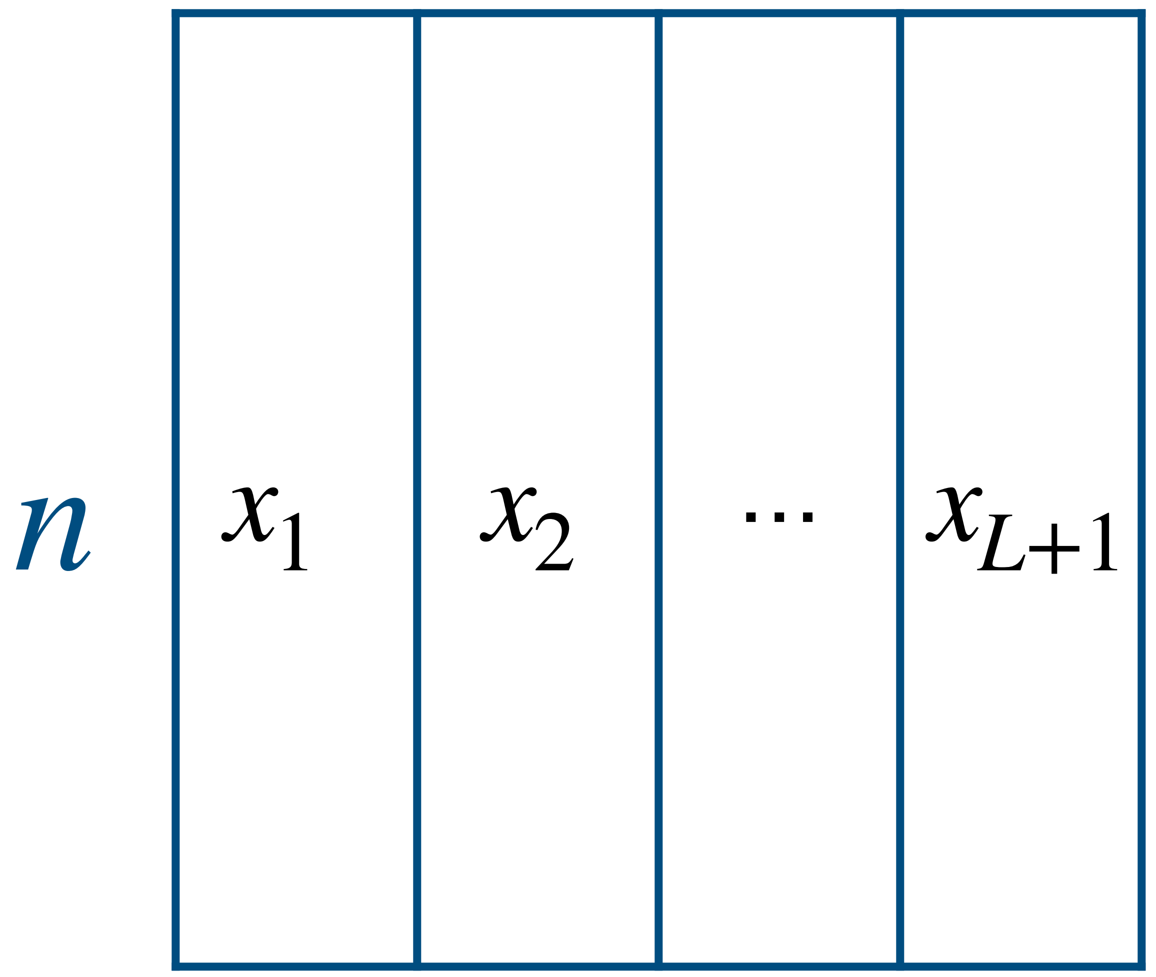

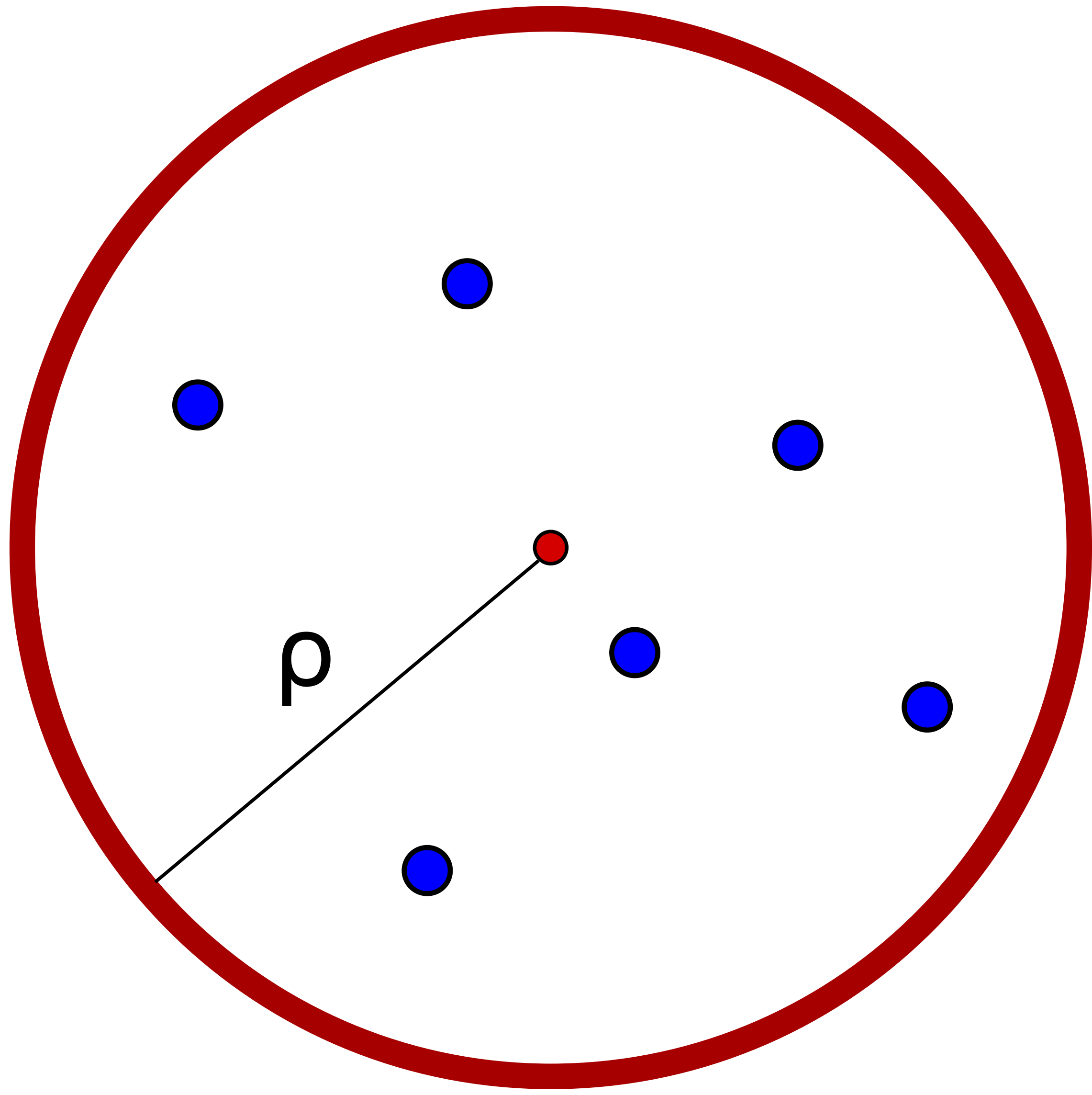

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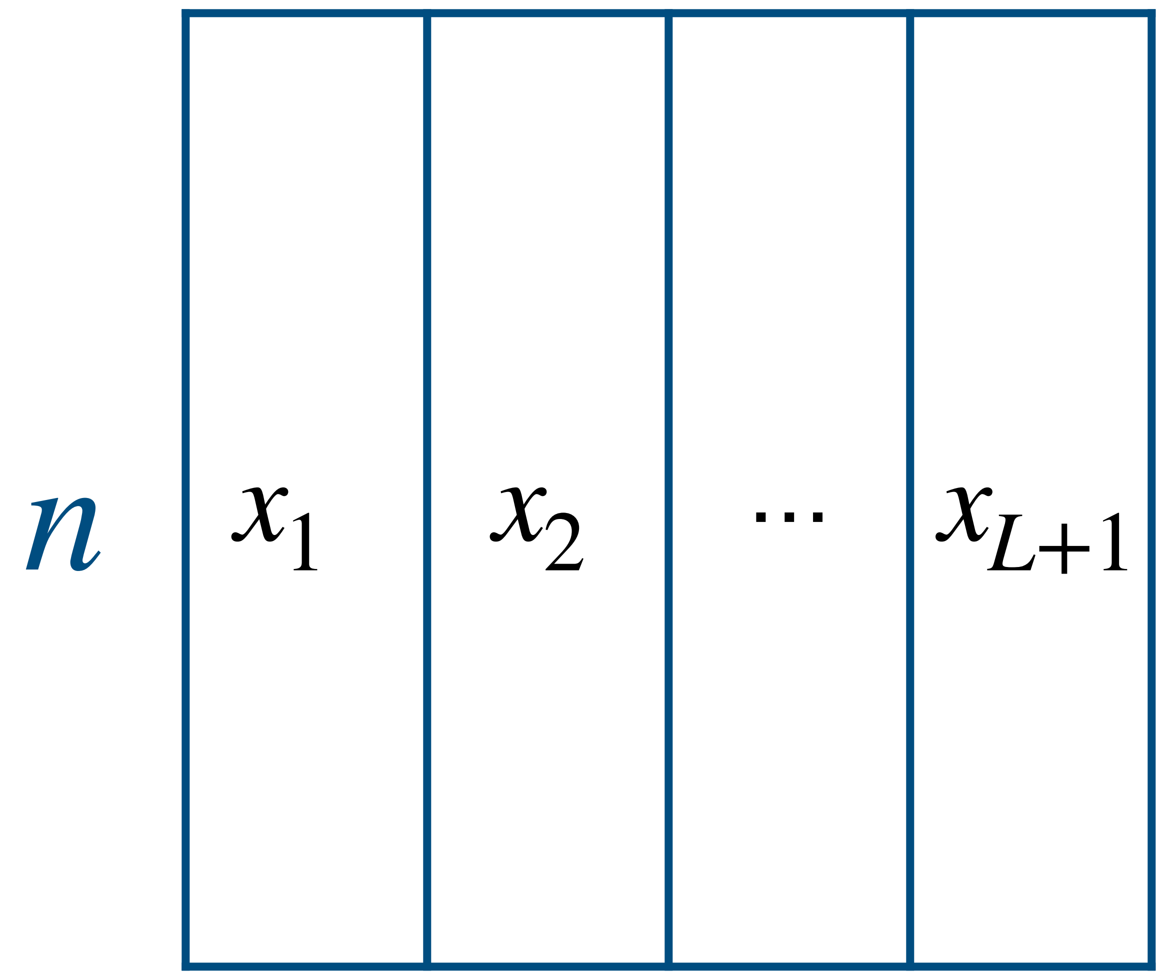

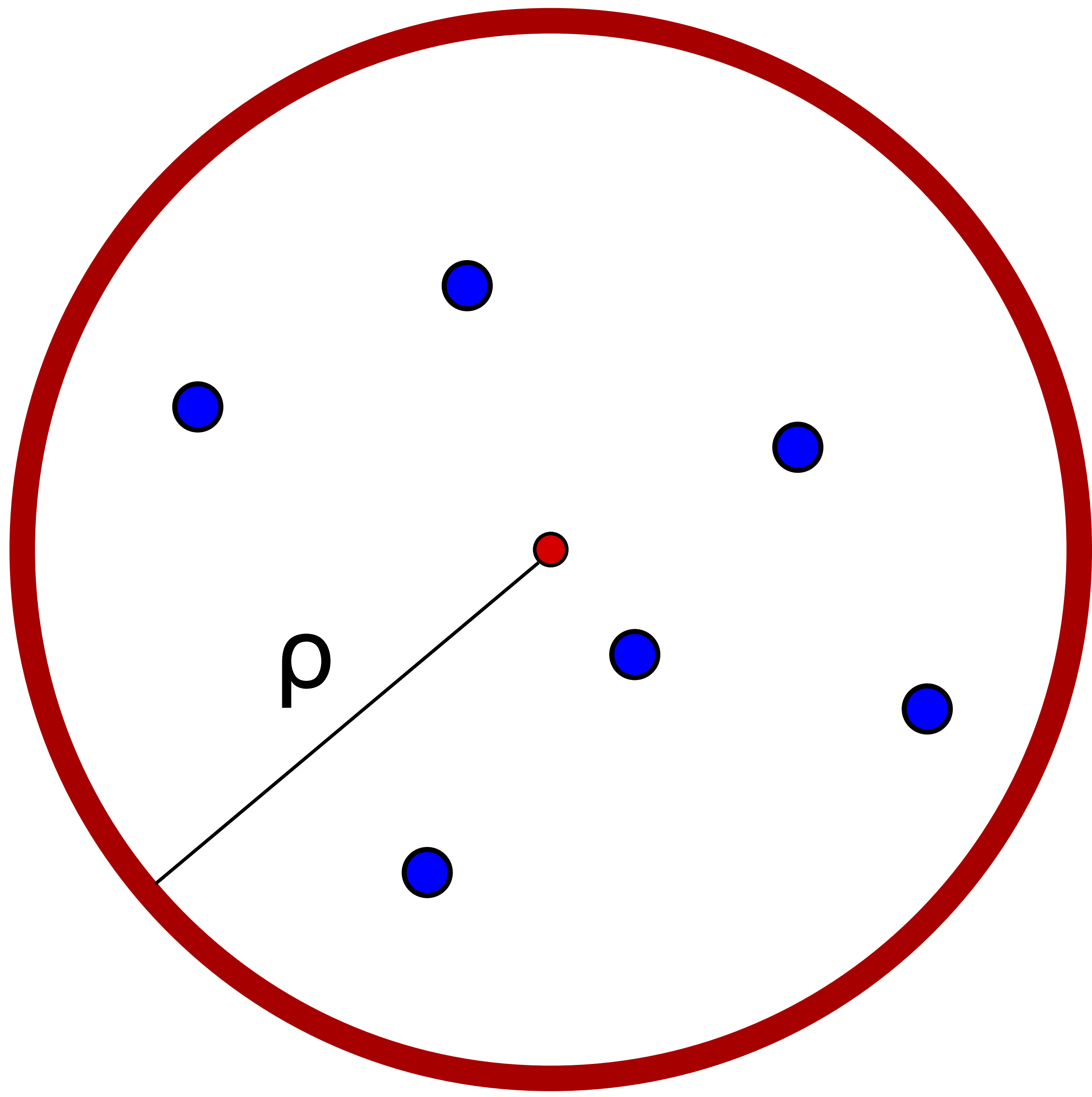

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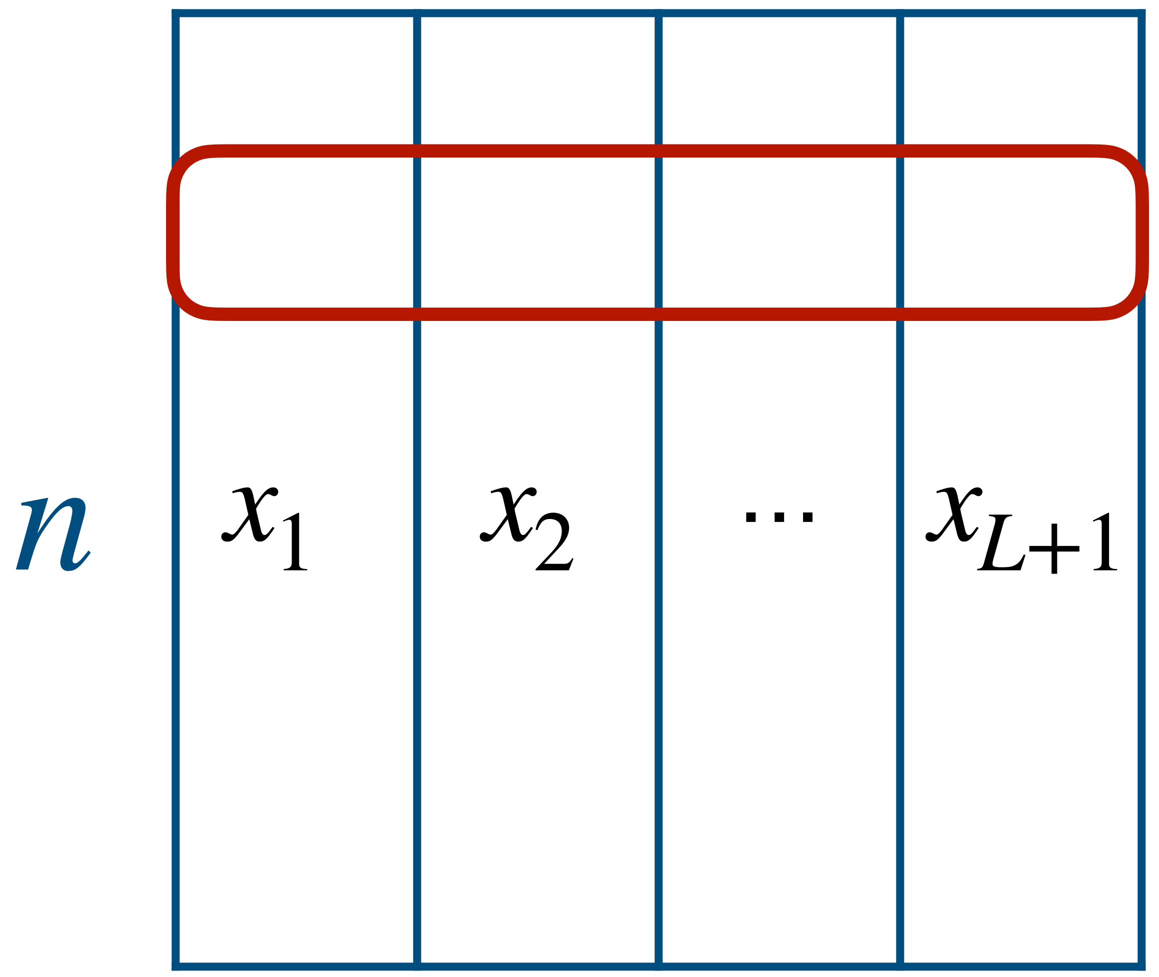

