Fast algorithms for separable linear programs

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Joint work with:
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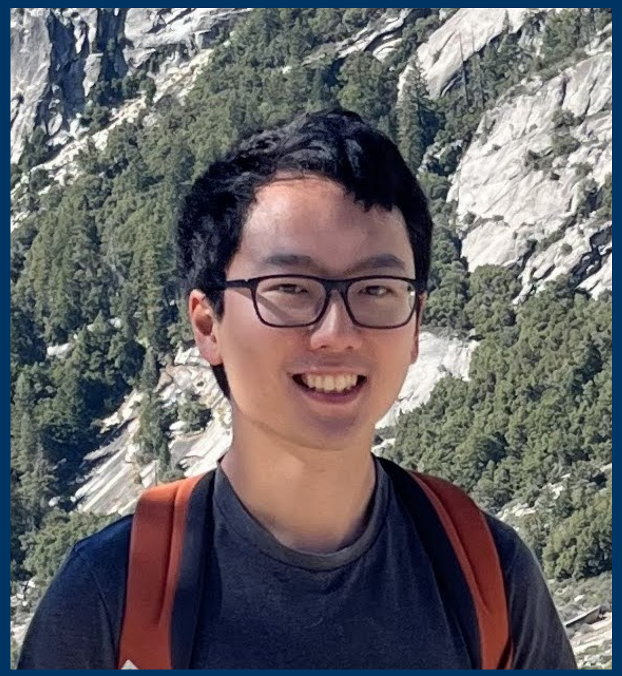
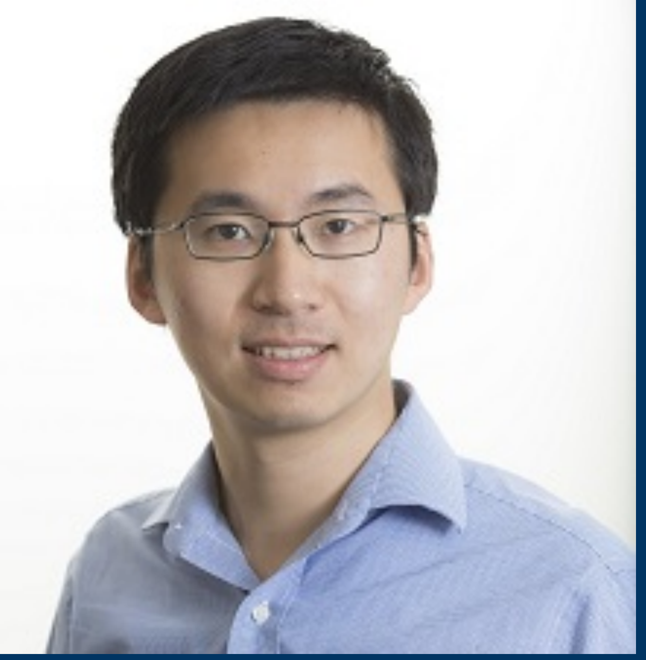
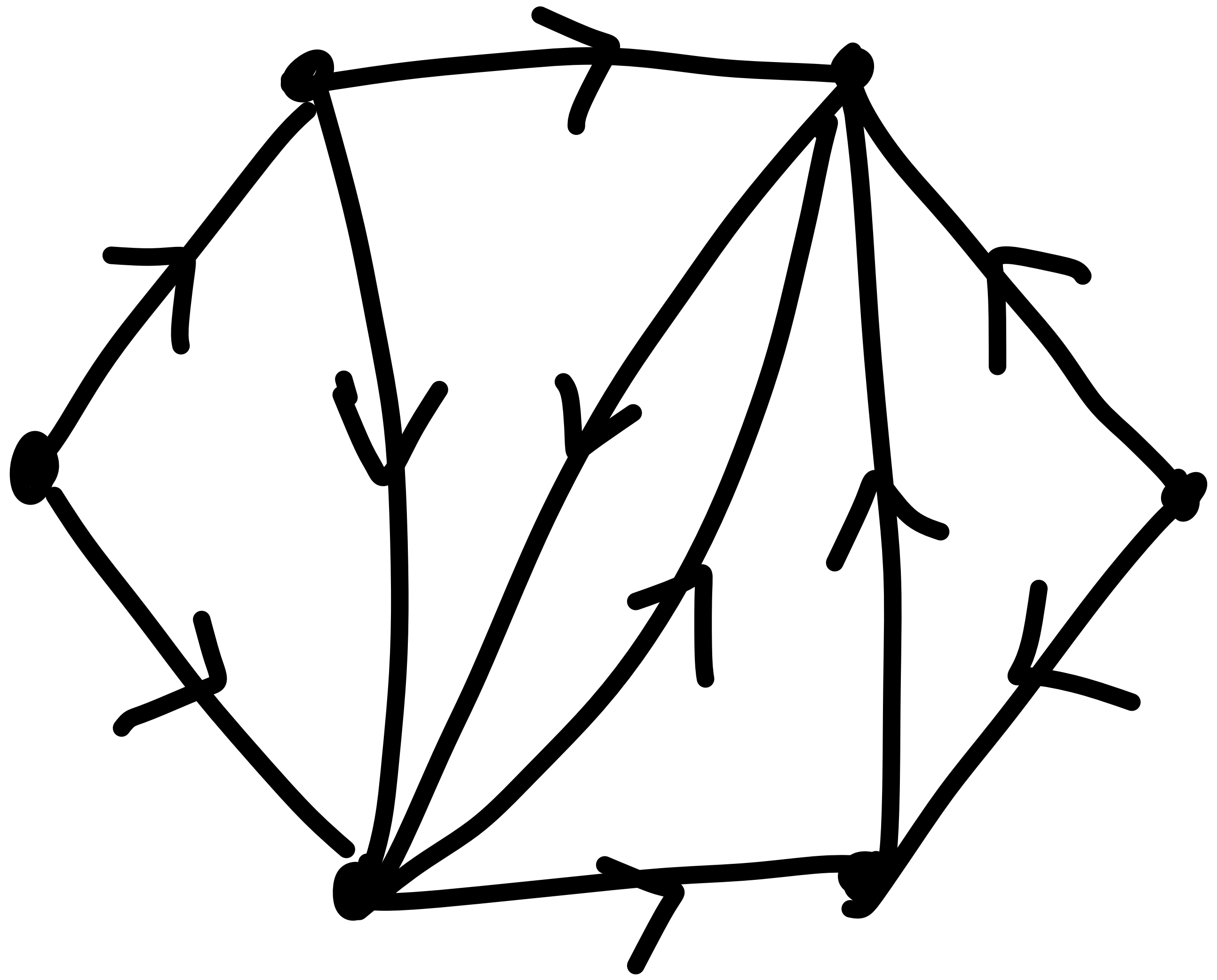
      

Min-cost flow

Input:

- graph $G = (V, E)$ with n vertices and m directed edges,



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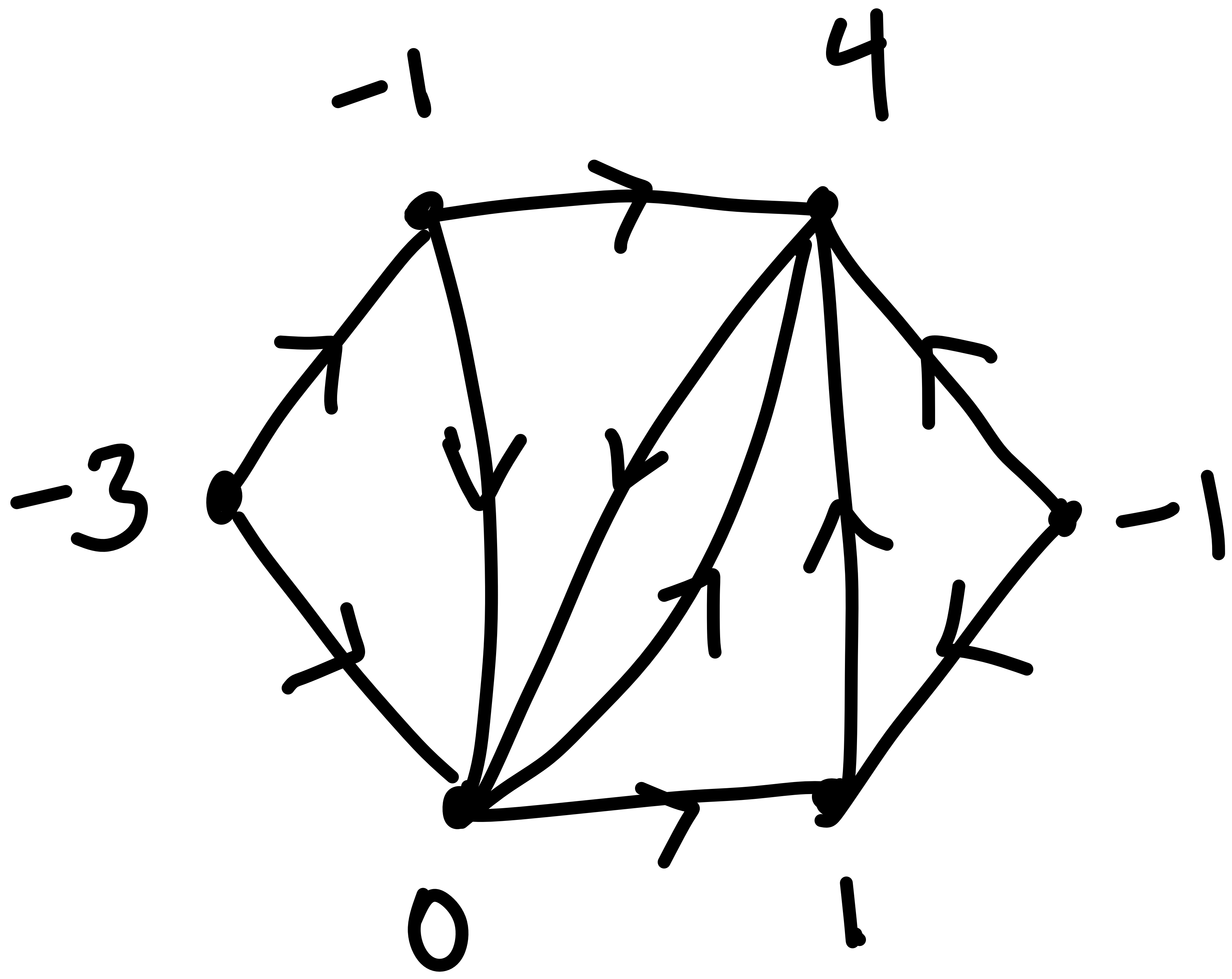
- vertex demands $\boldsymbol{d}$, such that

$$\sum_{v} d_v = 0.$$



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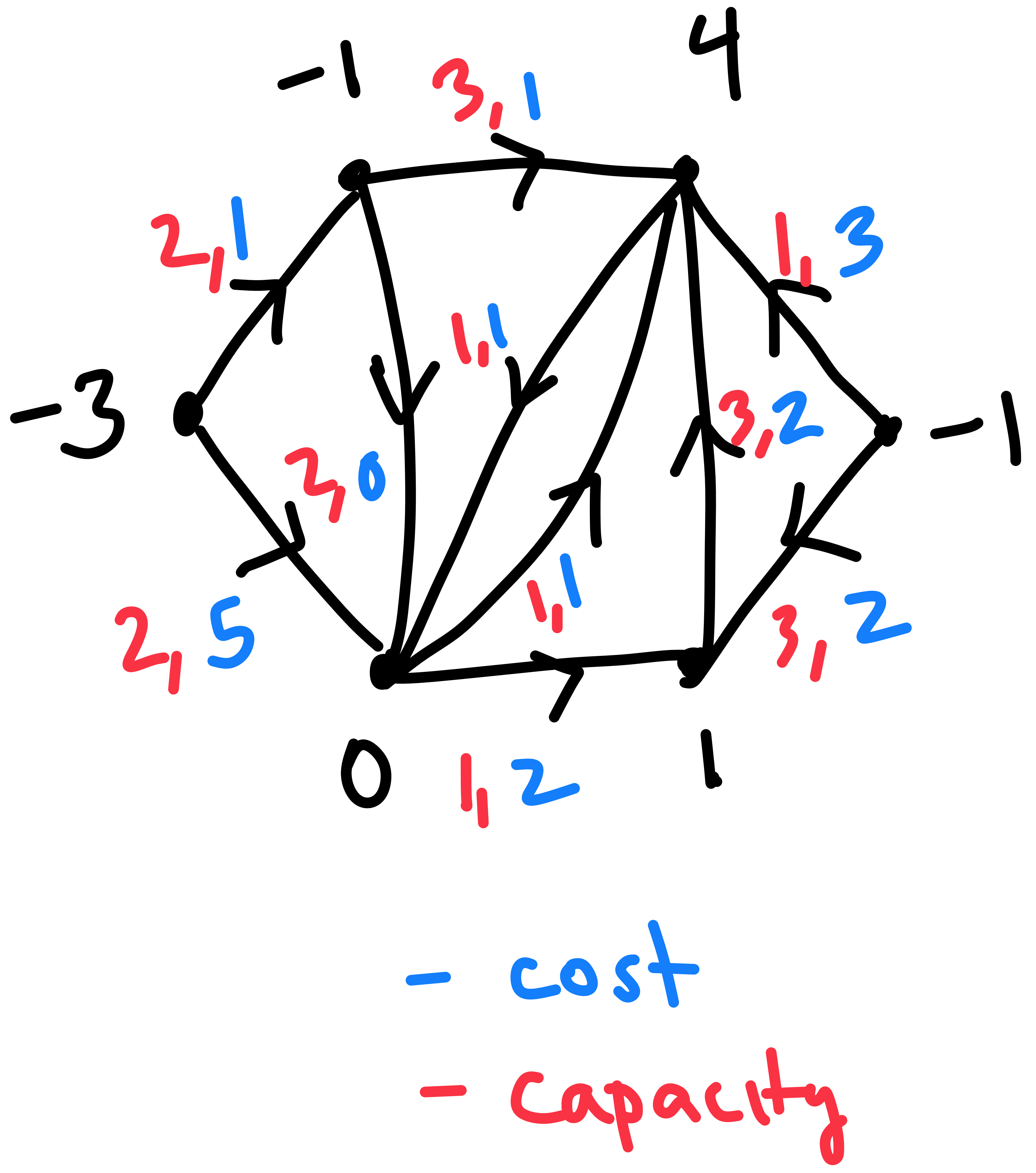

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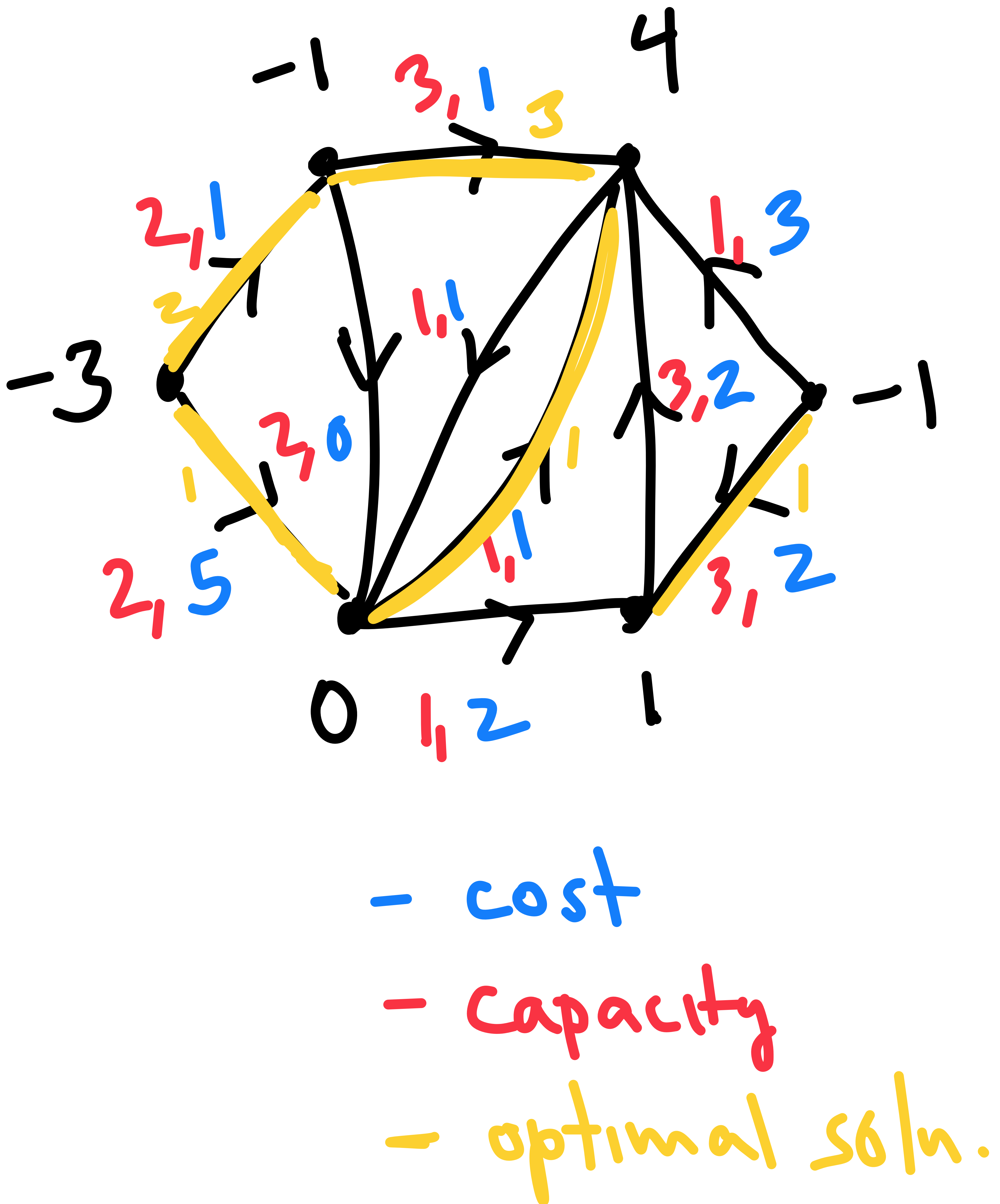
Output:

- Flow $\boldsymbol{f}$ minimizing $\boldsymbol{c}^\top \boldsymbol{f}$, and satisfying capacity constraints and demands.



General LP

$$
\begin{aligned}
\min\ & \boldsymbol{c}^\top \boldsymbol{x} \\
\text{s.t.}\quad & \mathbf{A}\boldsymbol{x} = \boldsymbol{b} \\
& \boldsymbol{x} \leq \boldsymbol{u} \\
& \boldsymbol{x} \geq \boldsymbol{\ell}
\end{aligned}
$$

where $\mathbf{A} \in \mathbb{R}^{n \times m}$.

Dual graph of general LP

$$
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Dual graph $G_{\mathbf{A}}$: n vertices, and each column of $\mathbf{A}$ is a hyper-edge (equiv. clique) on the set of vertices corresponding to rows with non-zero entries

Treewidth of LP

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Dual graph $G_{\mathbf{A}}$: n vertices, and each column of $\mathbf{A}$ is a hyper-edge (equiv. clique) on the set of vertices corresponding to rows with non-zero entries

Define treewidth of the LP to be the treewidth of $G_{\mathbf{A}}$.

Separable LP

$$\begin{aligned} & \min \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} \quad & \mathbf{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \leq \boldsymbol{u} \\ & \boldsymbol{x} \geq \boldsymbol{\ell} \end{aligned}$$

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Say LP is separable if $G_{\mathbf{A}}$ is separable.

Min-cost flow LP

$$
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& \min \boldsymbol{c}^\top \boldsymbol{f} \\
\text{s.t.} \quad & \mathbf{B}^\top \boldsymbol{f} = \boldsymbol{d} \\
& \boldsymbol{f} \le \boldsymbol{u} \\
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\end{aligned}
$$

where $\mathbf{B}^\top \in \mathbb{R}^{n \times m}$ is the transpose of the adjacency matrix of input graph G.

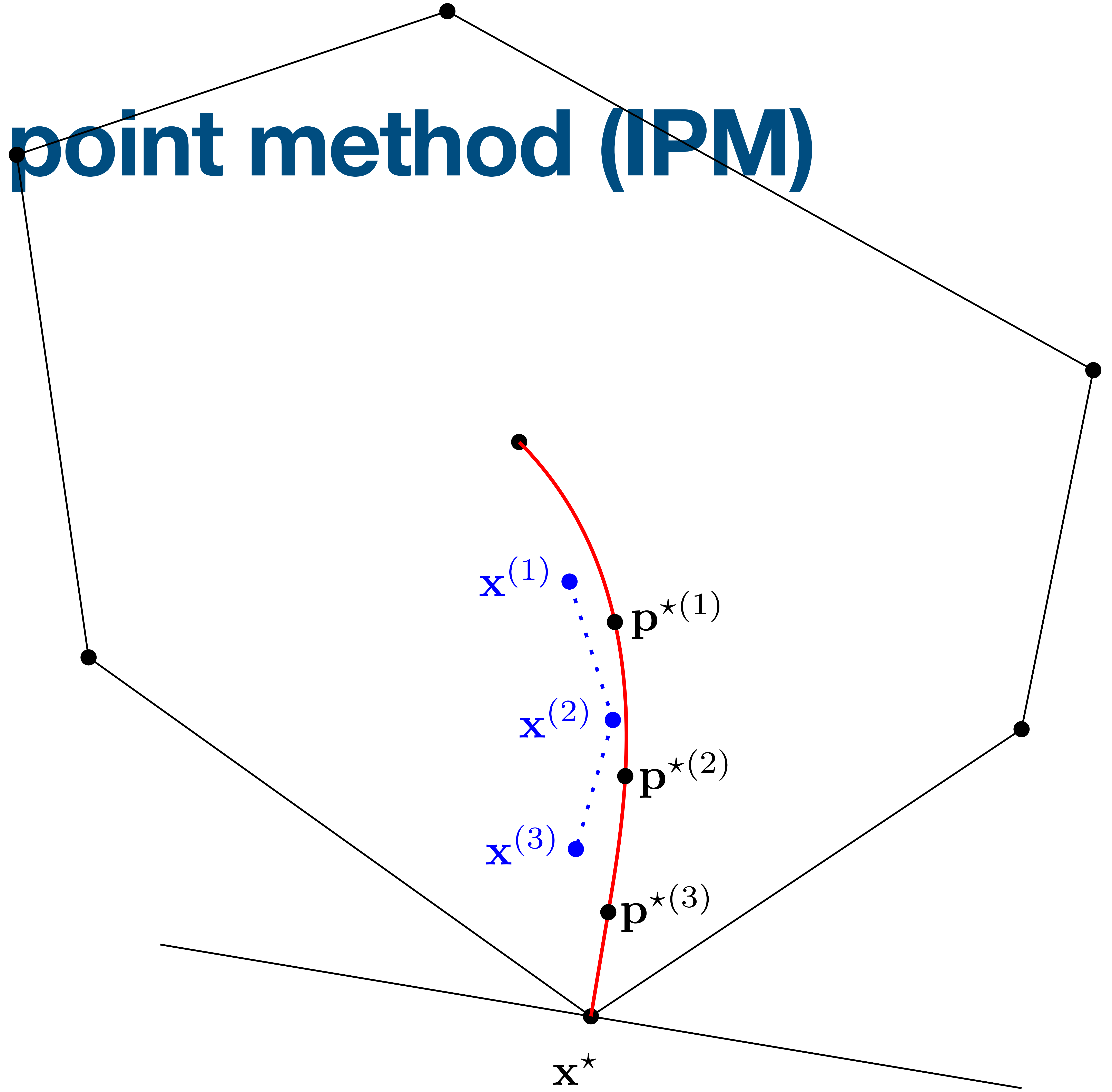
Current state of the art

Problem setting	Time	Reference
min-cost flow, strongly polytime	$O(mn + m^{31/16})$	[Orlin13]
min-cost flow, weakly polytime	$O(m^{1+o(1)})$	[CLKPPS22]
min-cost flow, planar graphs	$\tilde{O}(m)$	[**D**GGLPSY22]
min-cost flow, treewidth *t* graphs	$\tilde{O}(m\sqrt{t})$	[**D**Y23+]
k-commodity flow	$\tilde{O}(k^{2.5}\sqrt{m}n^{\omega-1/2})$	[BZ23]
k-commodity flow, planar graphs	$\tilde{O}(k^{2.5}n^{1.5})$	[**D**GLSY24]
general LPs	$\tilde{O}(m^{\omega})$	[CLS19]
LPs with treewidth *t*	$\tilde{O}(mt^{(\omega+1)/2})$	[GS22,**D**GLSY24]
α -separable LPs	$\tilde{O}(m^{1/2+2\alpha})$	[**D**GLSY24]