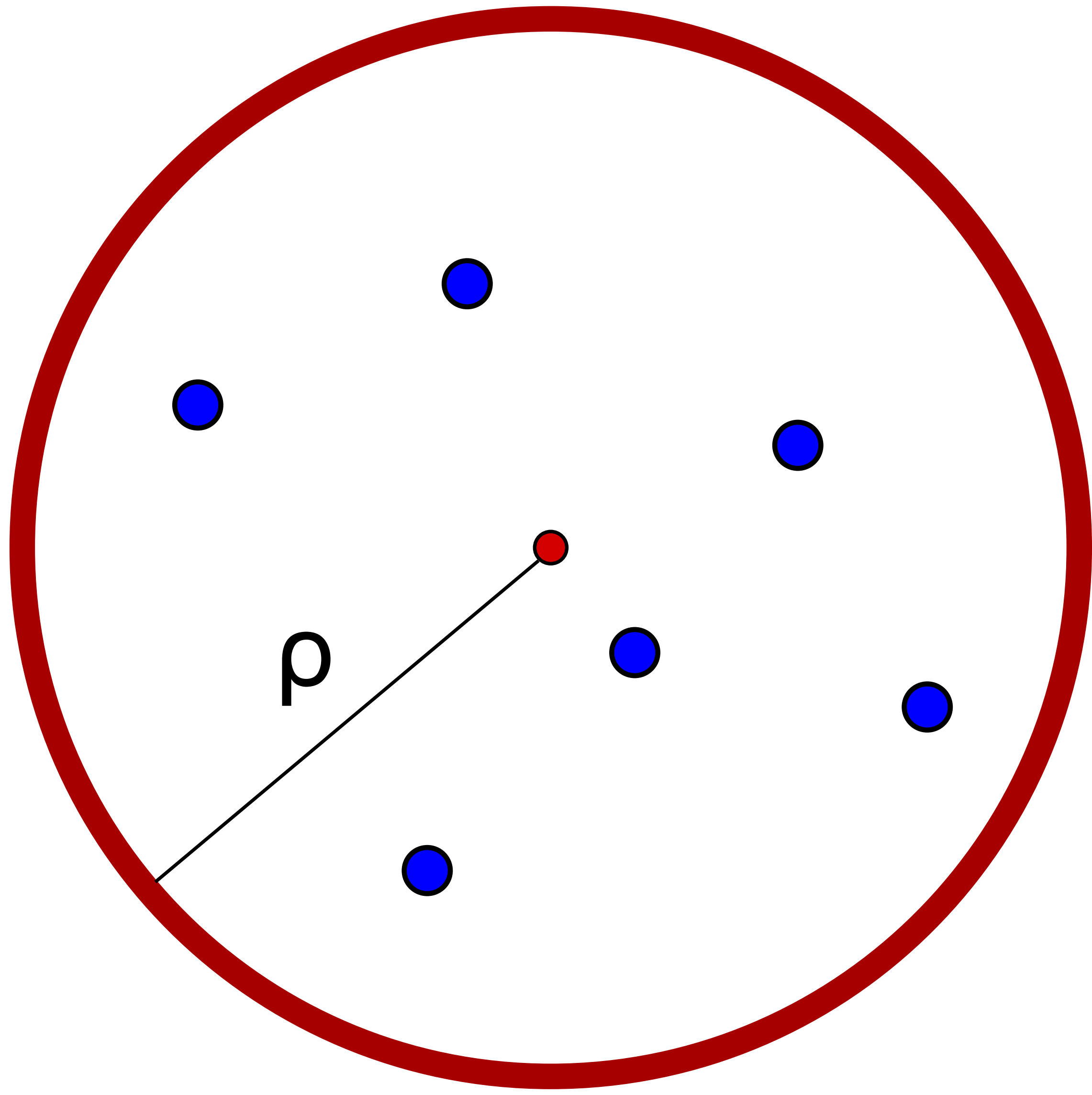

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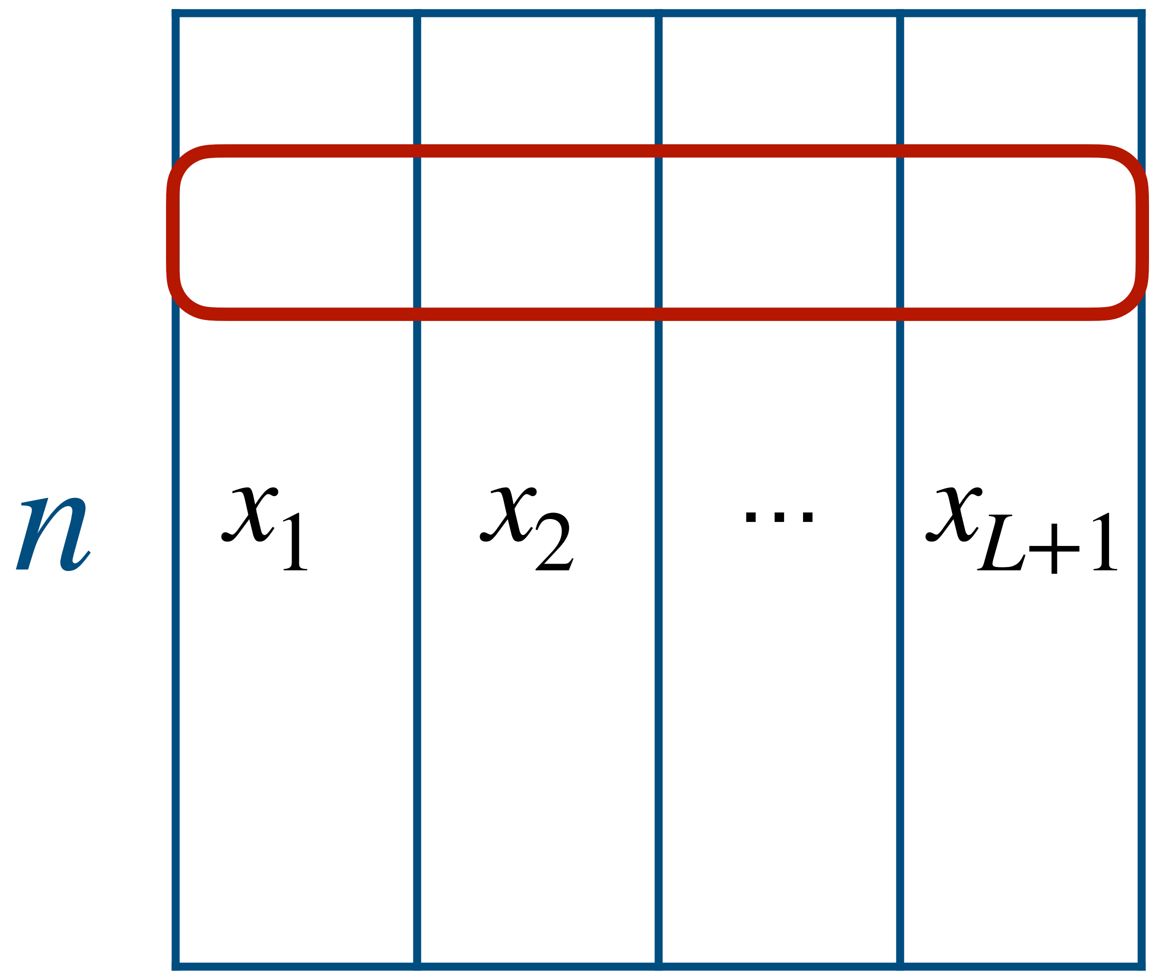

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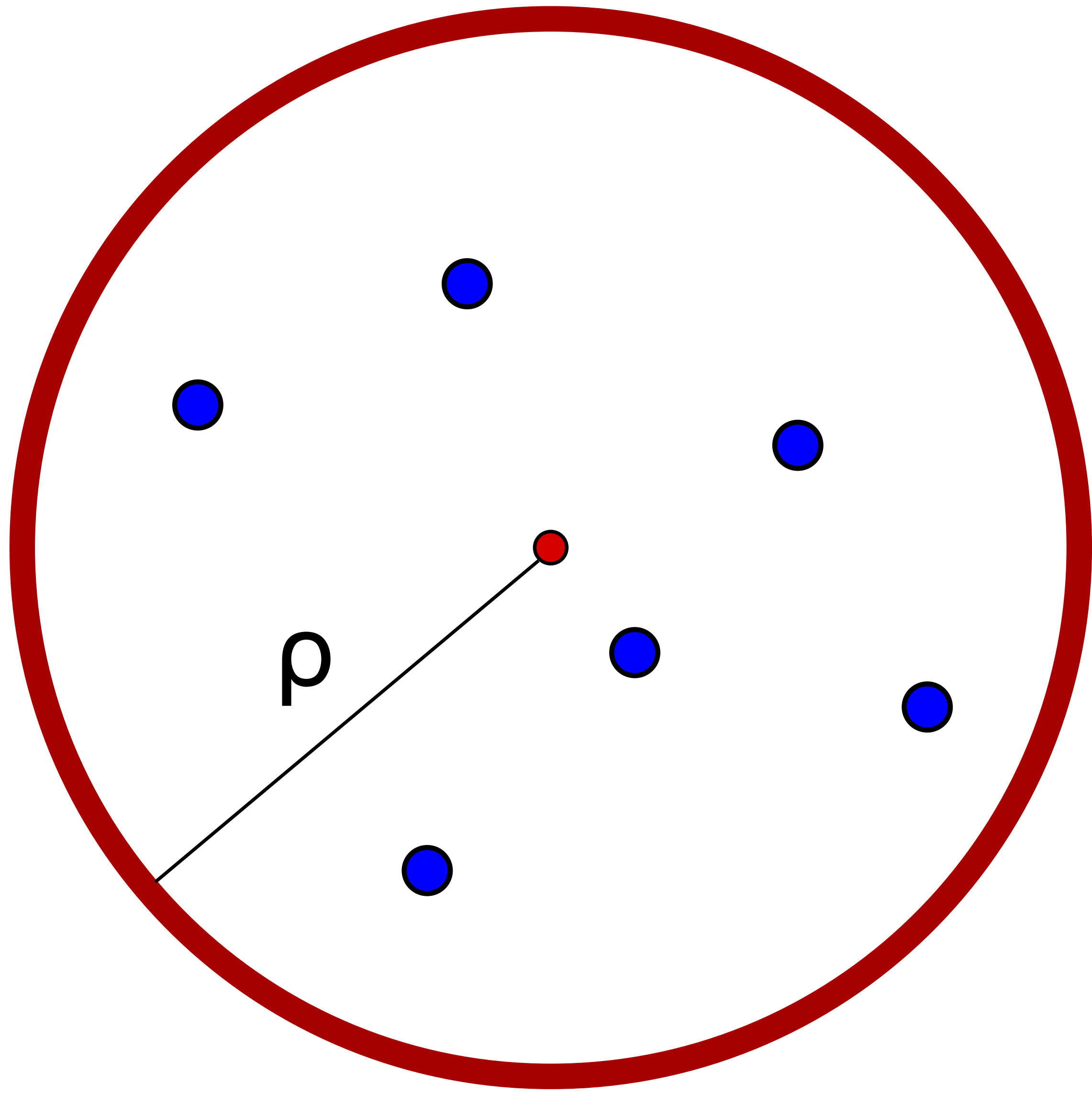

- There are at most $n^{2^{L+1}}$ **ρ-clustered distributions**. This is a **tiny** number so we can **treat each clustered distribution separately**.

Expectations in an RLC

- Let τ be a distribution over $\mathbb{F}_2^{L+1}$.
- How many τ-distributed matrices do we expect in an **RLC**?

$$\mathbb{E}(\tau\text{-distributed matrices in } C) = \#\tau\text{-distributed matrices} \cdot \Pr_{A\sim\tau}(A \subseteq C)$$
$$\approx 2^{nH(\tau)} \cdot 2^{-n(1-R)\cdot \dim\{x_1,\ldots,x_{L+1}\}}$$
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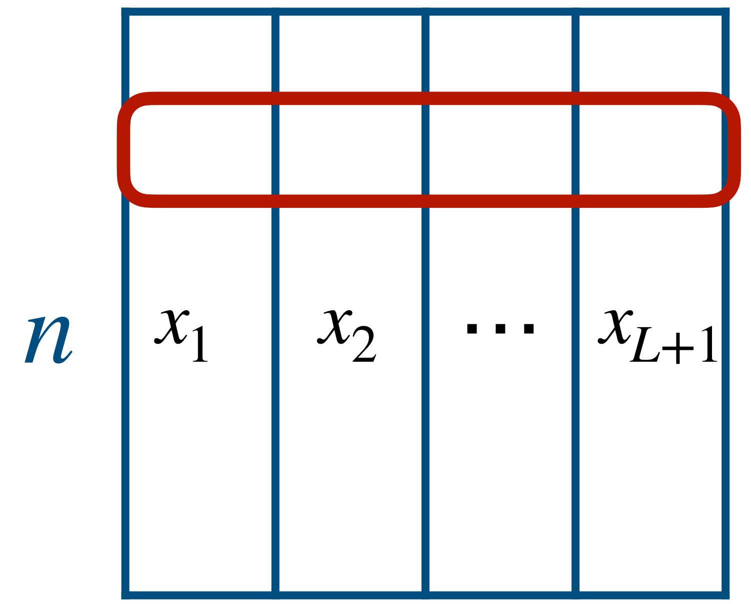

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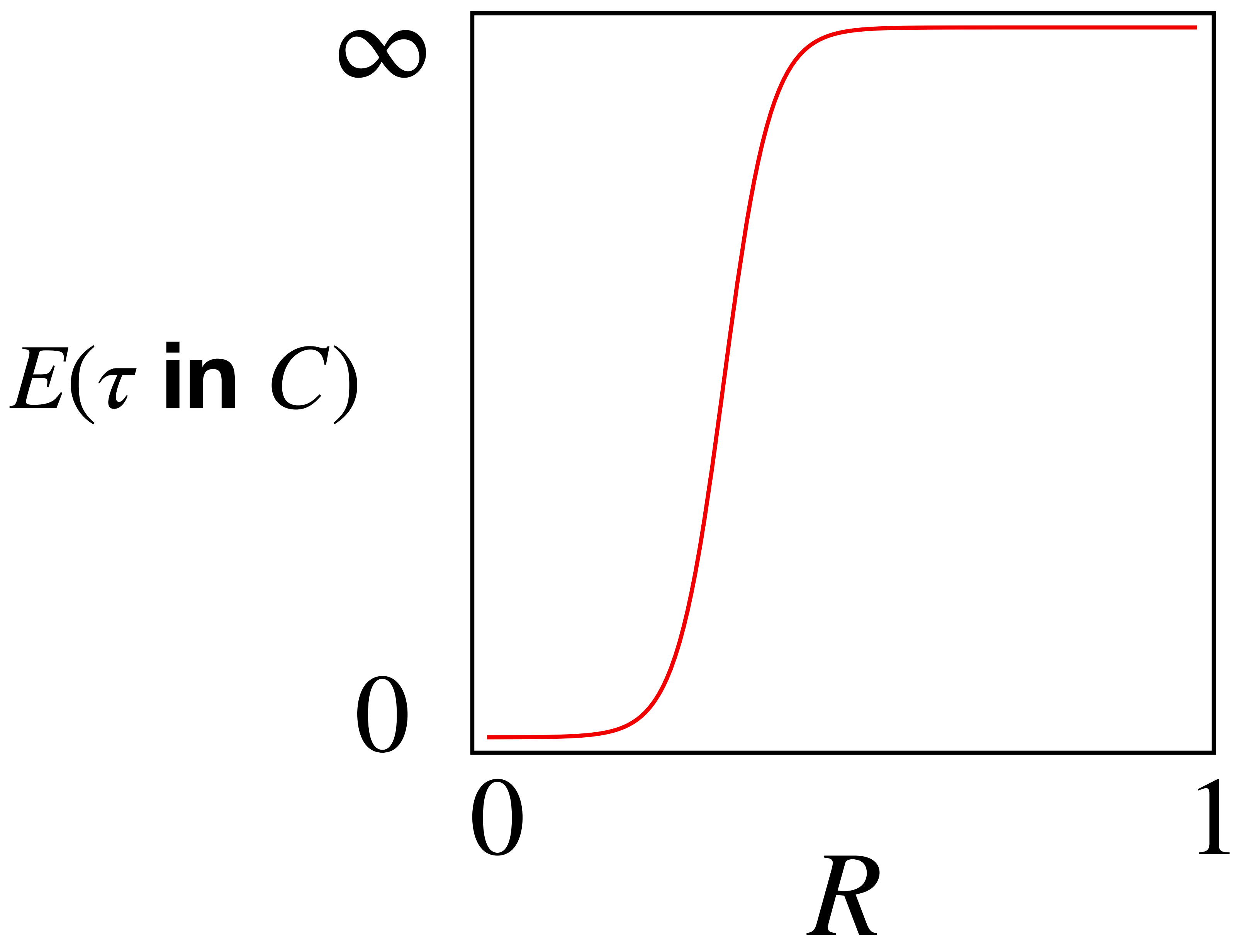

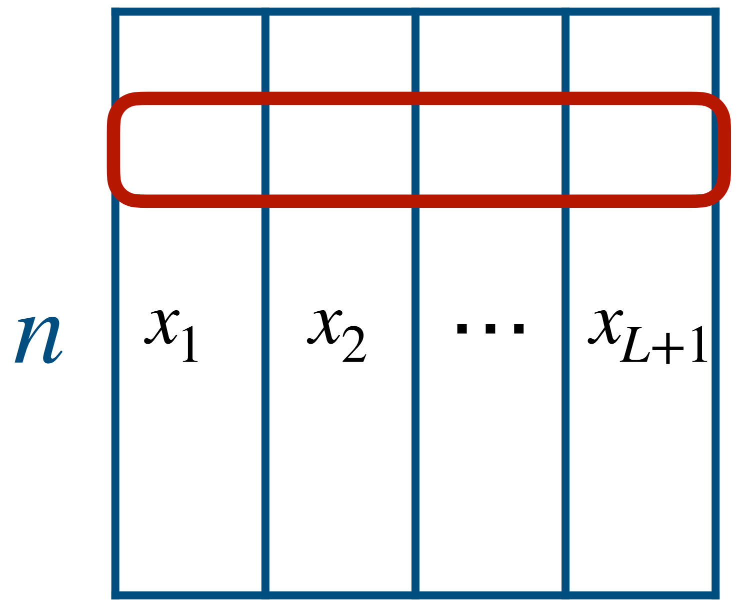

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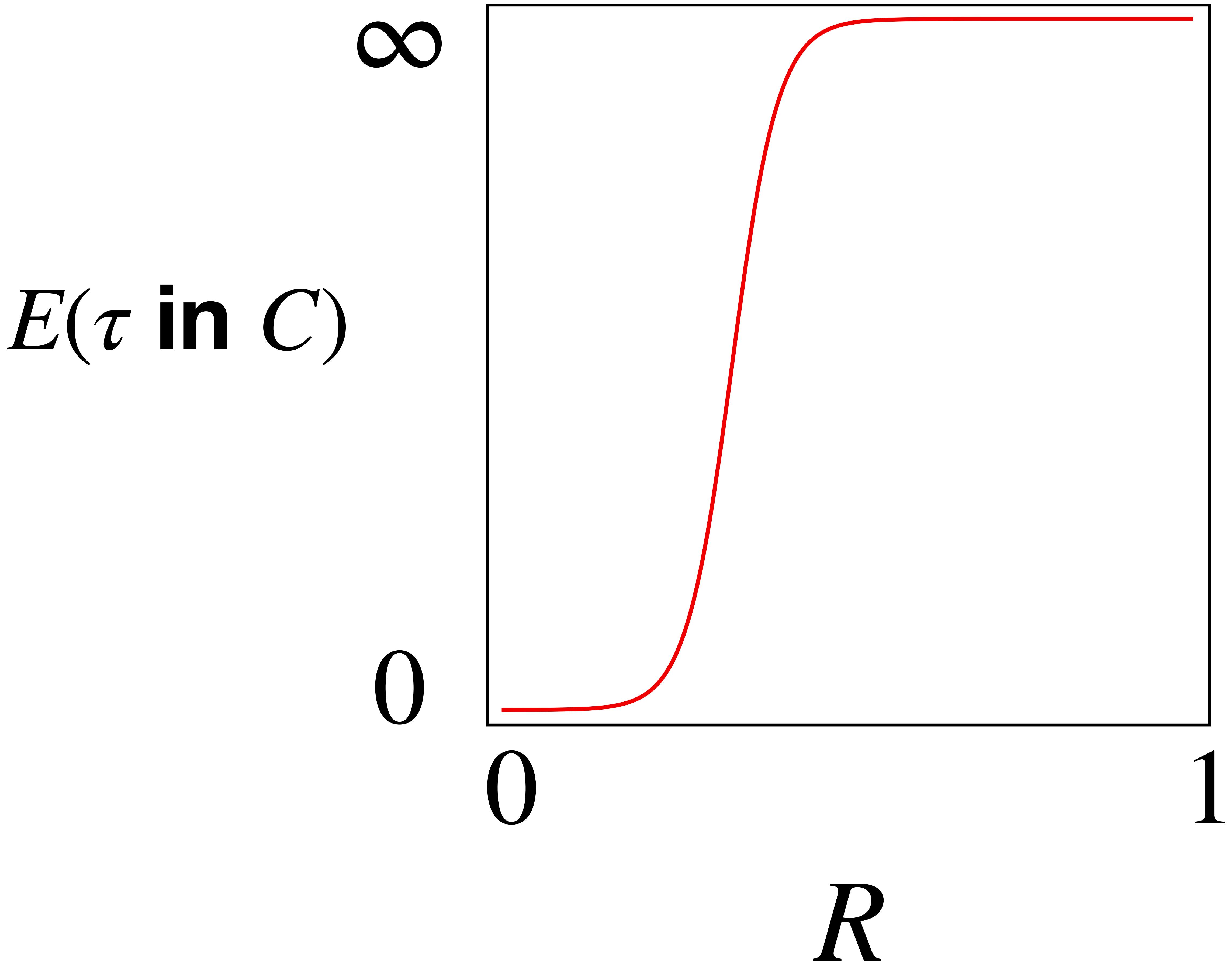