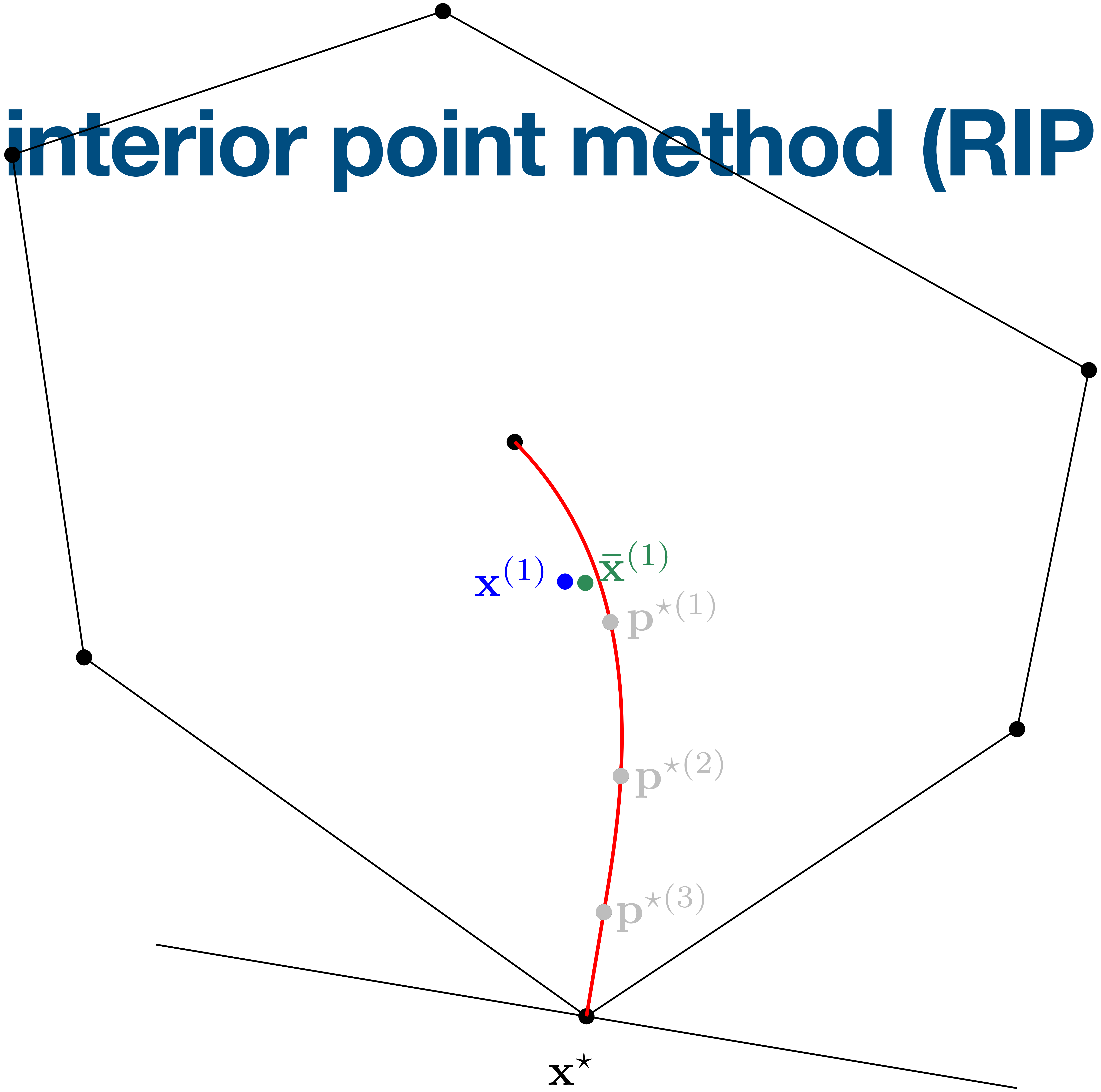
m variables; polynomially bounded entries and relative error

Interior point method for LPs

Interior point method (IPM)



Robust interior point method (RIPM)



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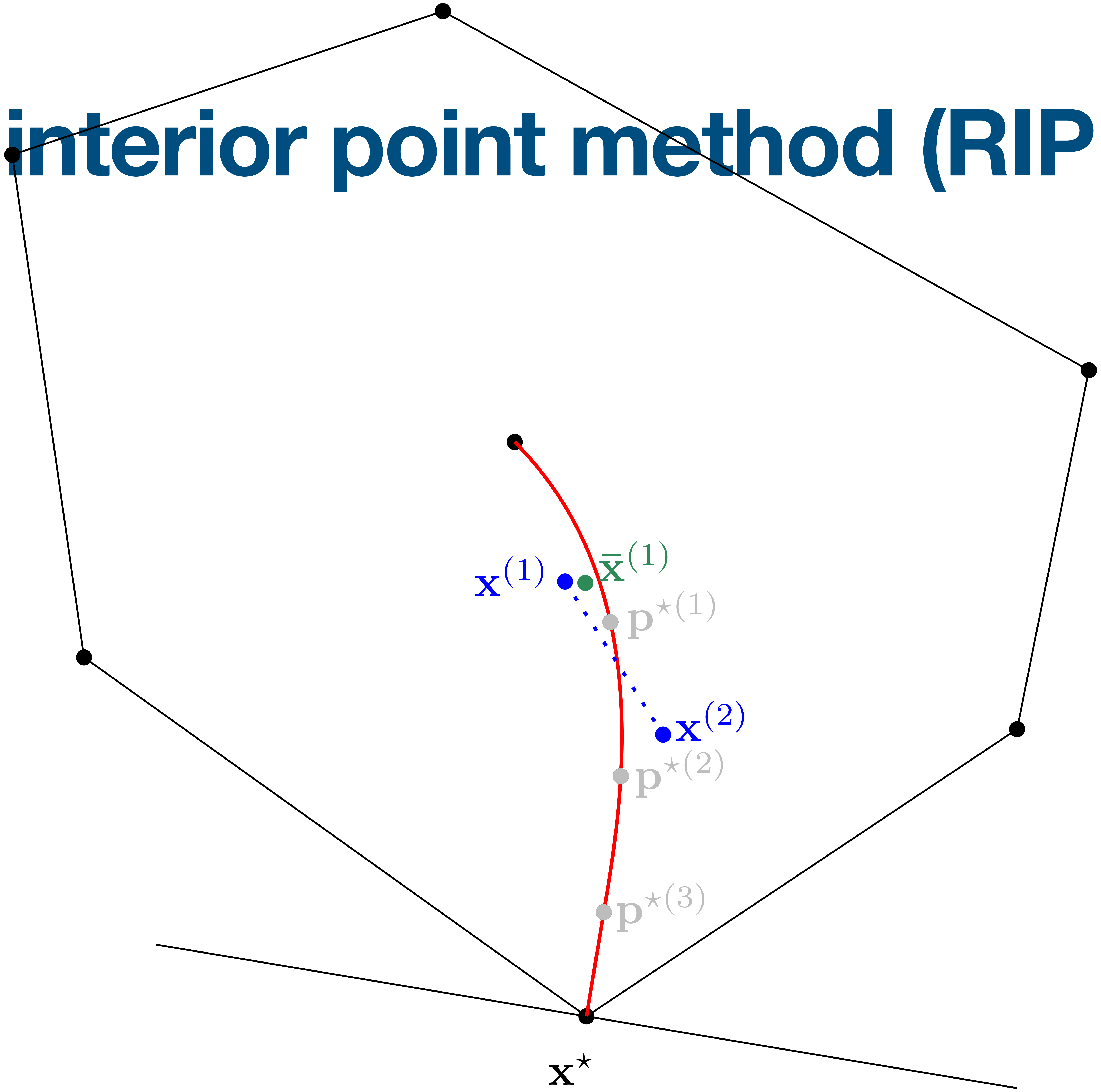

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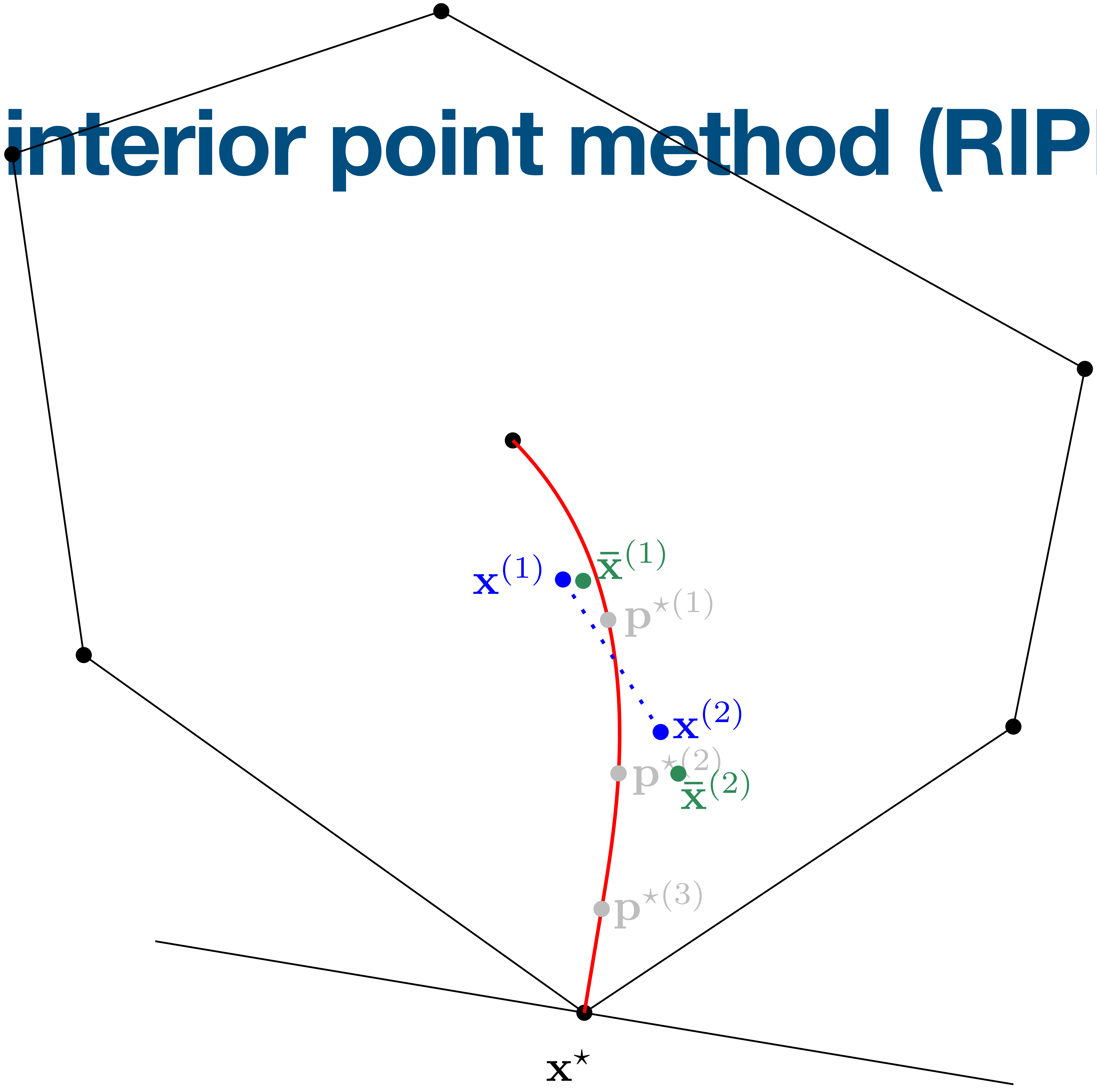

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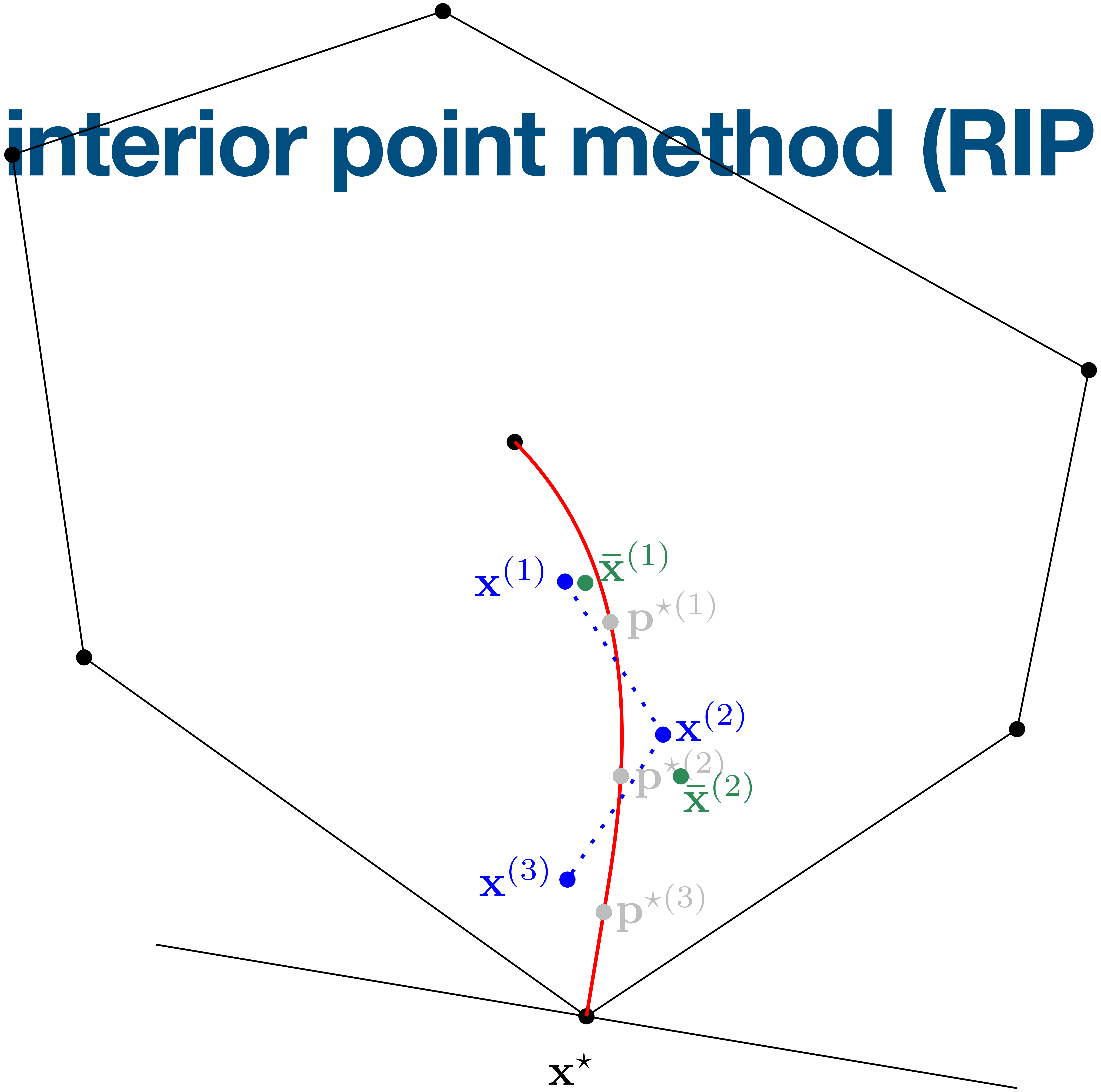