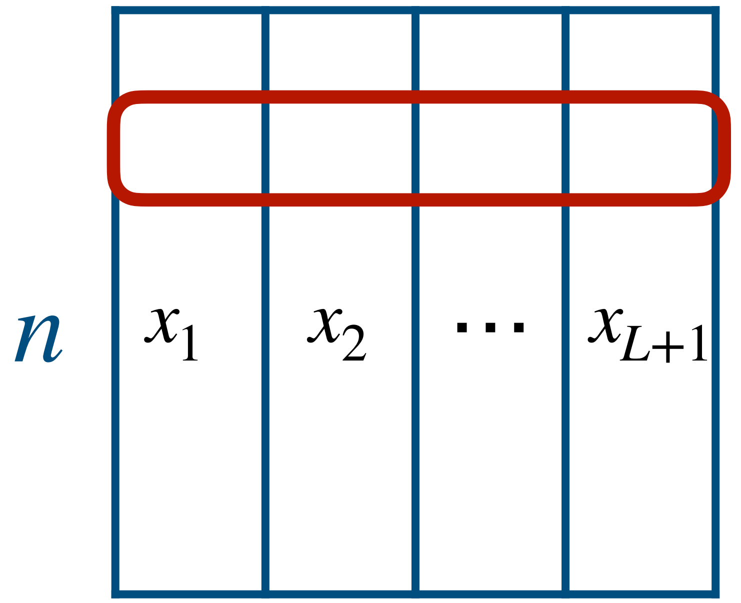

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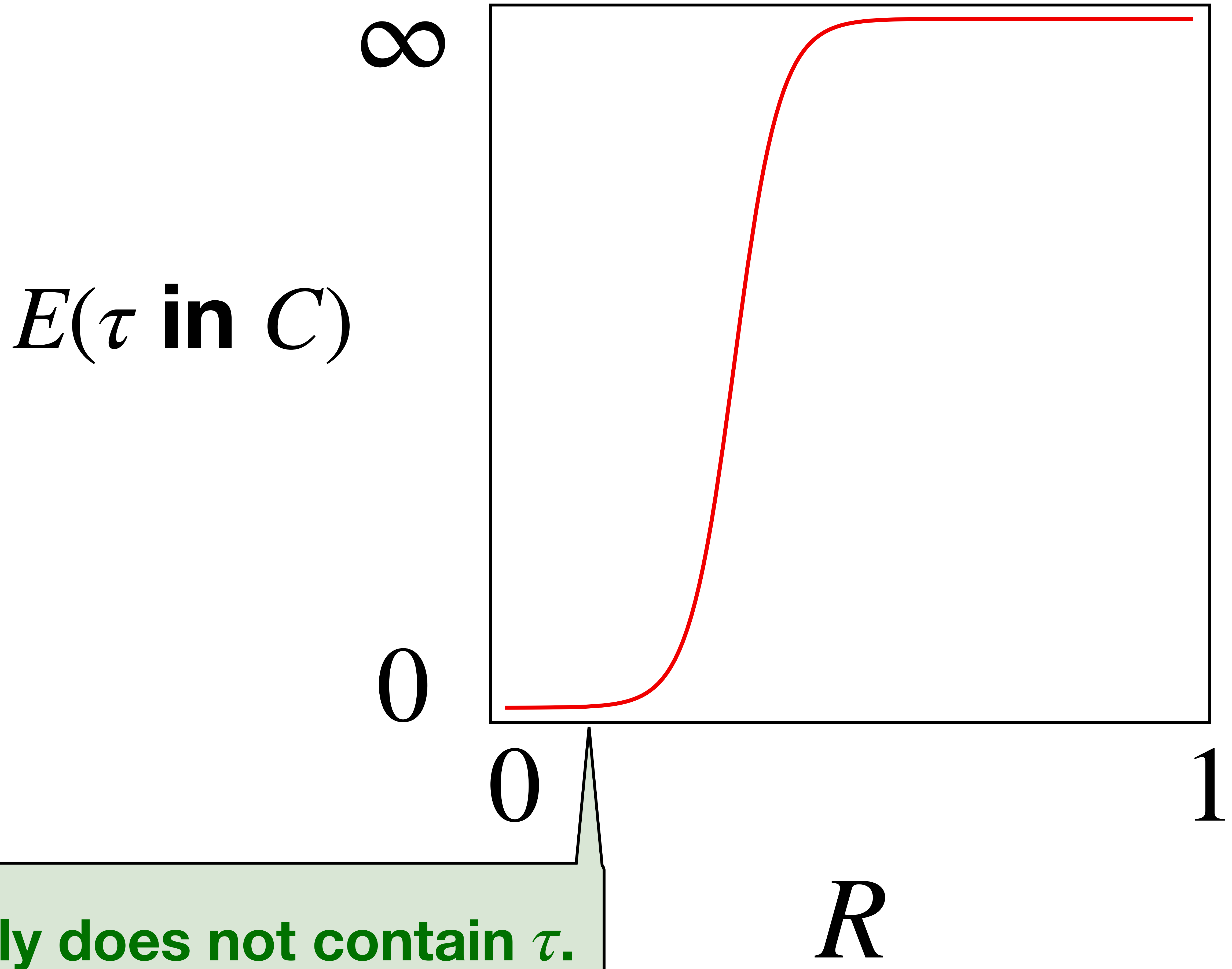

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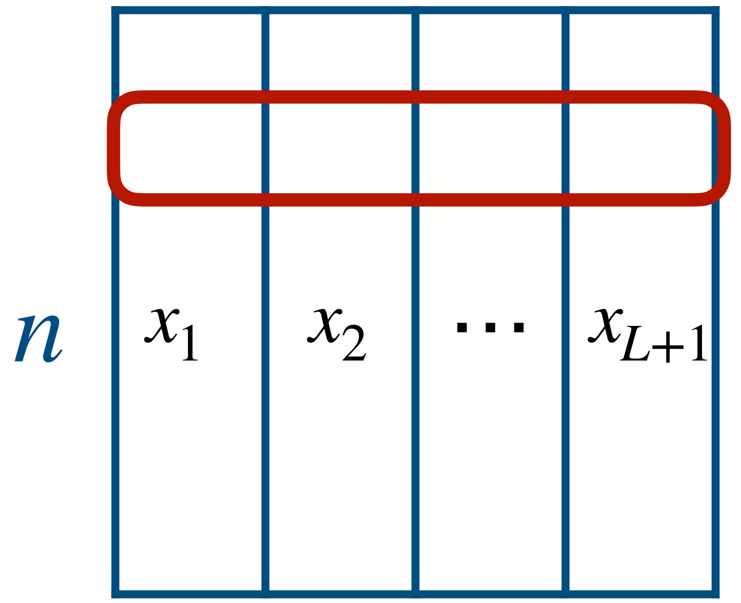

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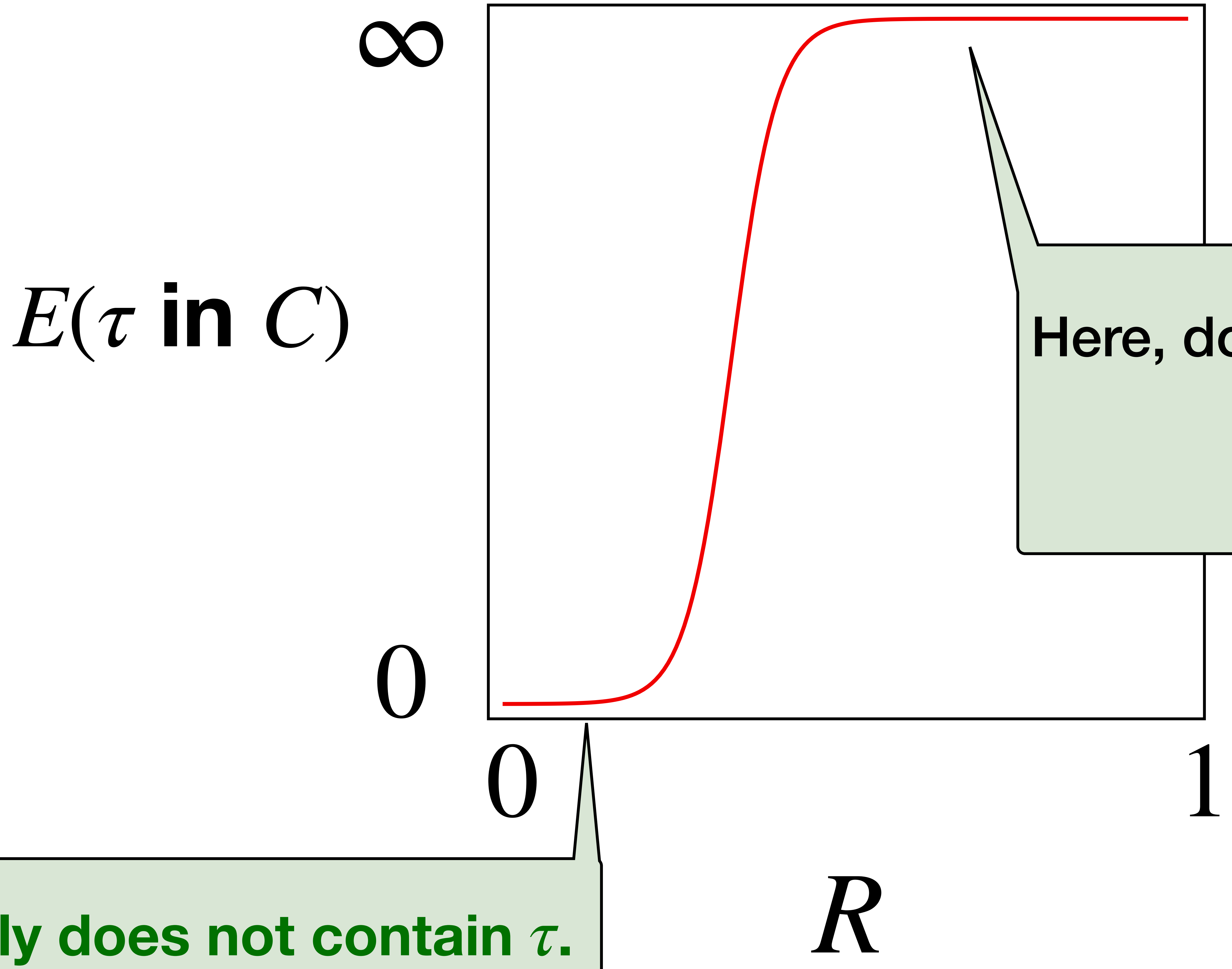
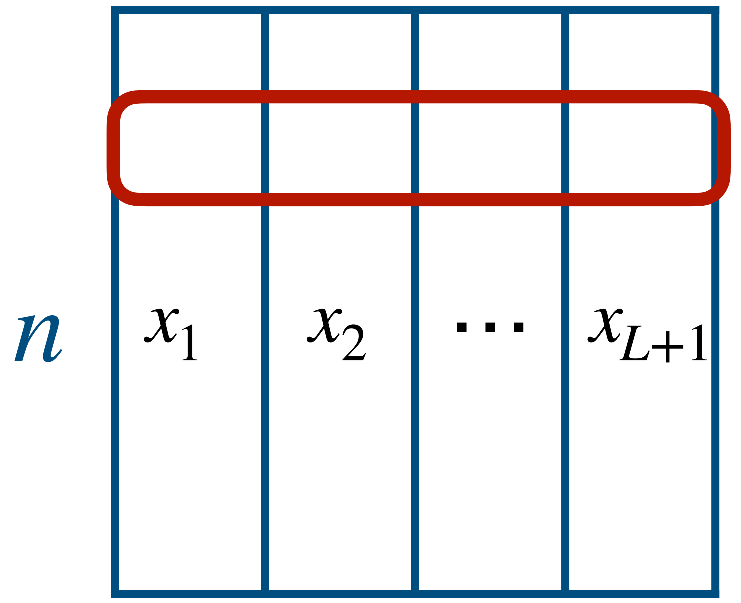
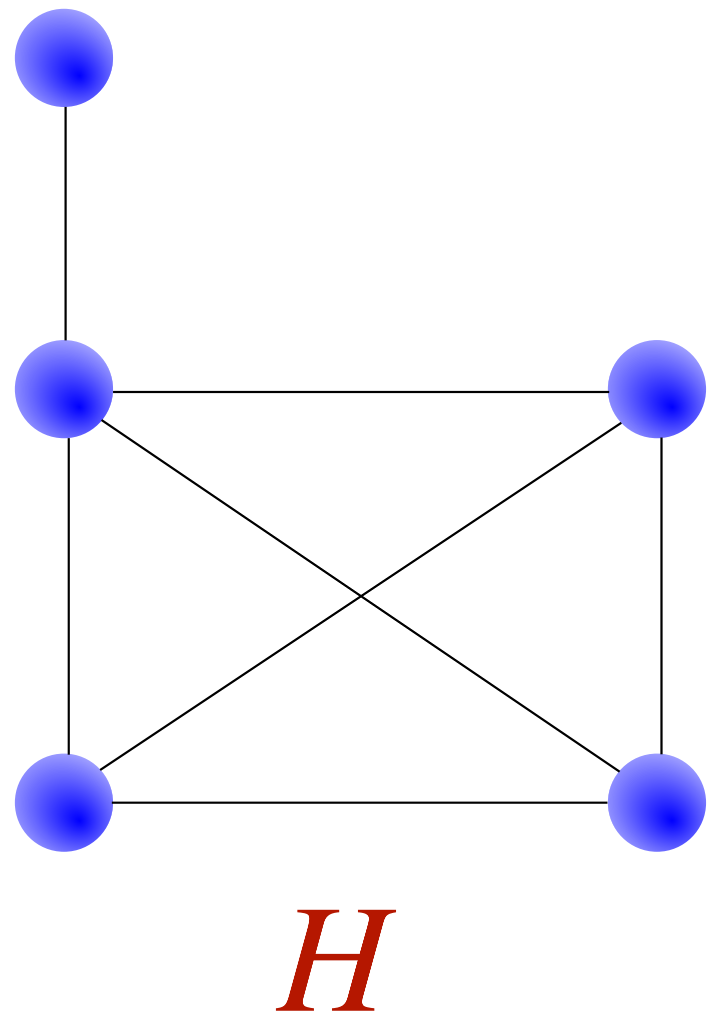


Here, does C **almost surely contain** τ?

Not necessarily!

Here, C **almost surely does not contain** τ.





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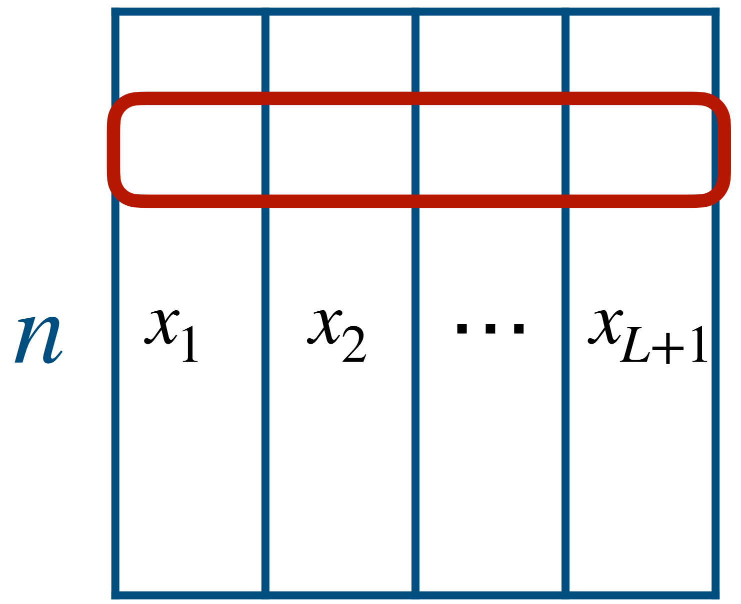

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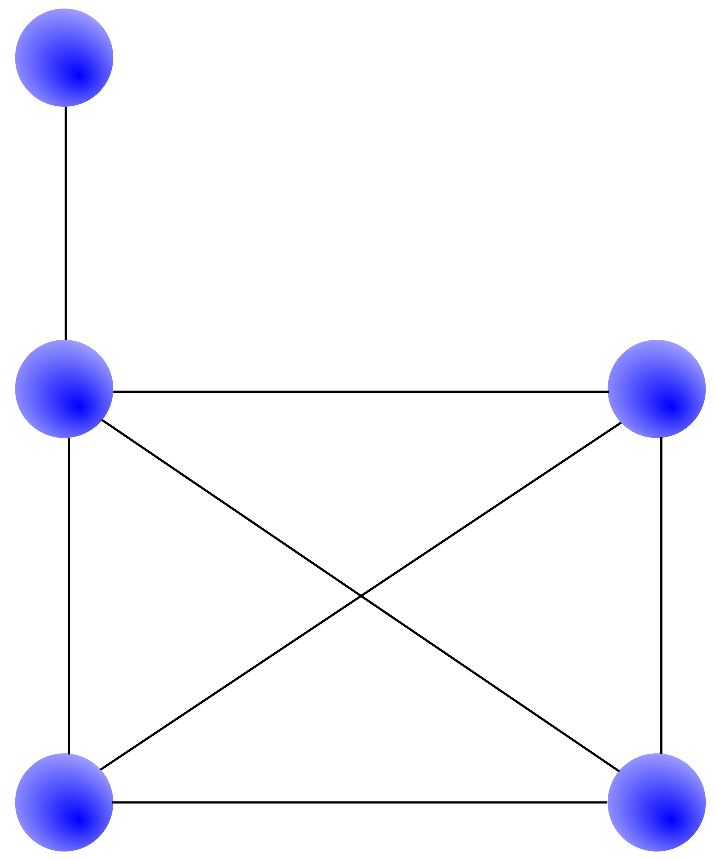
- What about **subgraphs** of H?



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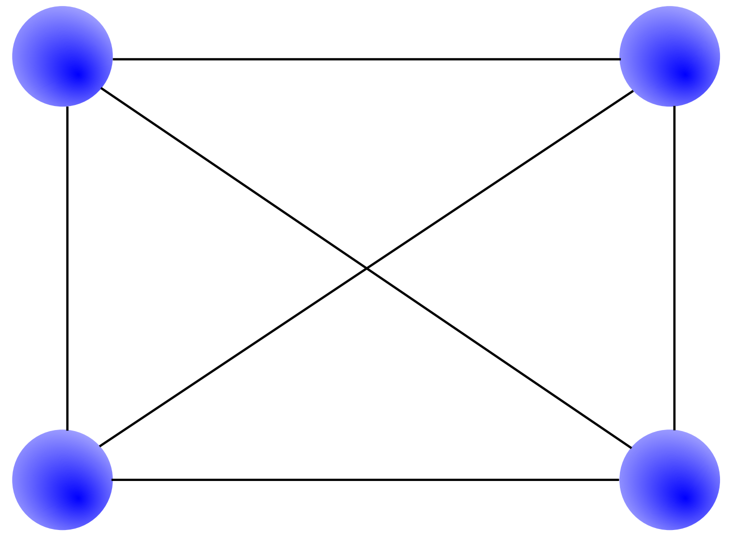

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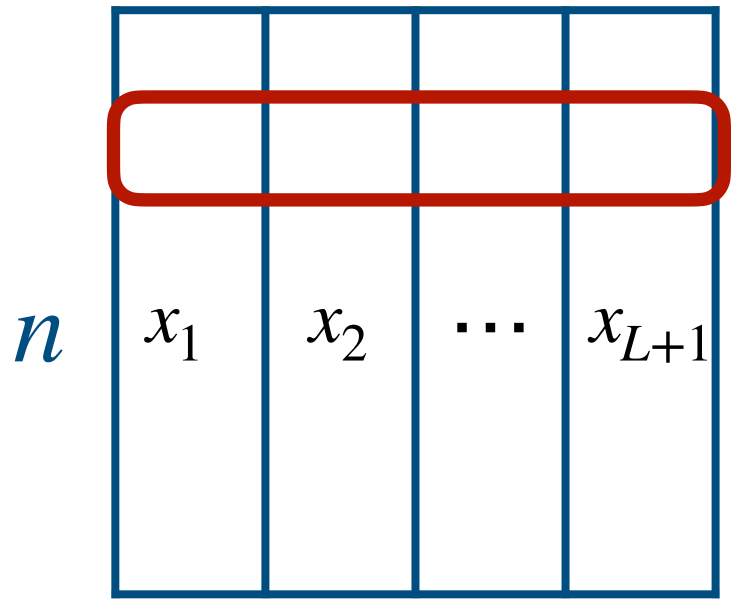

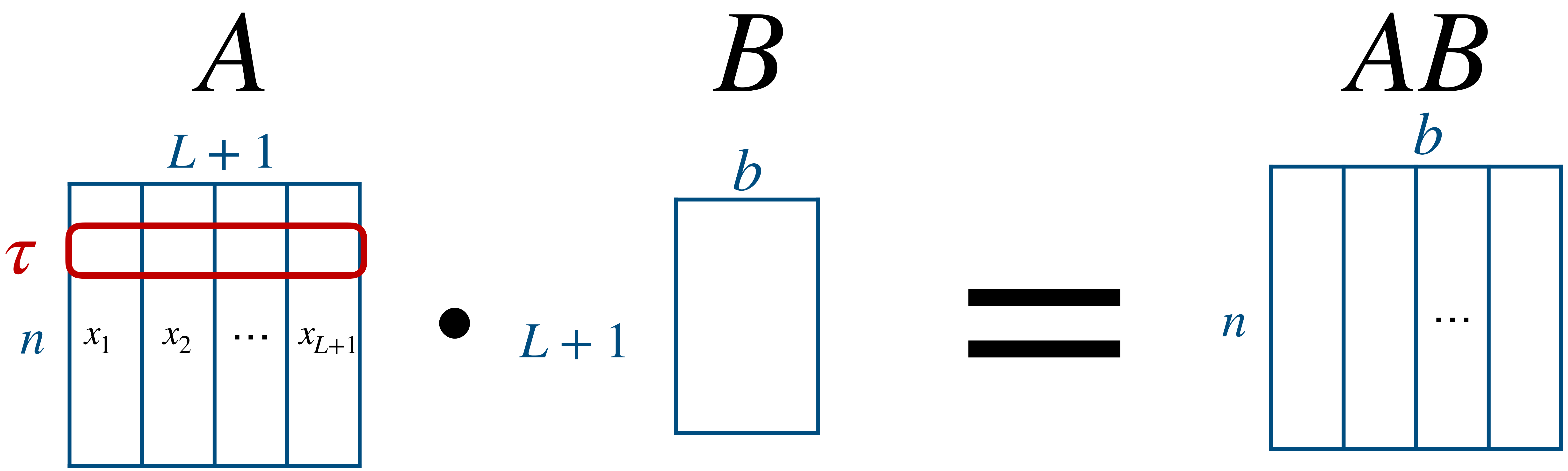

A
B
AB
L + 1
b
b
τ
n
x1
x2
⋯
xL+1
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n
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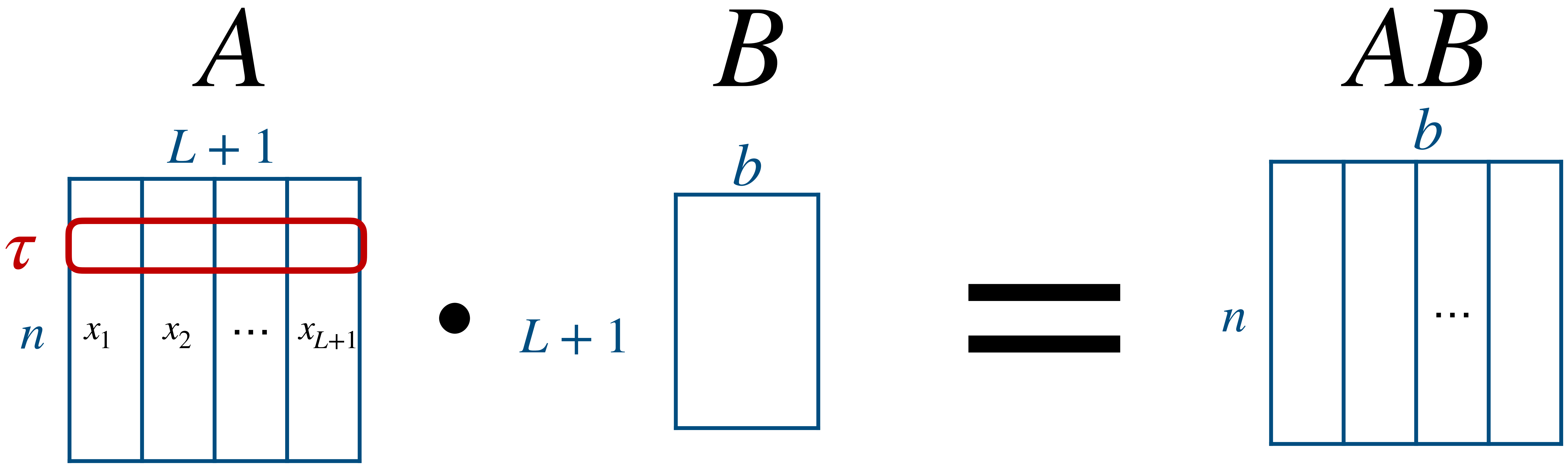

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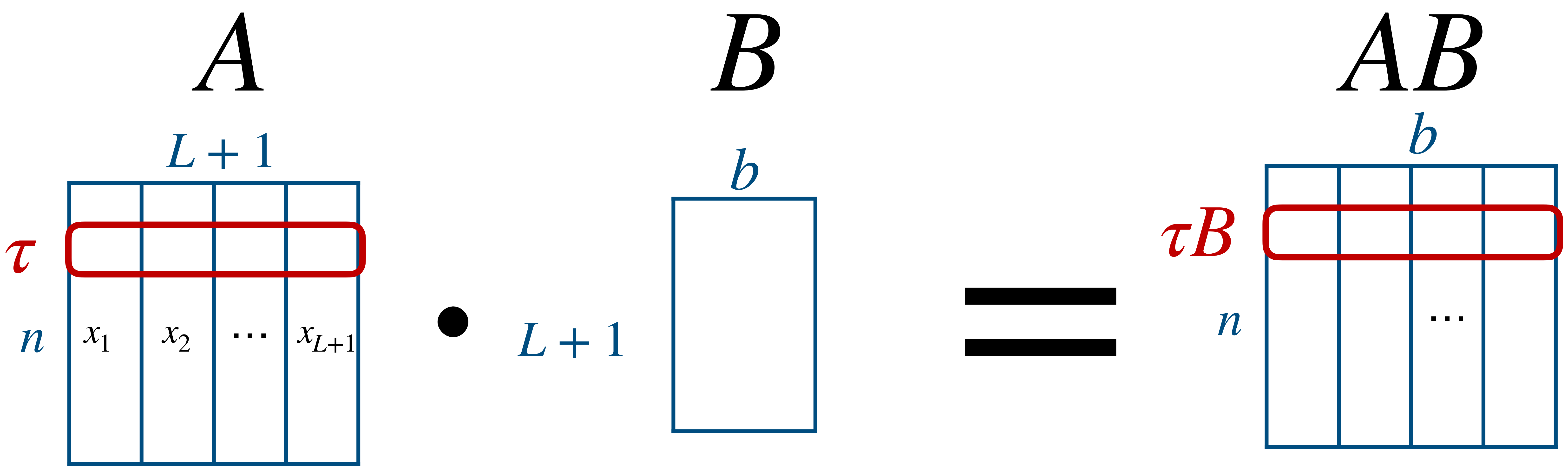
- A uniformly random row of AB is distributed like zB where $z \sim \tau$.

- We denote this distribution τB



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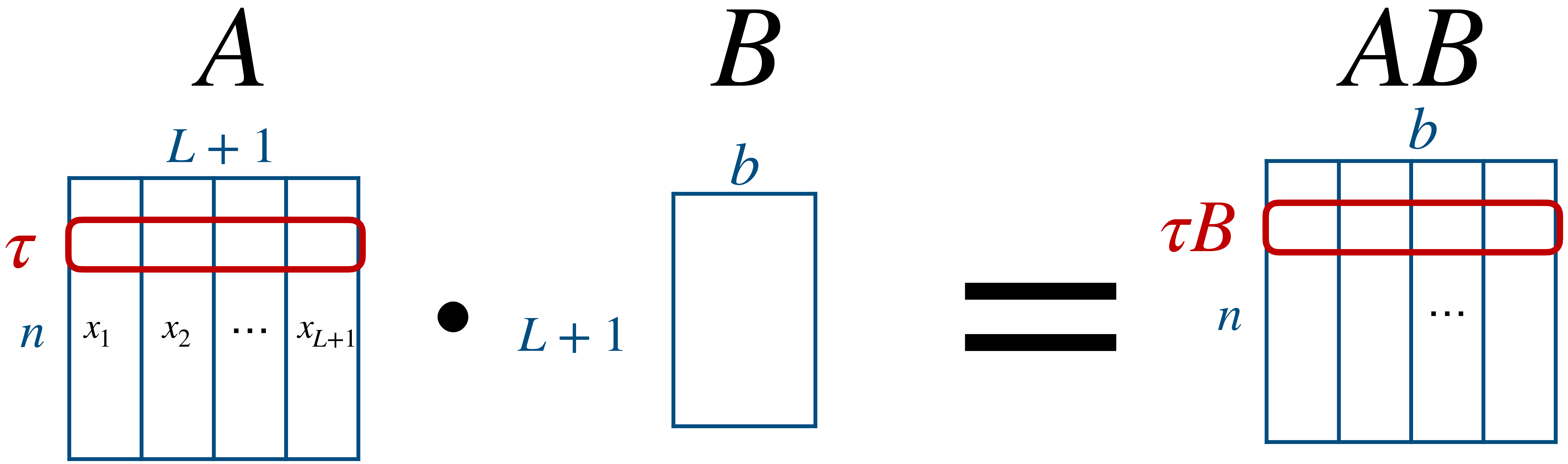
- In order to **contain τ**, a **linear code** must **contain τB**.



Theorem (thresholds for RLCs):

An **RLC** of rate R is likely to **contain a τ distributed matrix** if and only if

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An **RLC** of rate R is likely **(ρ, L)-list-decodable** if and only if

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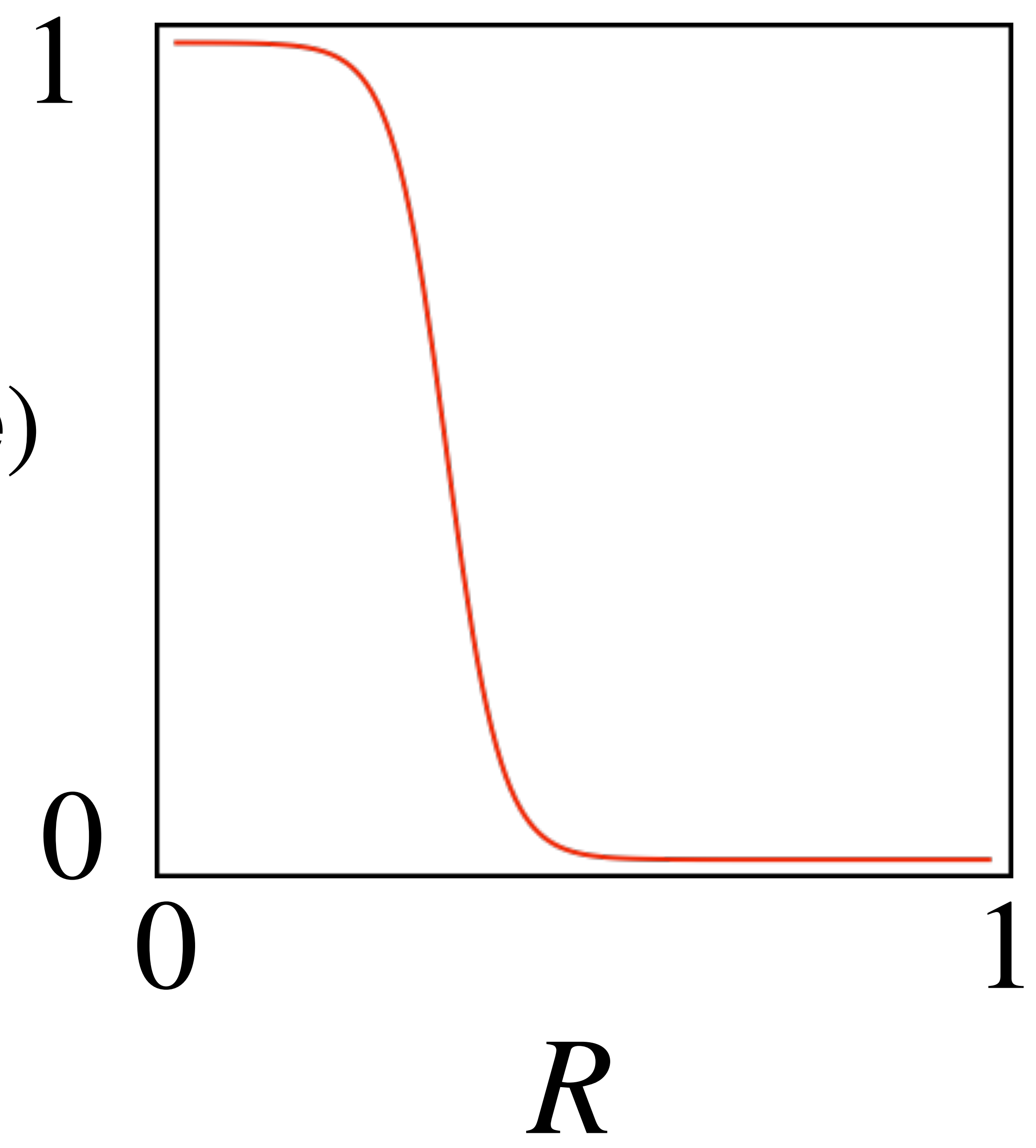

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- This holds for more than just **list-decodability**.
  - Any **property** characterized by **“foribdden distributions”** has such a characterization.
  - For example, **list-recoverability**!
  - In general, any **monotone, local and symmetric property**.

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- But what is this good for? we already know (through a long line of works) that **RLCs achieve the list-decoding GV-bound**.
- But now these results tell us something about **expectations**!

Definition: A random code ensemble $C \subseteq \mathbb{F}_q^n$ is **locally-similar** to an RLC of rate R if

$$\Pr\left[\{v_1, \ldots, v_k\} \subseteq C\right] \approx 2^{-(1-R)\cdot n \cdot \dim\{v_1,\ldots,v_k\}}$$

for all $v_1, \ldots, v_k \in \mathbb{F}_q^n$.

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Let D be an **RLC of rate R**. We know from previous works that an **D almost surely achieves the list-decoding GV-bound**.

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But

$$\begin{aligned}\mathbb{E}\left[\#\tau B\text{-distributed matrices in } C\right] &\approx \#\tau B\text{-distributed matrices} \cdot 2^{-(1-R)n\cdot\dim(\mathrm{supp}(\tau))} \\ &= \mathbb{E}\left[\#\tau B\text{-distributed matrices in } D\right] \le o(1)\end{aligned}$$

So C **is unlikely to contain τB and thus unlikely to contain τ**.

Theorem: If C is **locally-similar** to an **RLC of rate R** then it **achieves the list-decoding GV-bound** with high probability.

The same argument works for **list-recovery** or any other **local symmetric property:**

Theorem: If C is **locally-similar** to an **RLC of rate R** then it **achieves the same list-recovery parameters** as an RLC.