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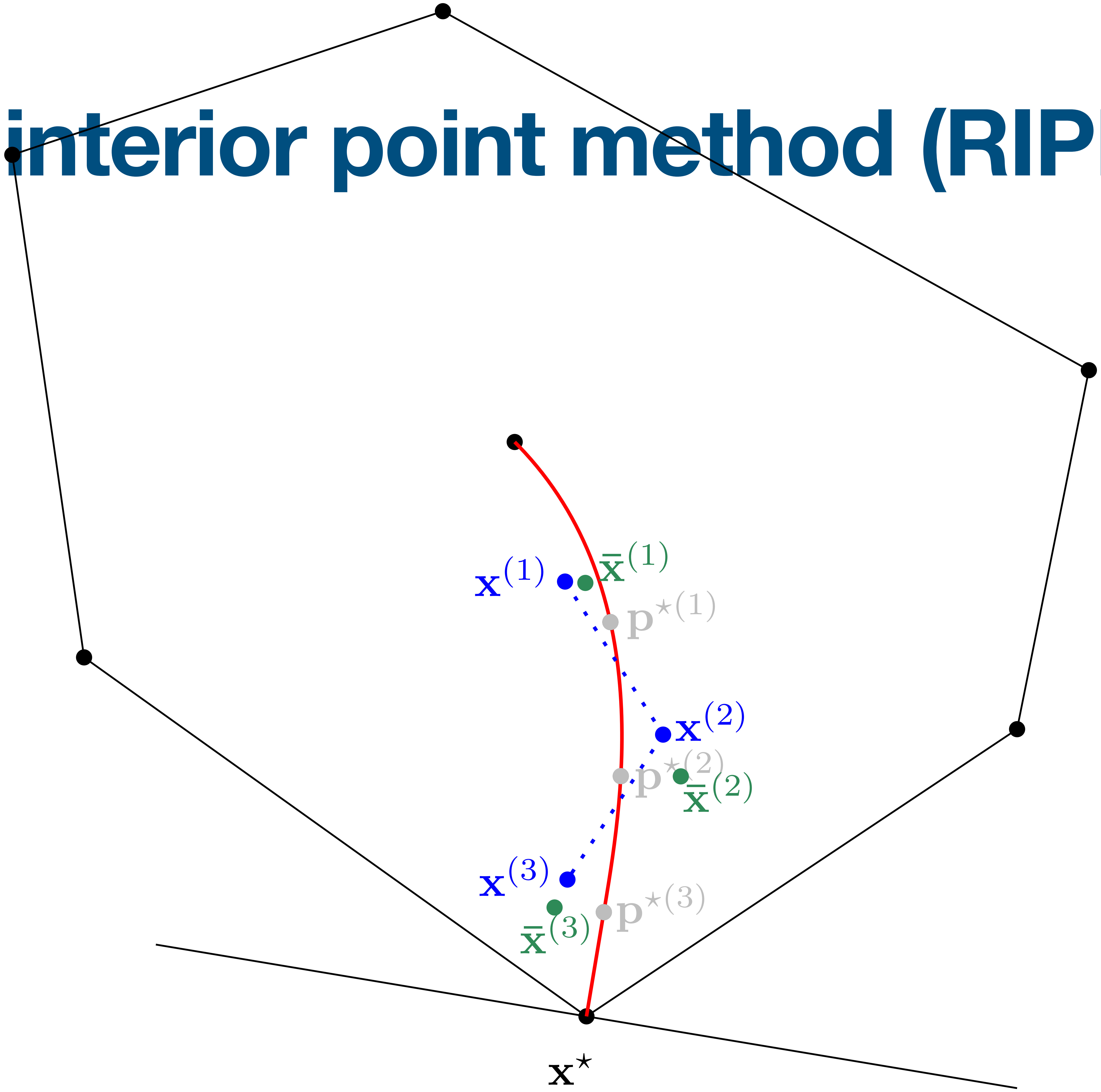
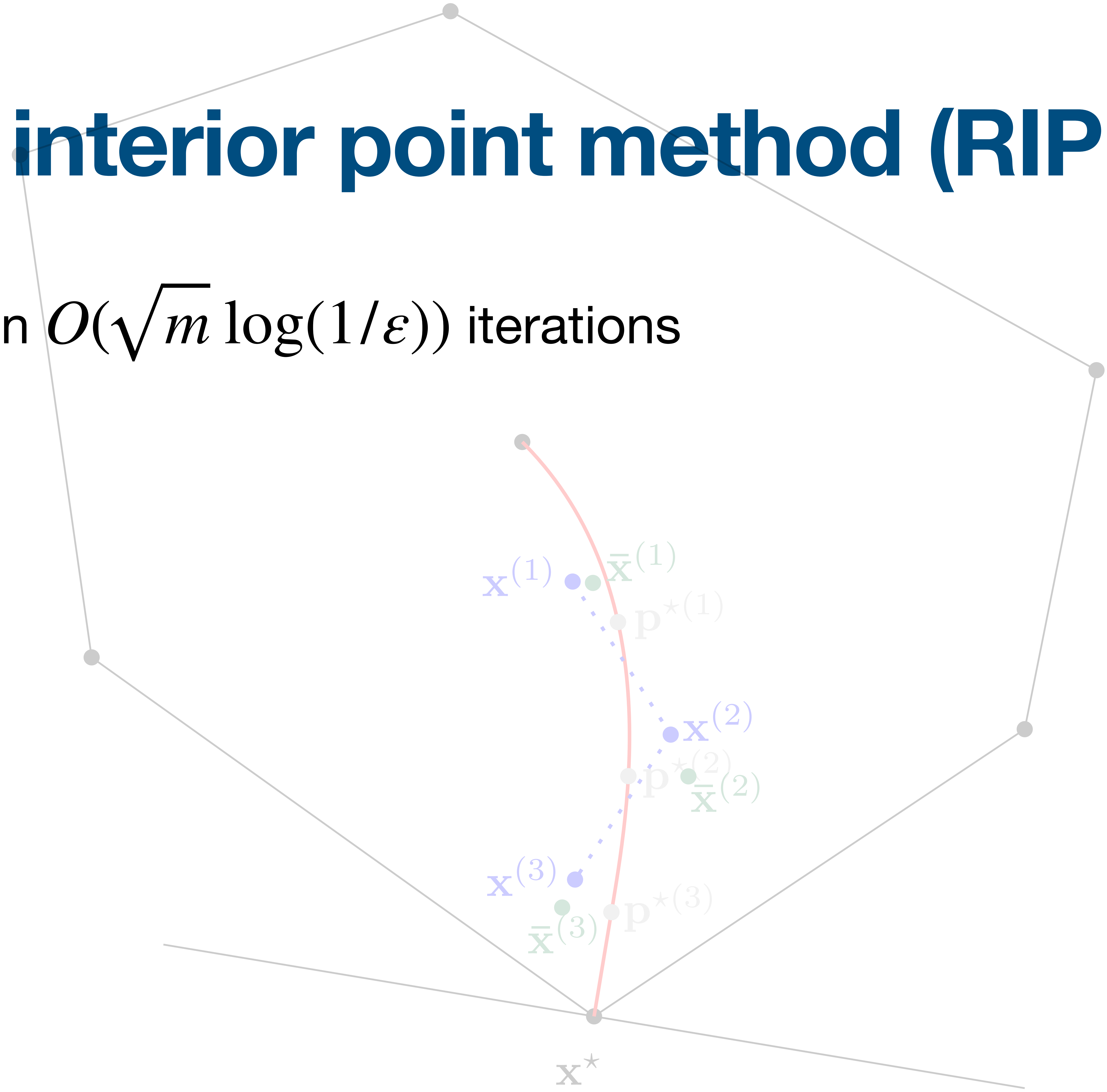


Robust interior point method (RIPM)

- converges in $O(\sqrt{m}\log(1/\varepsilon))$ iterations



Robust interior point method (RIPM)

- converges in $O(\sqrt{m}\log(1/\varepsilon))$ iterations
- guarantee: steps have bounded 2-norm

RIPM for LPs reduces to 2 problems…

1) Dynamic algorithm to maintain the current solution $\boldsymbol{x}$
   - at every step, update $\boldsymbol{x} \leftarrow \boldsymbol{x} + \delta_{\boldsymbol{x}}$

2) Dynamic algorithm to maintain coordinate-wise approximation $\overline{\boldsymbol{x}}$

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$$\mathbf{P}_{\boldsymbol{w}} \approx \mathbf{W}^{1/2}\mathbf{A}^\top(\mathbf{A}\mathbf{W}\mathbf{A}^\top)^{-1}\mathbf{A}\mathbf{W}^{1/2}$$

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"multiscale representation" [DLY21]

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tree operator, inverse tree operator (function of $\boldsymbol{w}$)

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efficiency depends on Δ, ∇

Defining the tree operators

Balanced separators

Given graph $G = (V, E)$, $b \in (0,1)$, and a weight assignment $\boldsymbol{p}$ to the vertices.

A vertex set S is a (b-)balanced separator of G (with respect to $\boldsymbol{p}$) if $G \backslash S$ gives disconnected components A, B, both containing at most b-fraction of the total weight.

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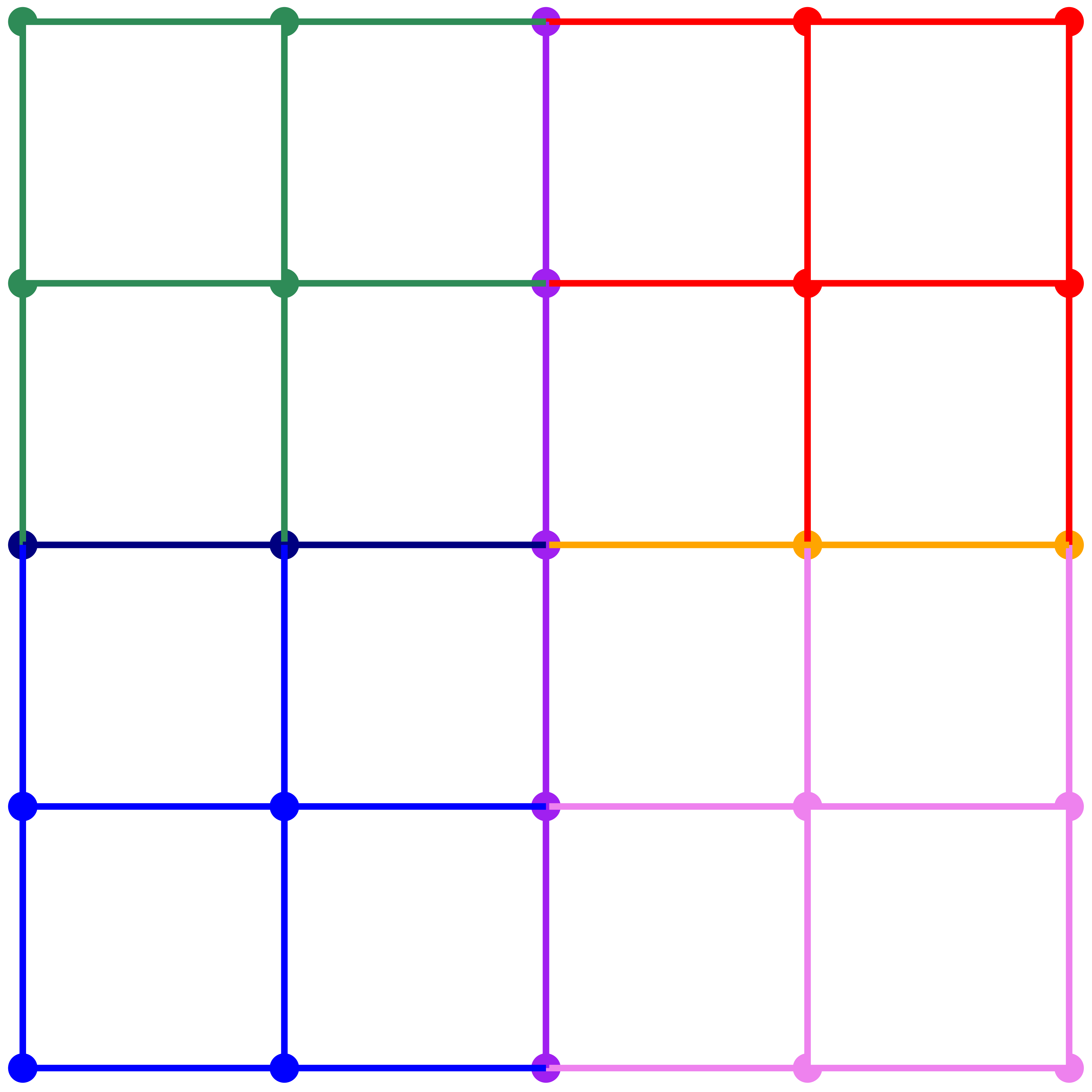
- Treewidth t graphs have a size t separators

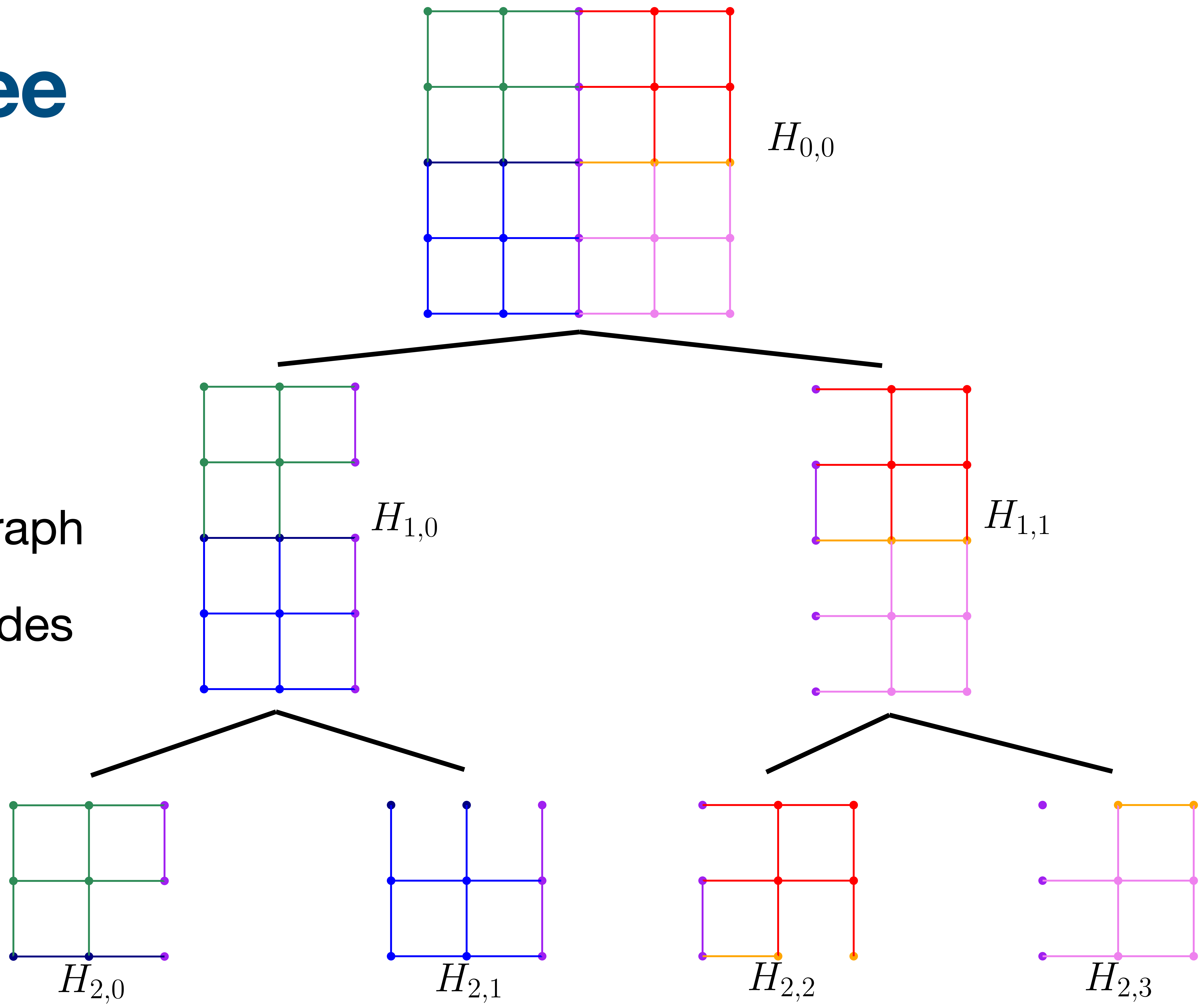
Use balanced separators to decompose graph

- planar graph G on n vertices
- recursively use balanced separator
- decompose until there is no more non-trivial balanced separator