The reduction paradigm

1. Choose a **random code ensemble** C.

2. Show that C is **locally-similar** to an **RLC**.

3. Conclude that C has all the local symmetric properties of an RLC, including **achieving the list-decoding GV-bound**.

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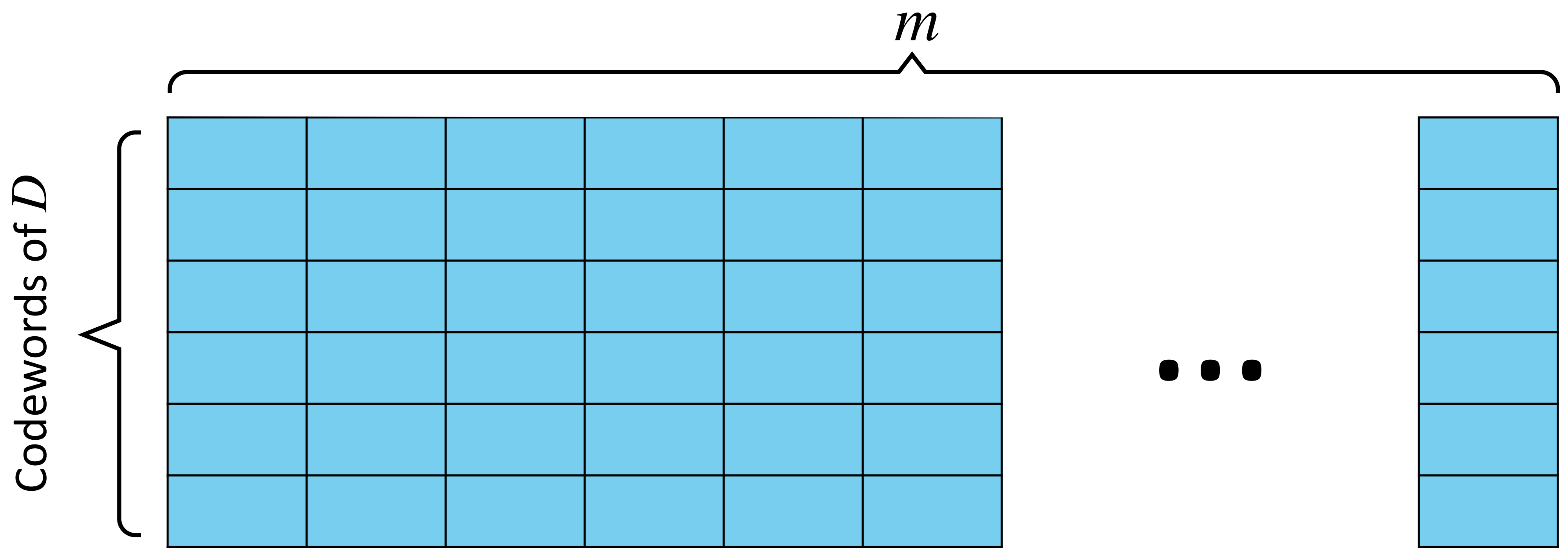
Puncturing of Codes



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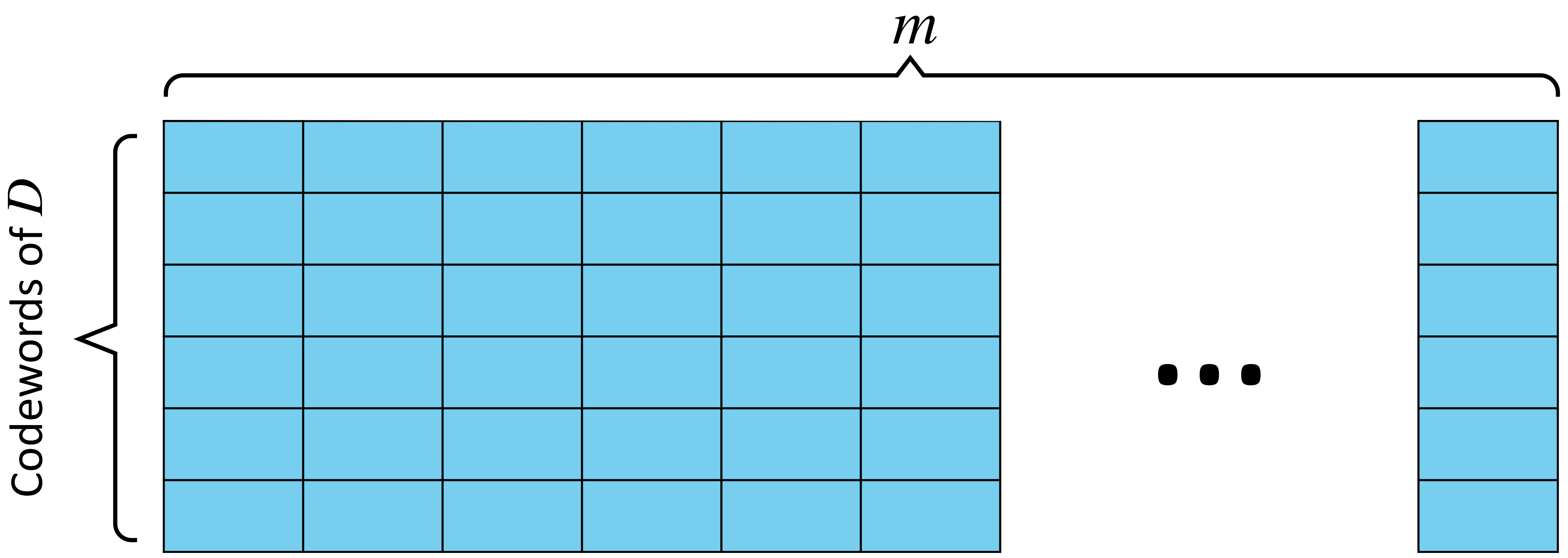

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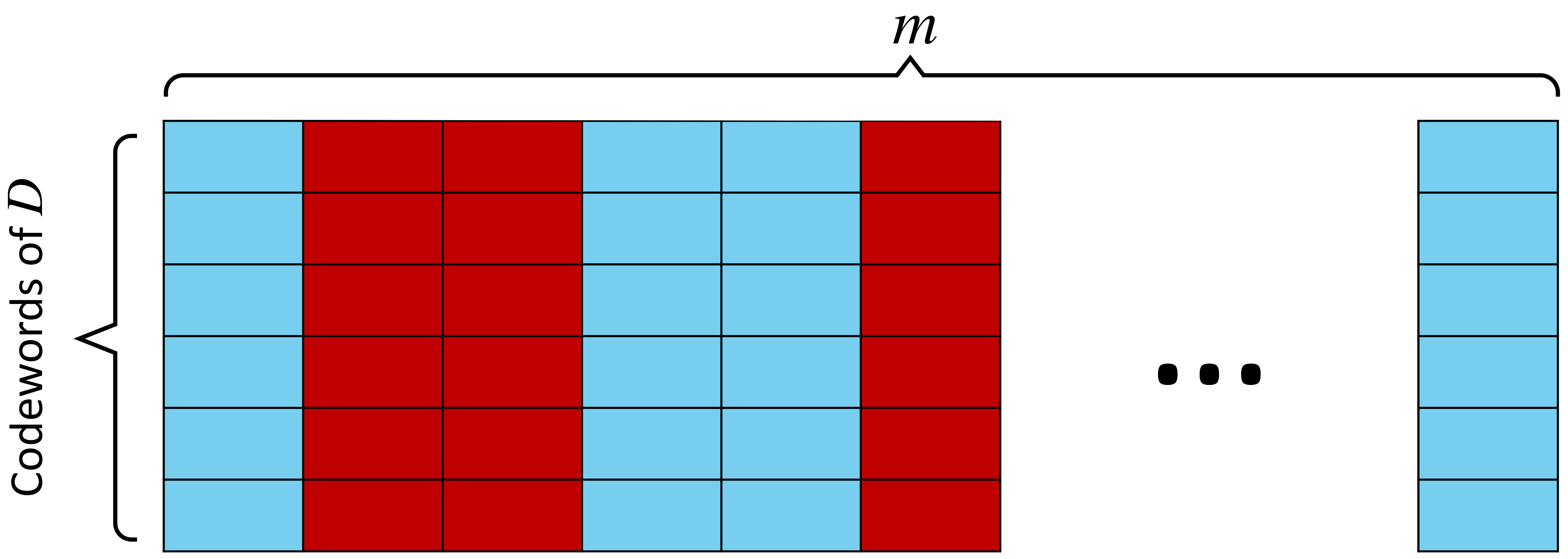

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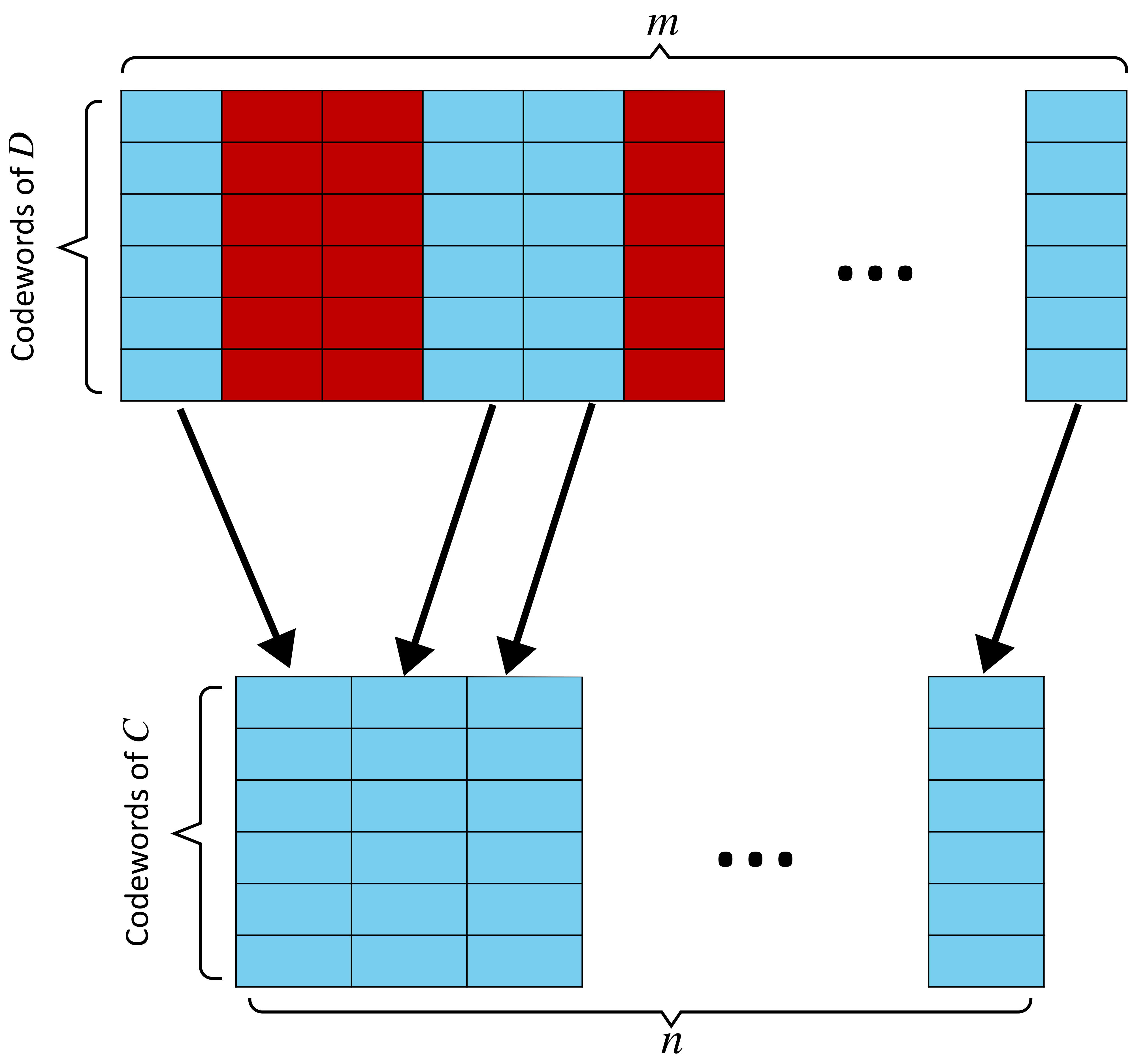

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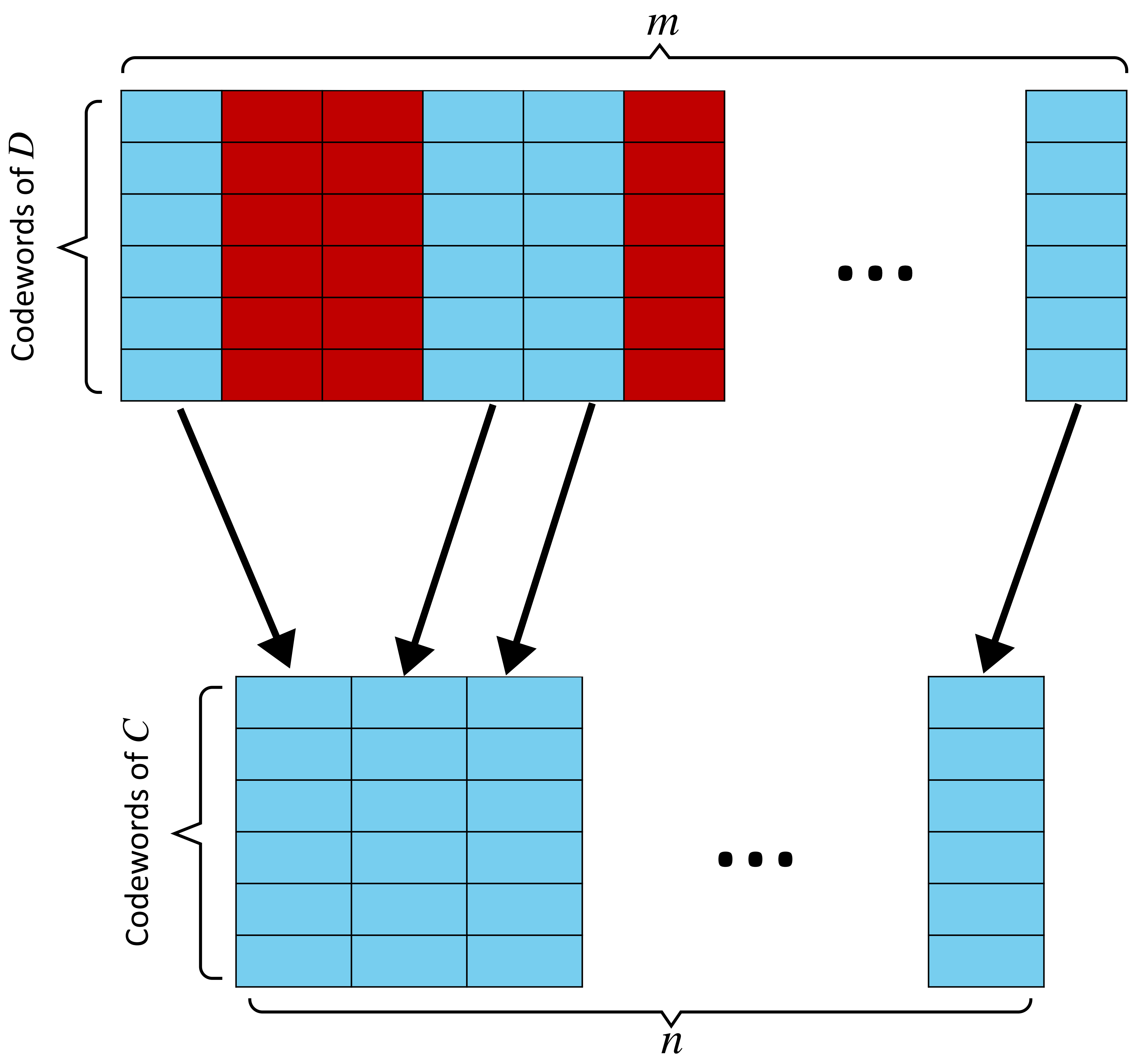

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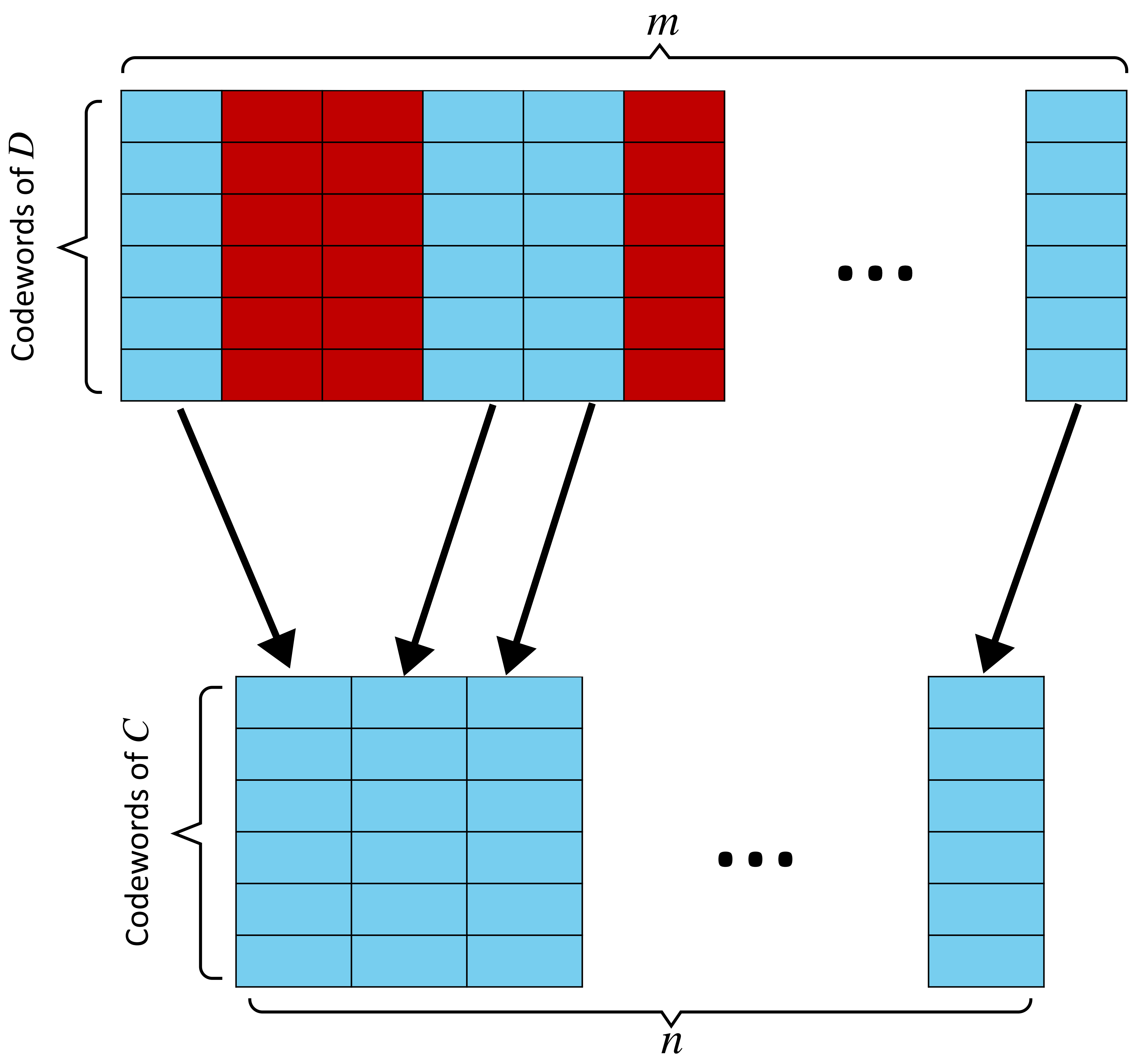

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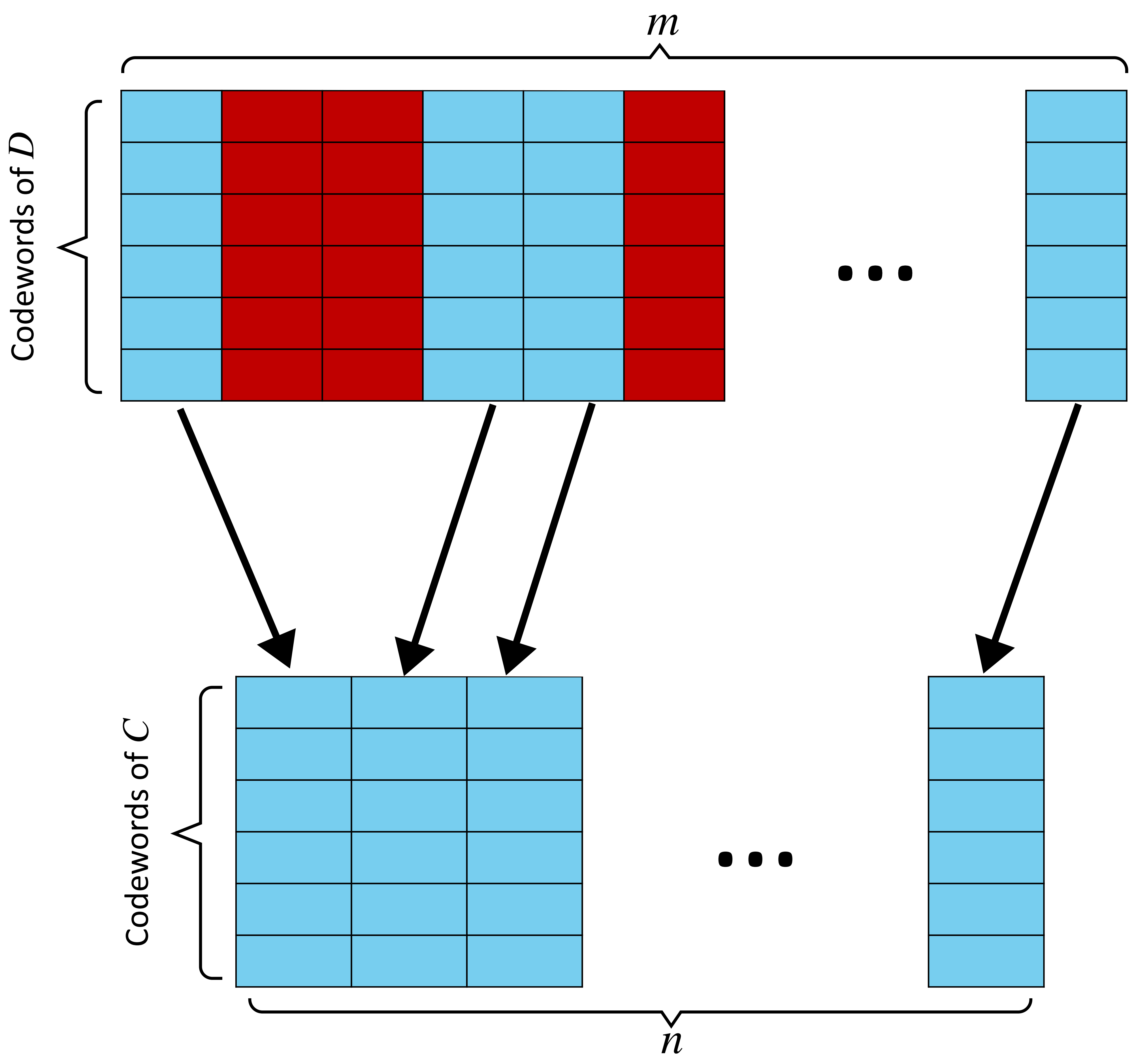
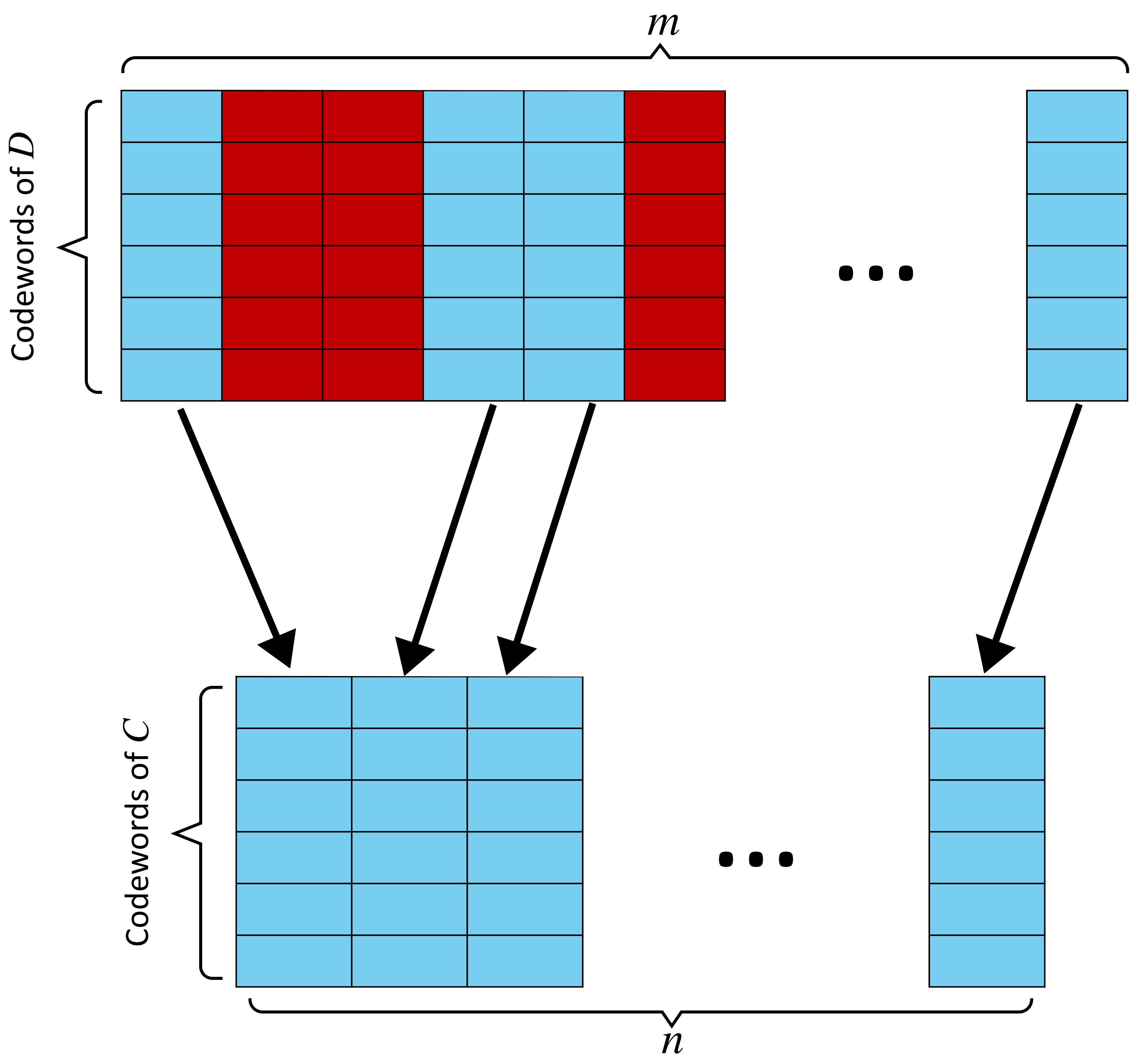
- A **Reed-Solomon code over a random evaluation set** is a random puncturing of the **full Reed-Solomon code.**



Puncturing of low-bias codes



Puncturing of low-bias codes

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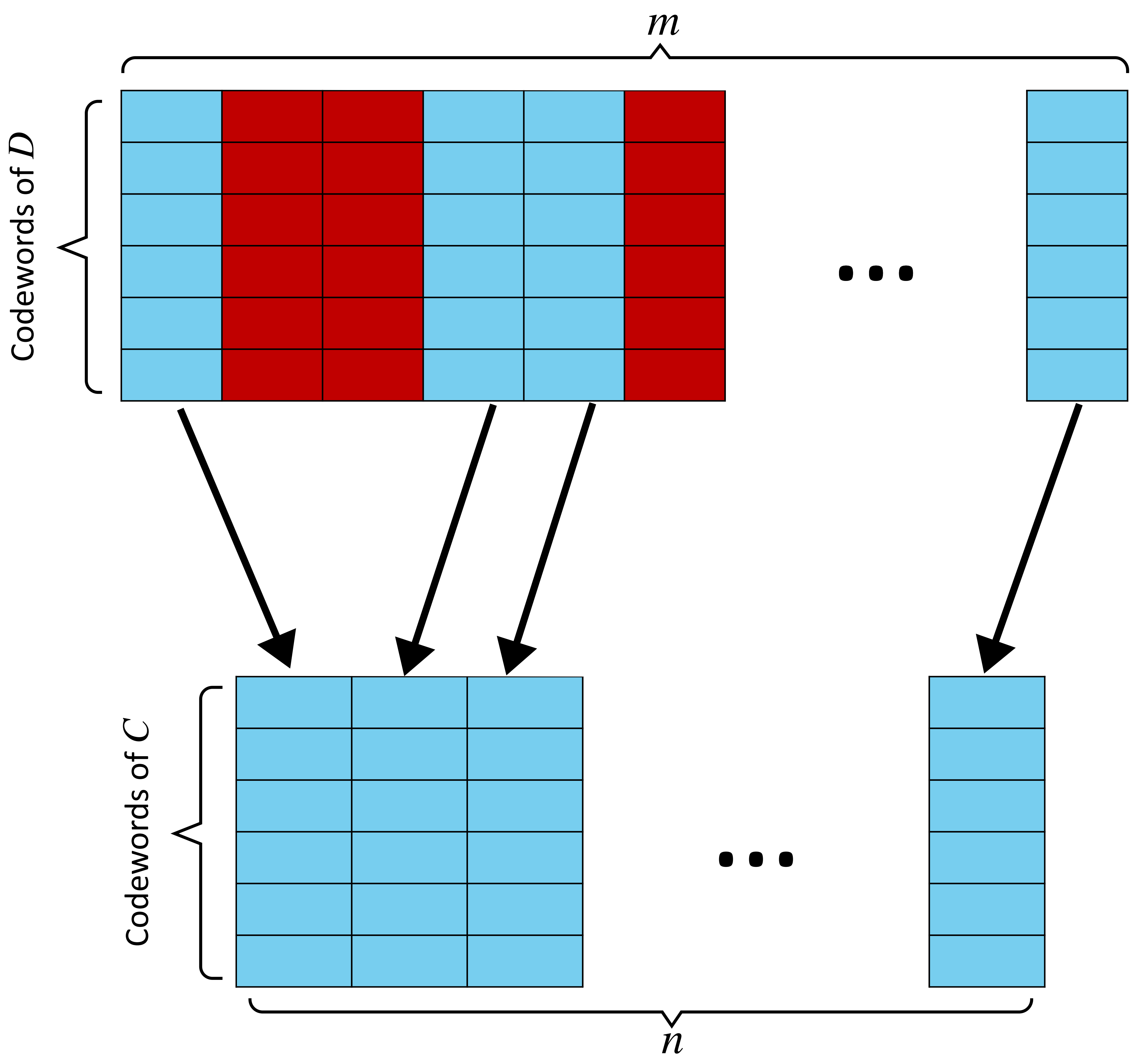

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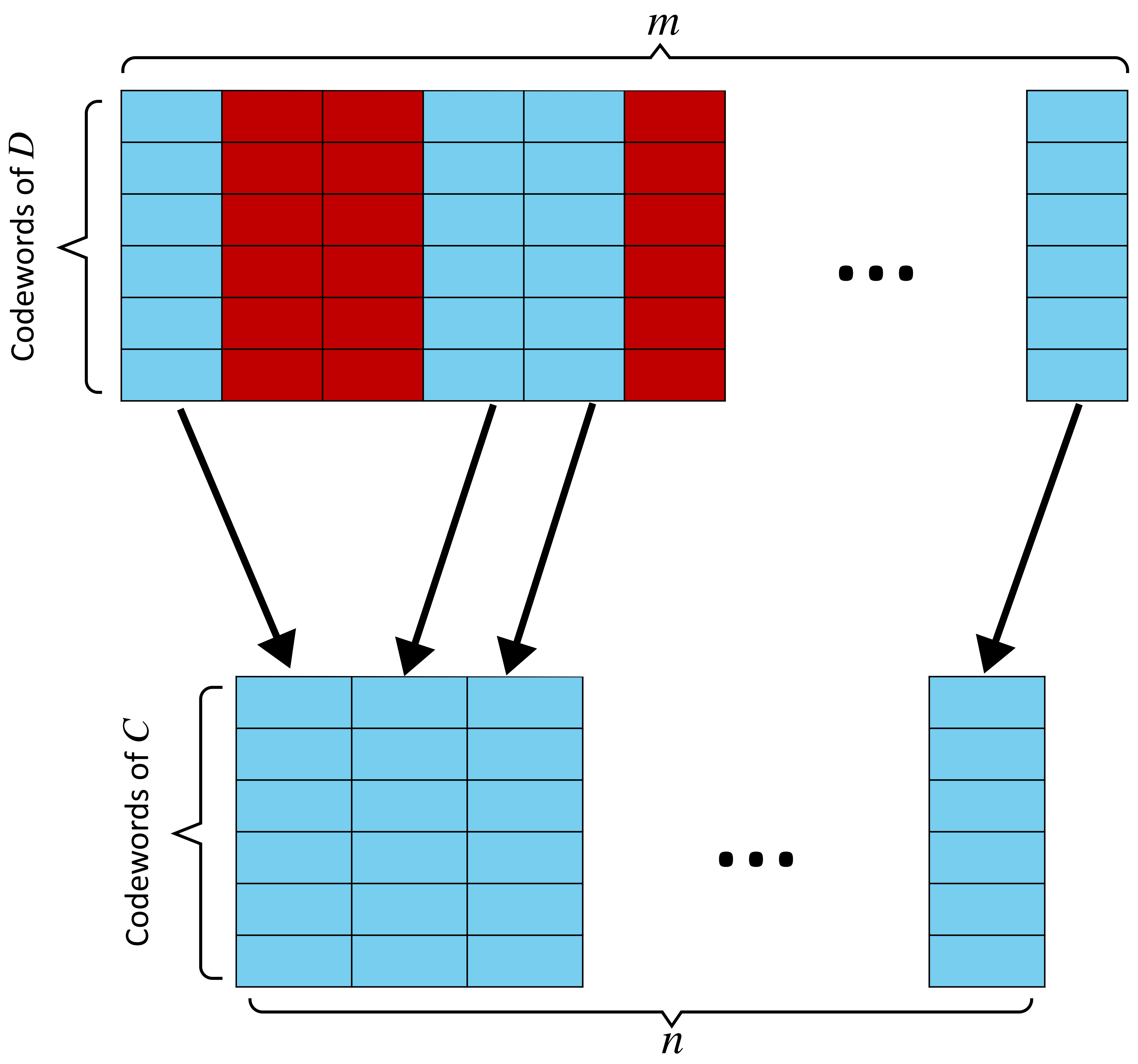

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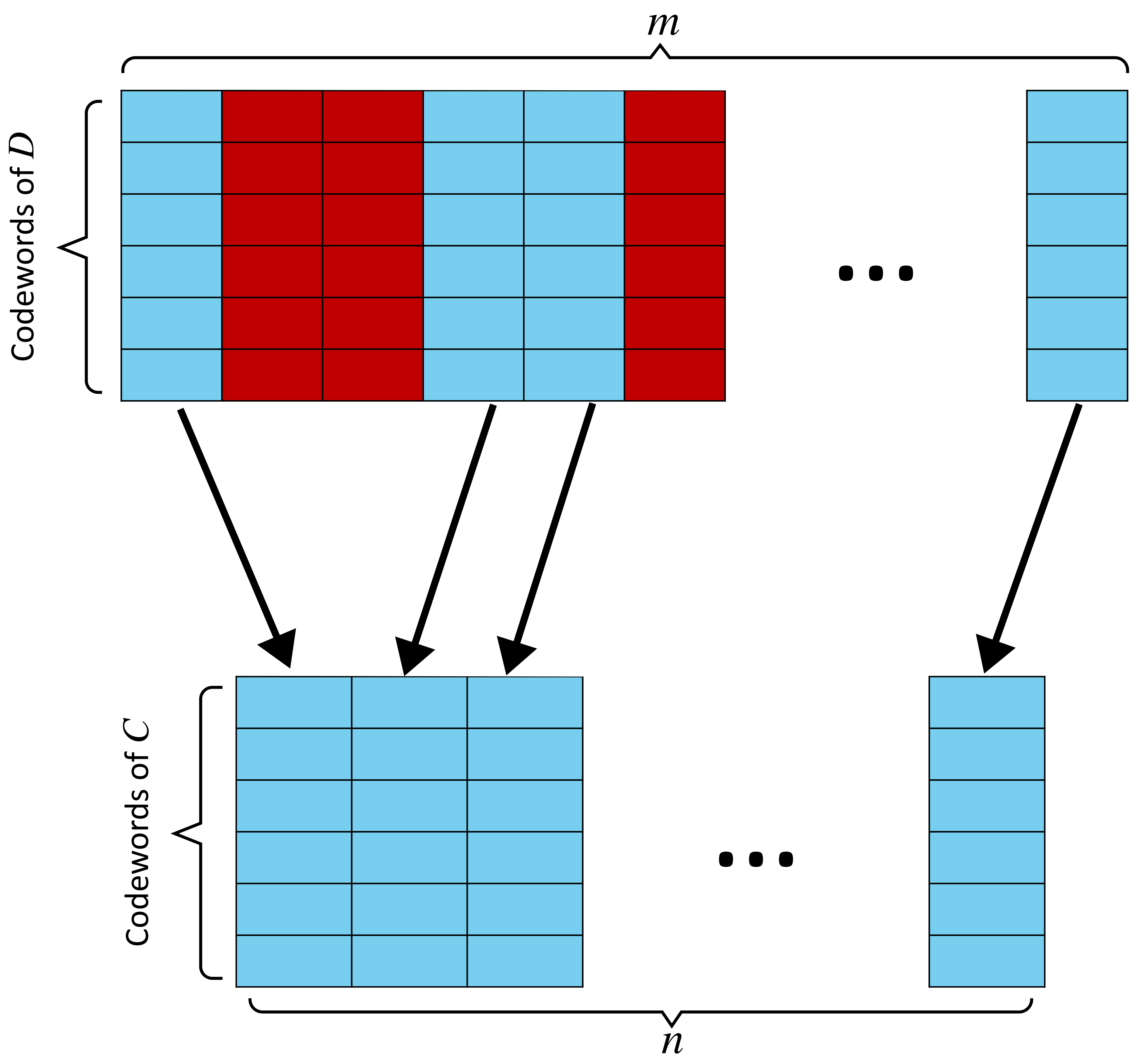

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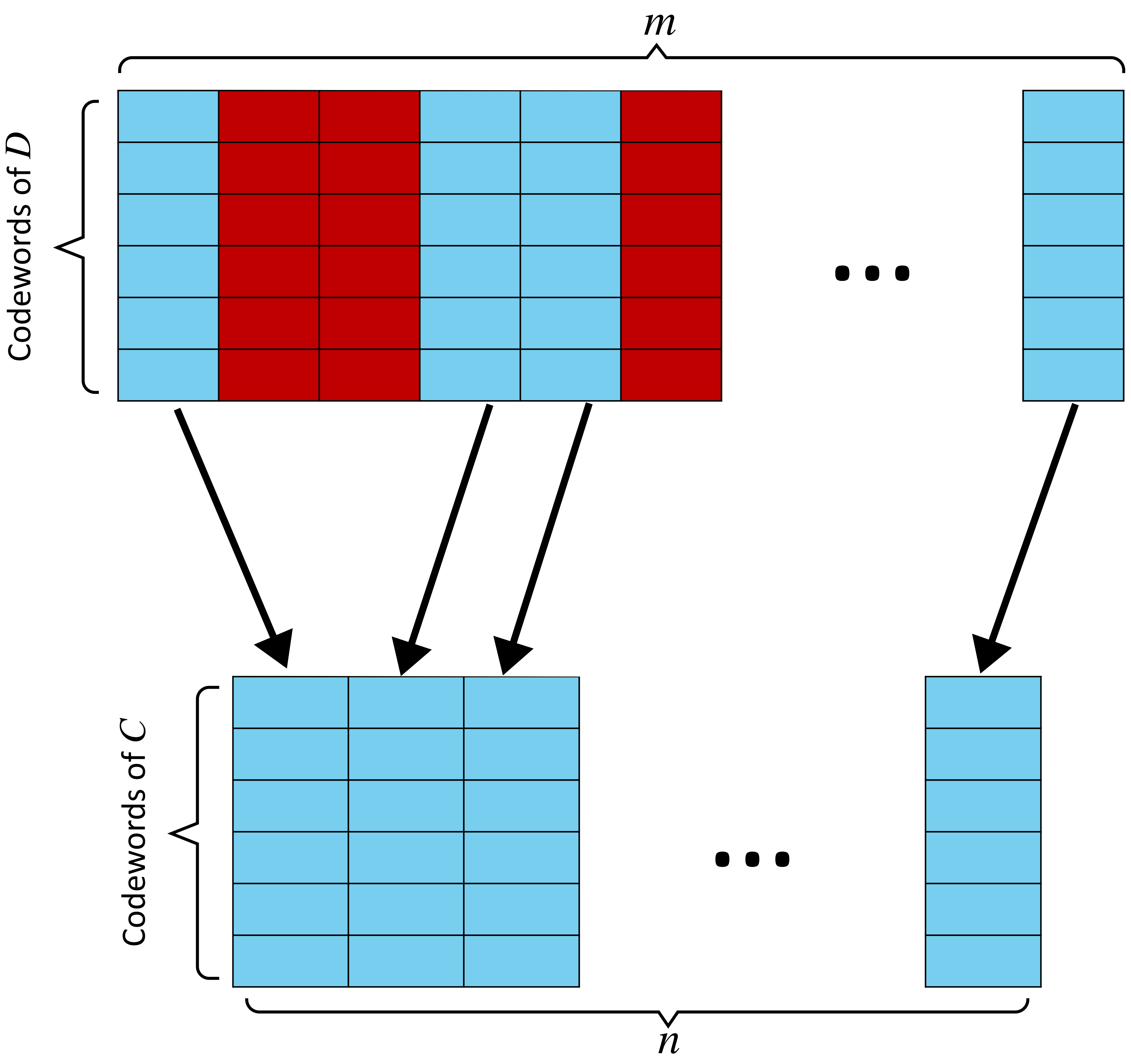

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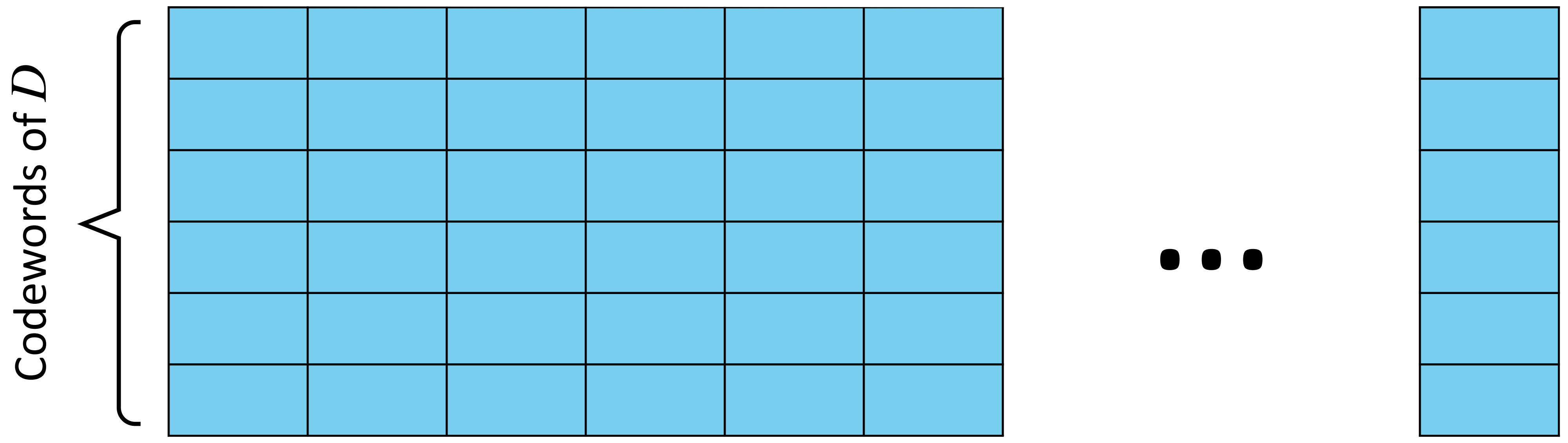

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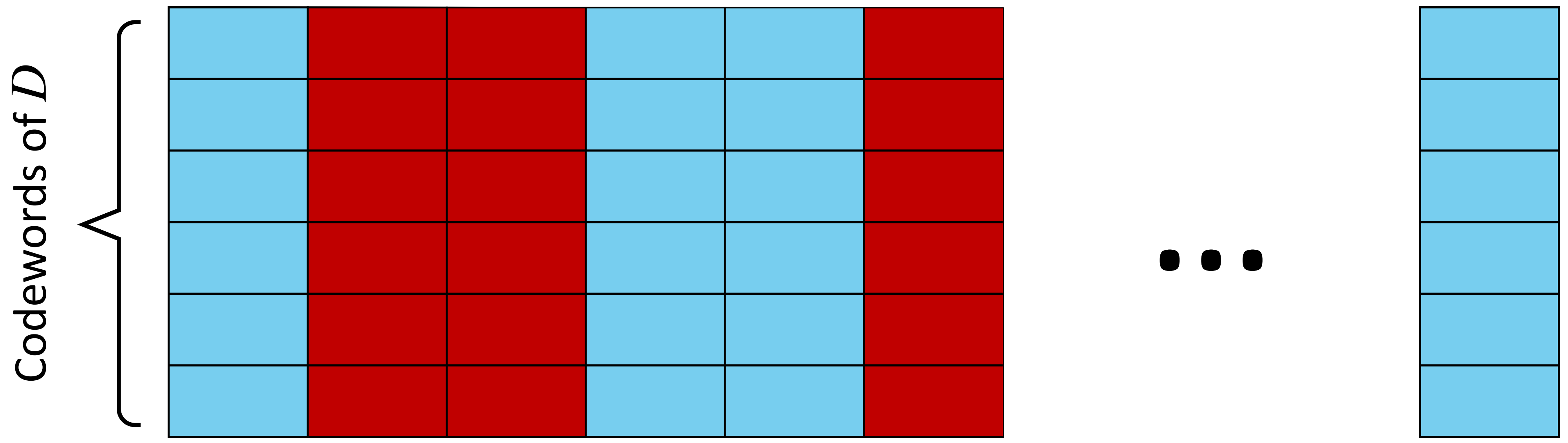

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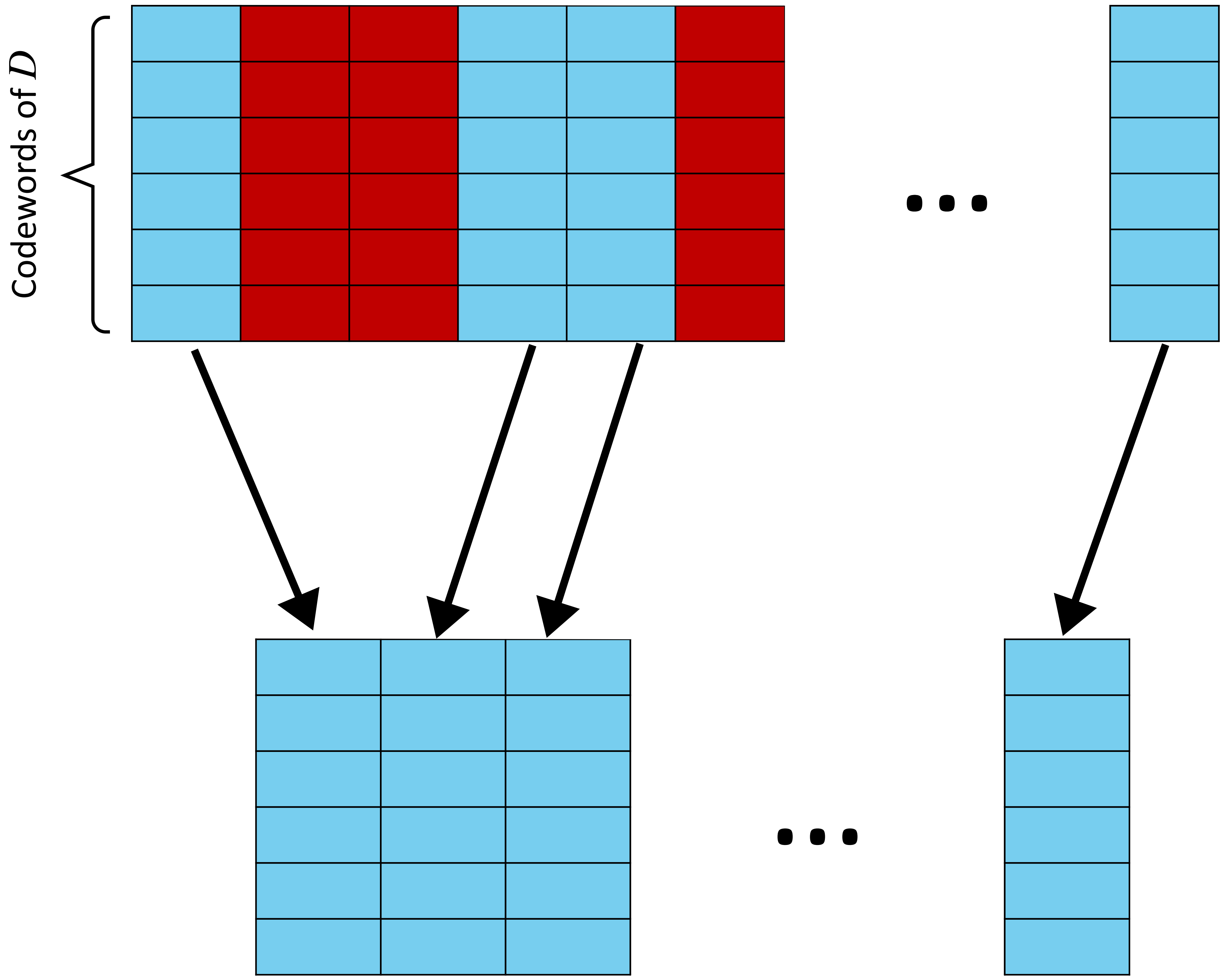

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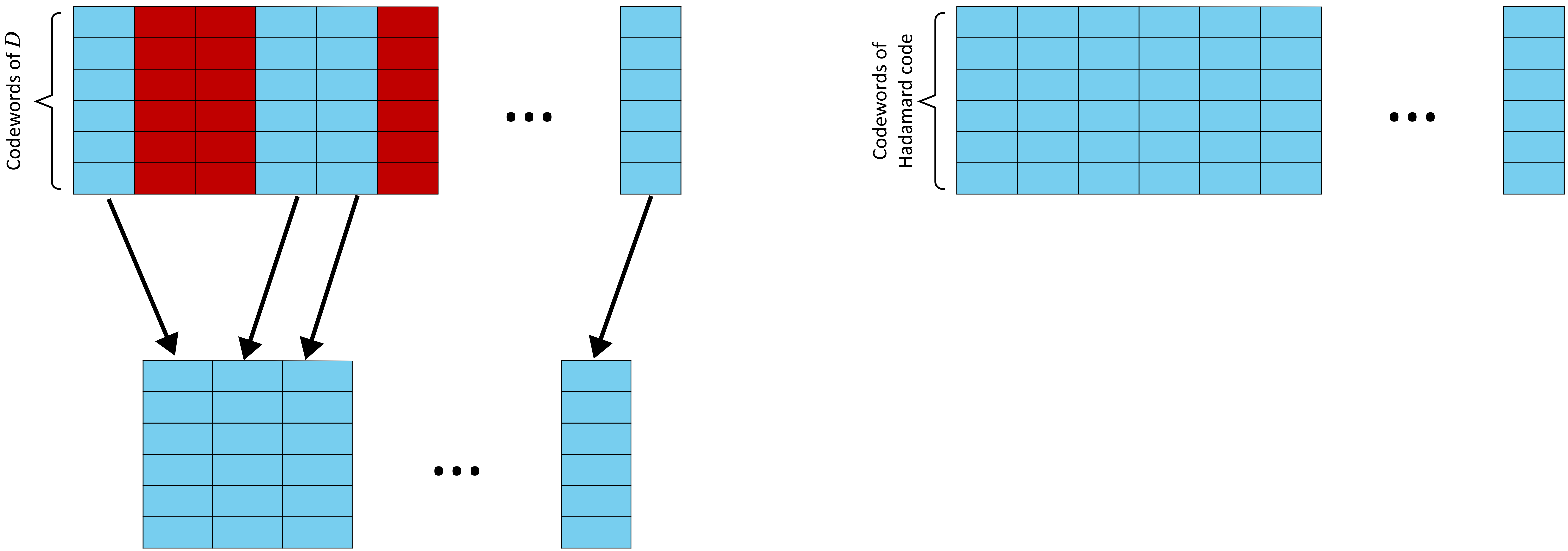

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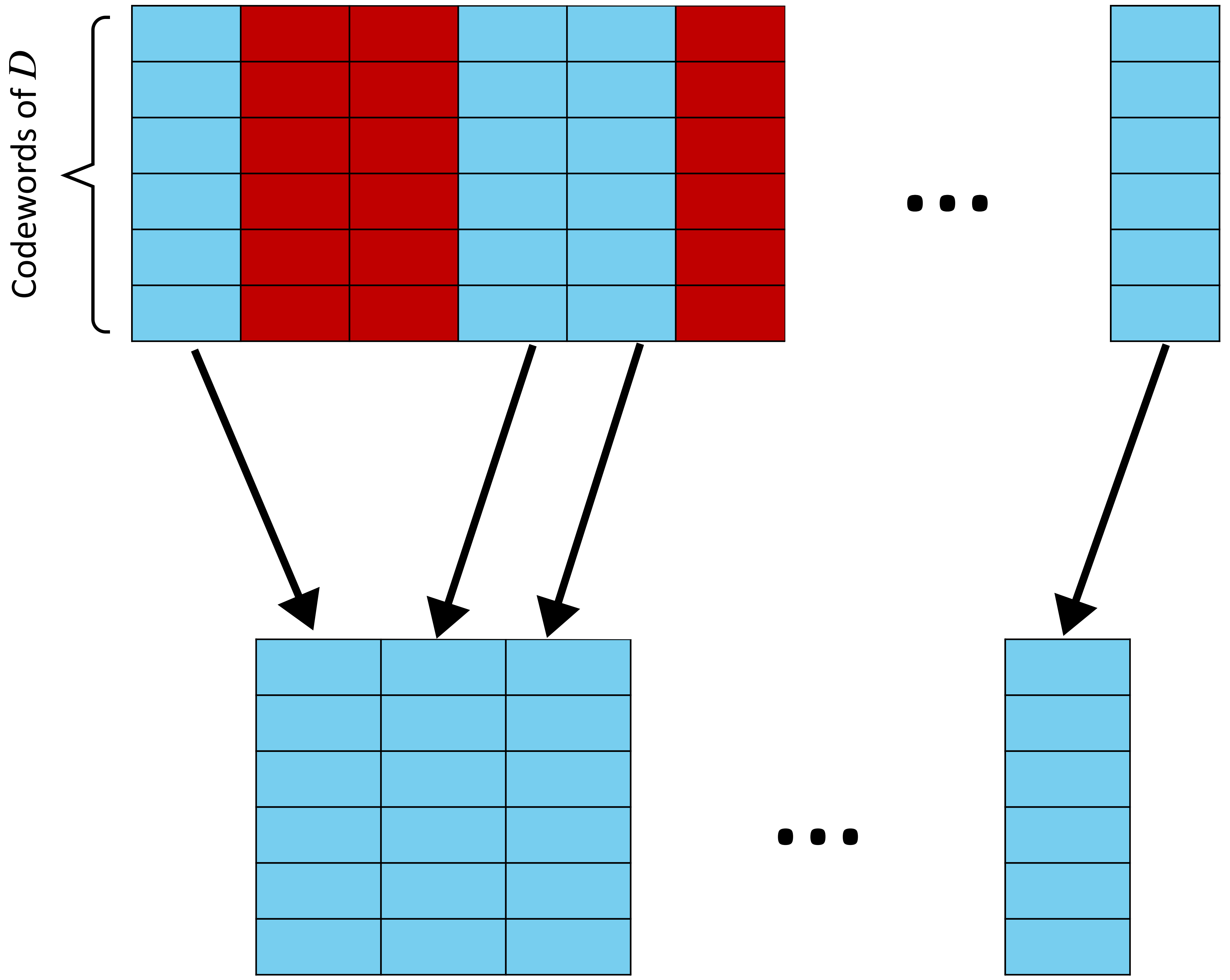

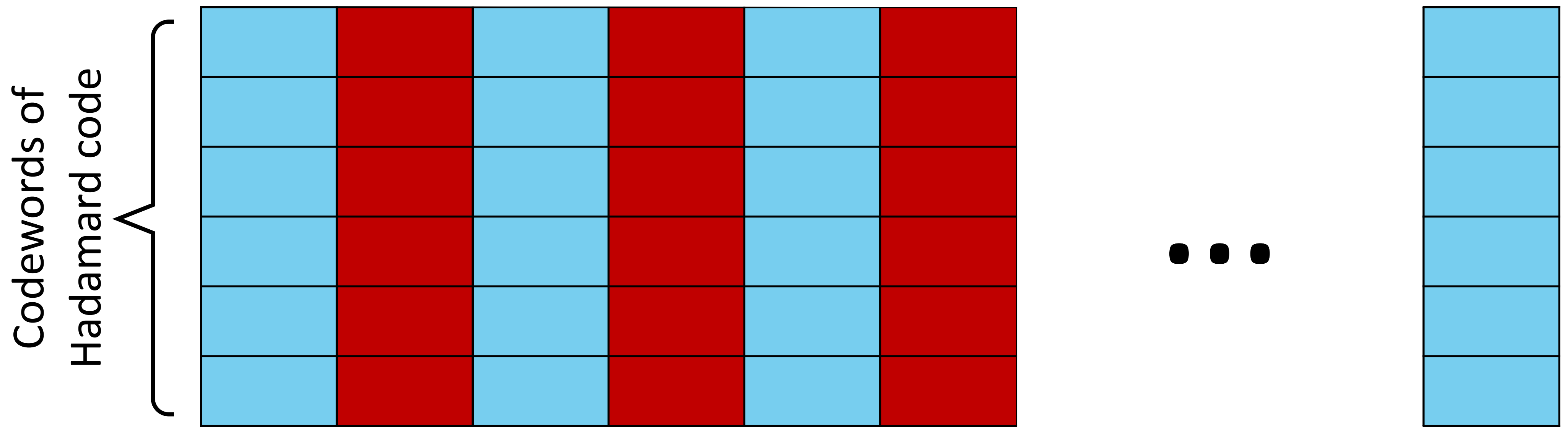

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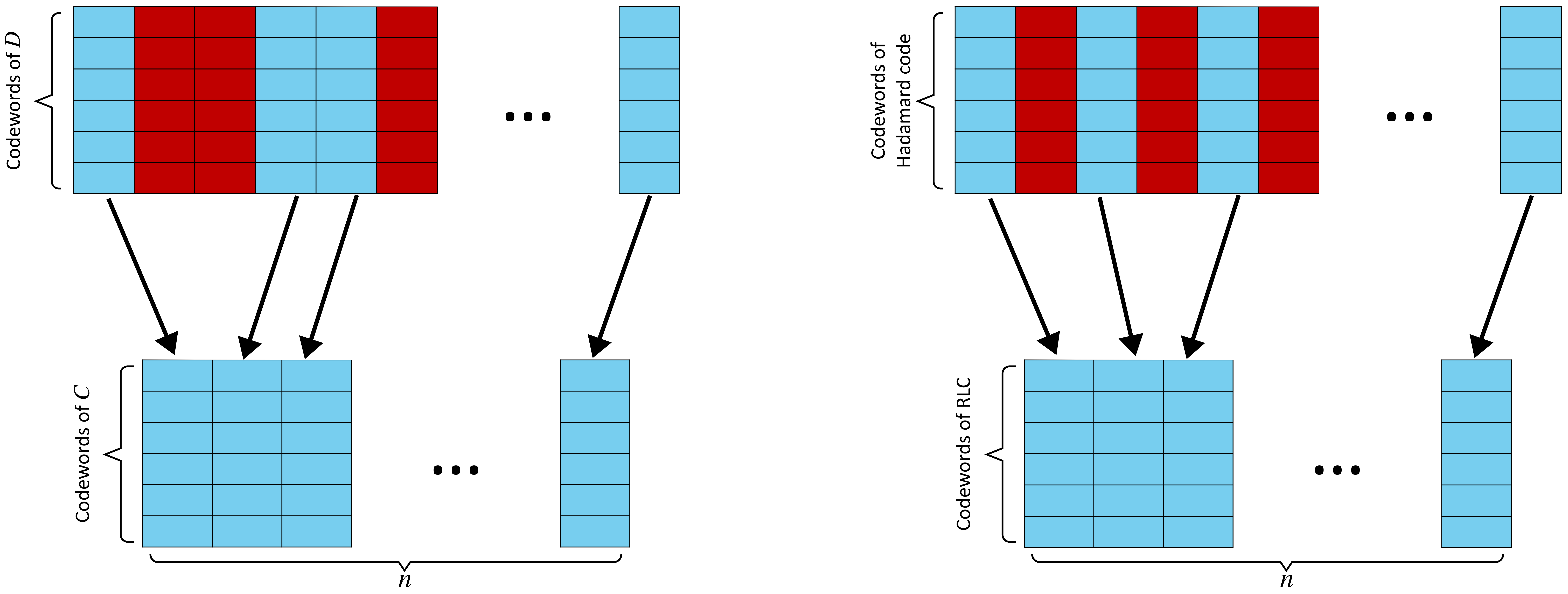

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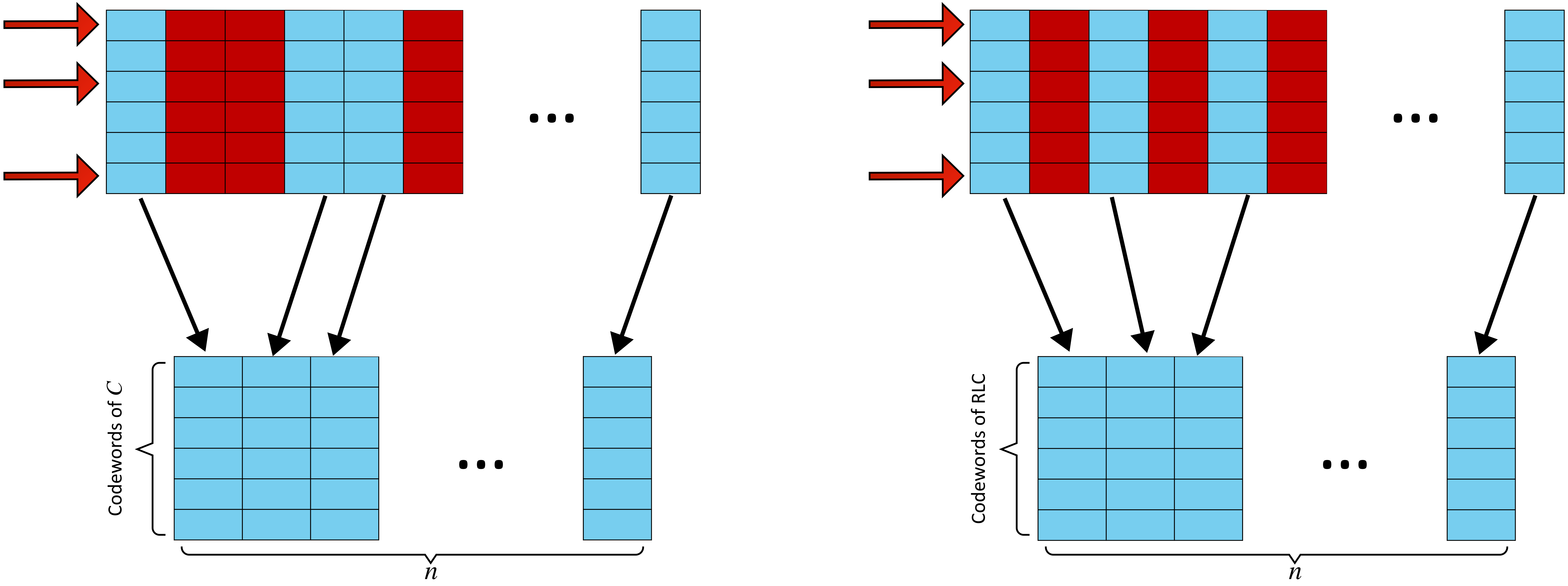

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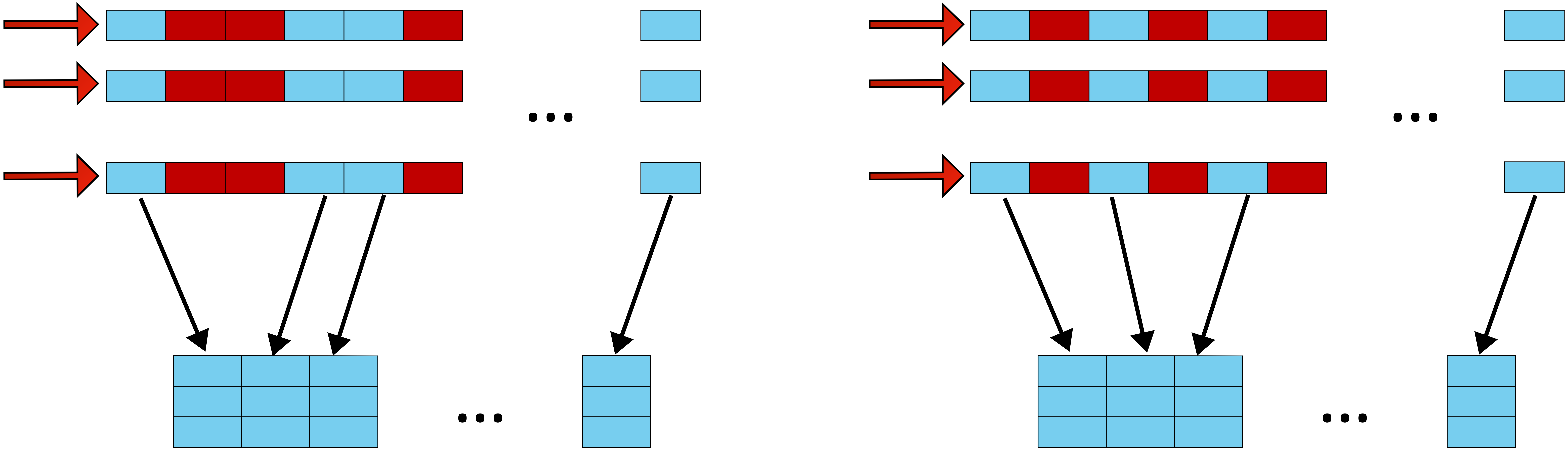

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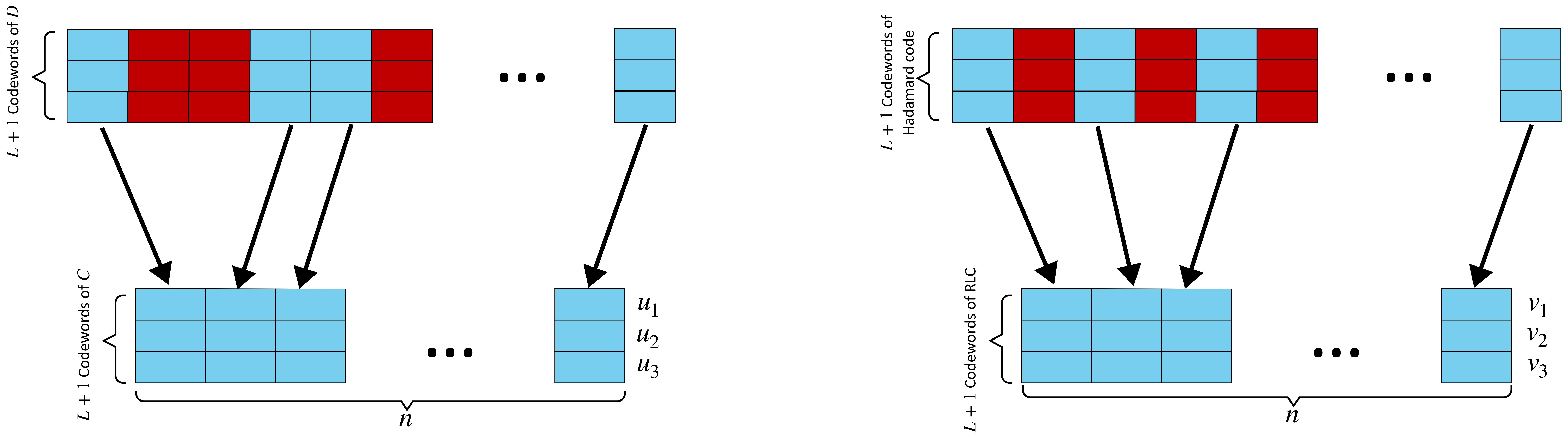

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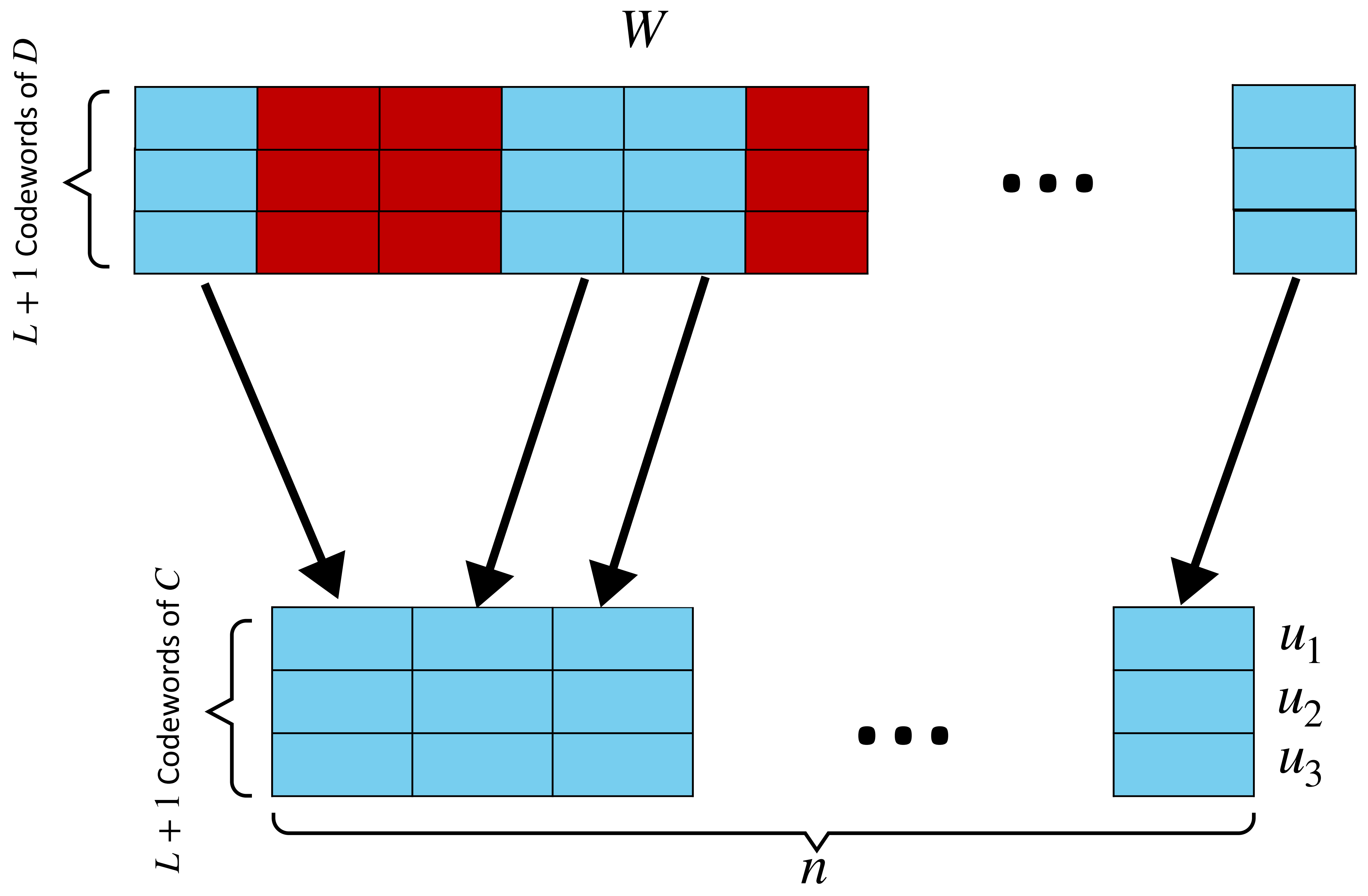

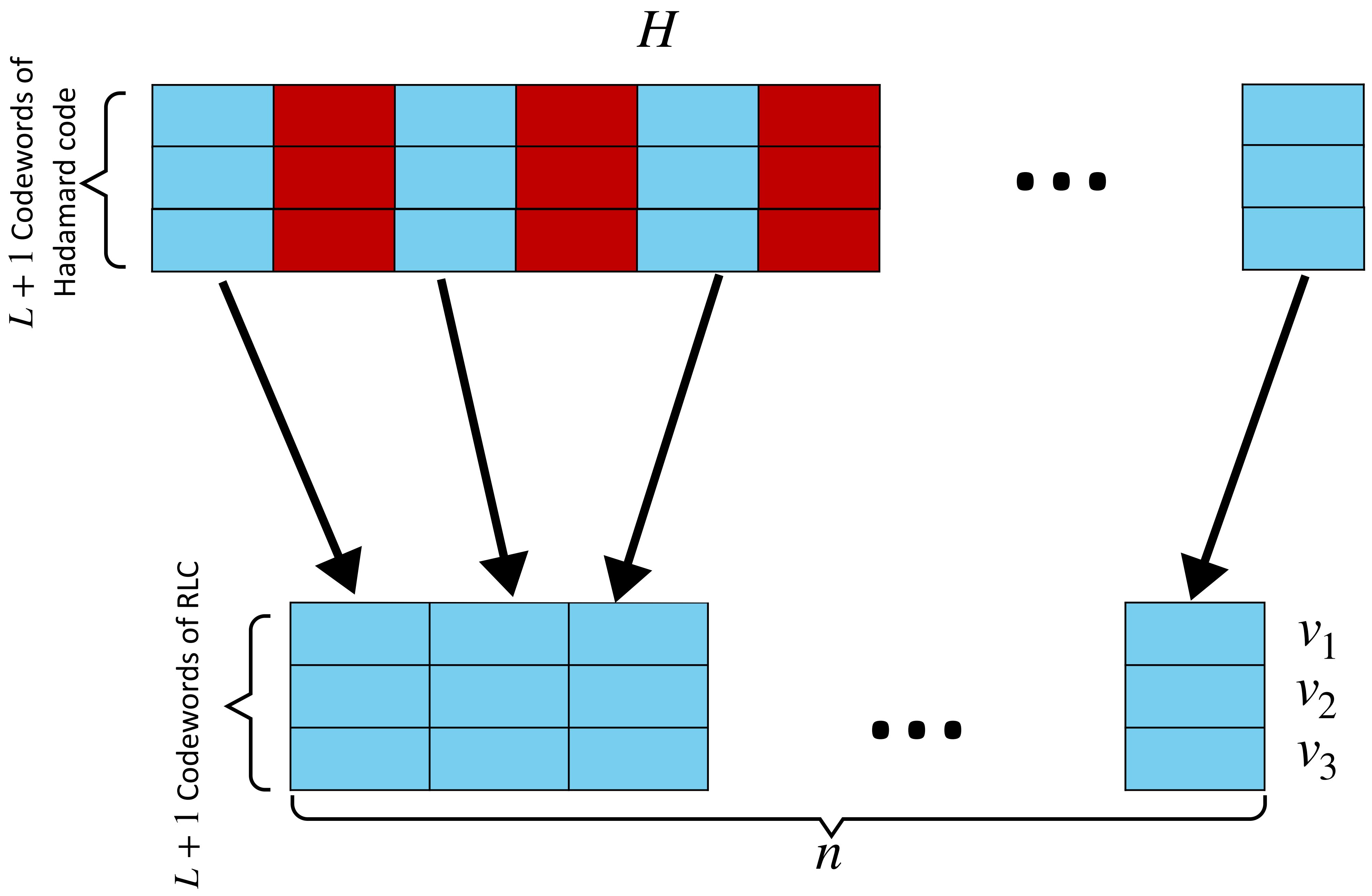

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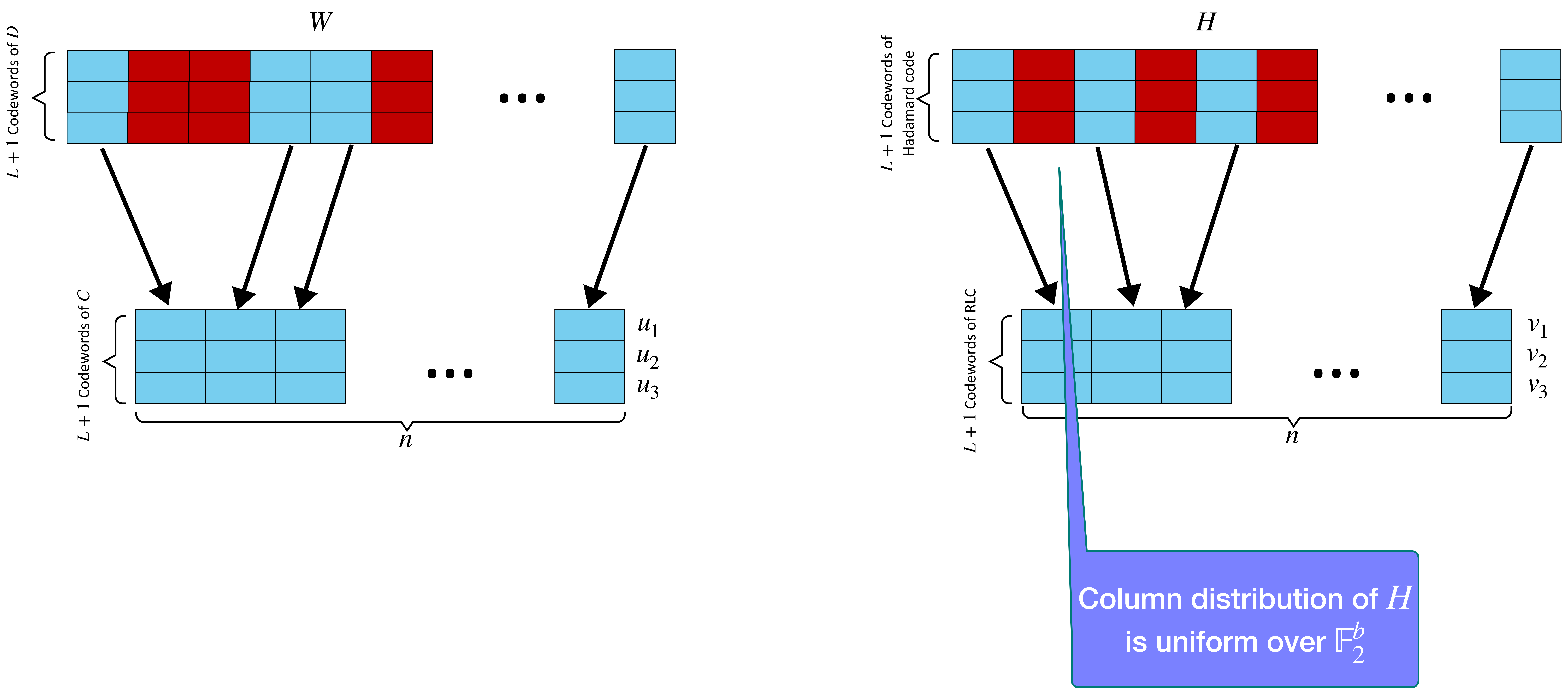

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