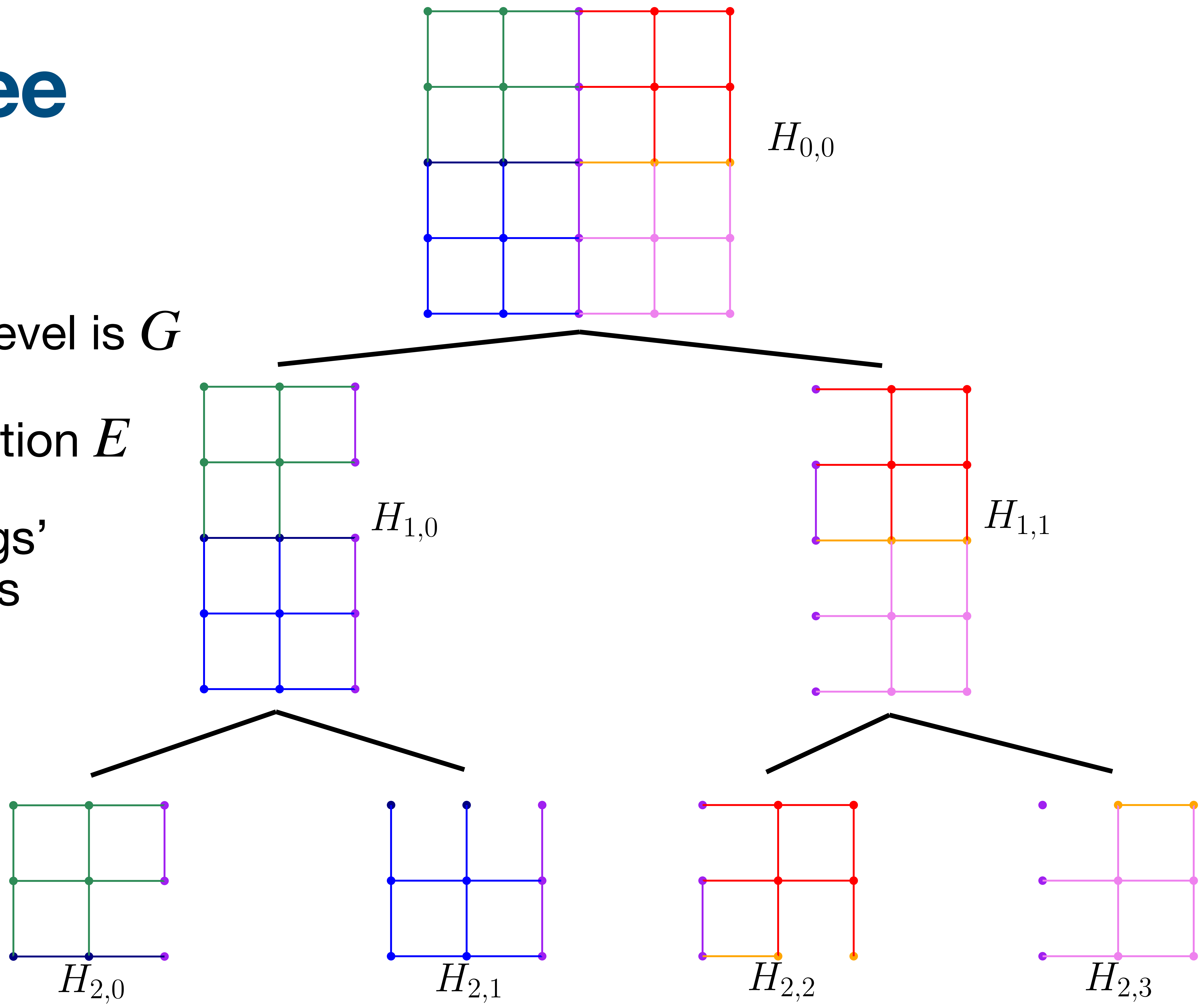
Separator tree

- height $\eta = O(\log n)$
- constant degree
- each node is a subgraph
- constant size leaf nodes



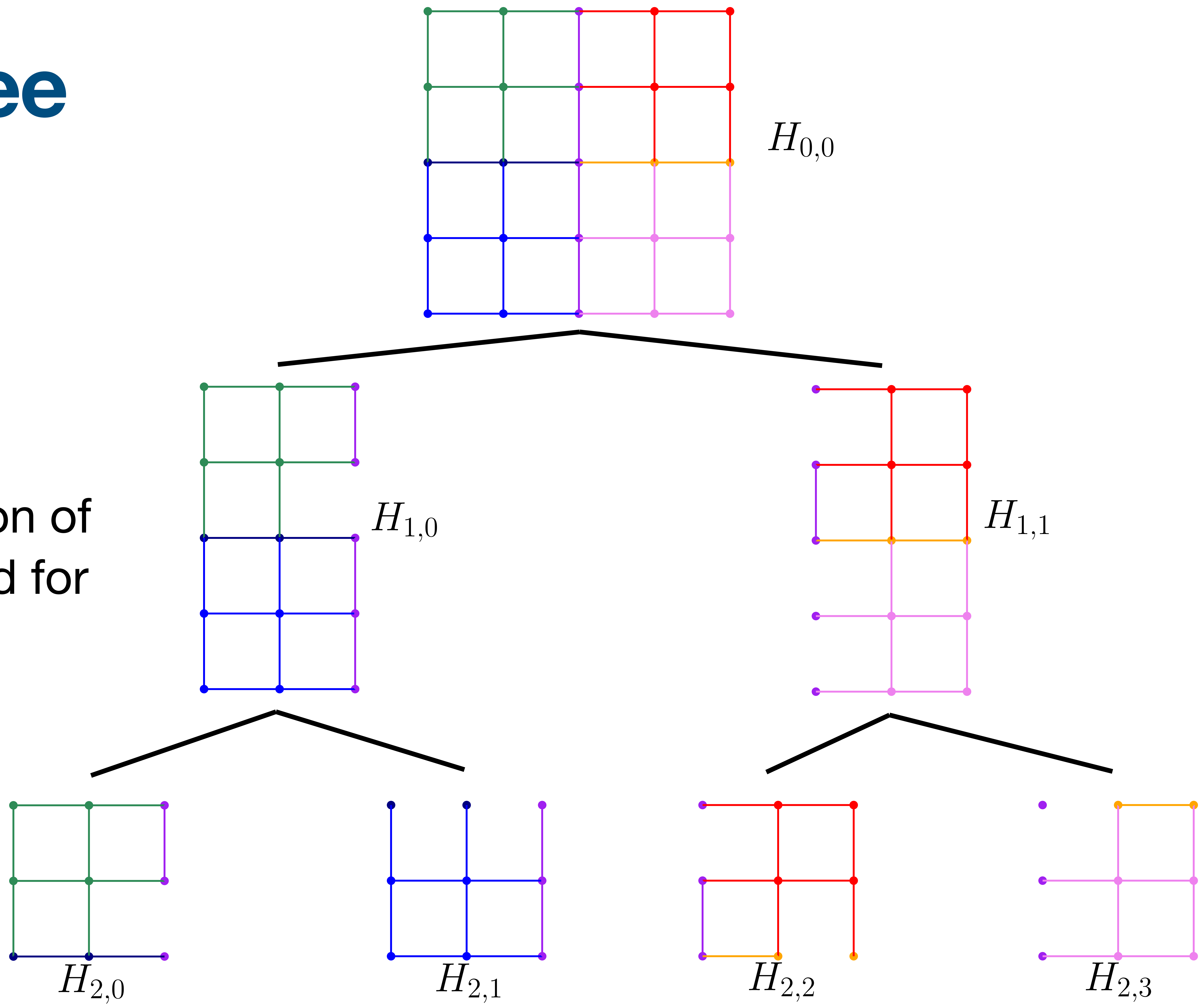
Separator tree

- union of nodes at a level is G
- nodes at a level partition E
- intersection of siblings' vertex sets is parent's separator



Separator tree

- boundary set
- separator
- gives natural definition of $V_{i,j}$'s and E_i's needed for the tree operator



Recursive Cholesky decomposition

- Recursively factor the symmetric matrix $\mathbf{L} = \mathbf{A}\mathbf{W}\mathbf{A}^\top$ based on separator tree

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- Rows and columns of $\mathbf{L}$ are indexed by vertices of $G_{\mathbf{A}}$
- Partition vertex set into F, C, then

$$\mathbf{L} = \begin{pmatrix} \mathbf{L}_{FF} & \mathbf{L}_{FC} \\ \mathbf{L}_{CF} & \mathbf{L}_{CC} \end{pmatrix} = \begin{pmatrix} \mathbf{I} & 0 \\ \mathbf{L}_{CF}\mathbf{L}_{FF}^{-1} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{L}_{FF} & \mathbf{0} \\ \mathbf{0} & \mathbf{Sc}(\mathbf{L}, C) \end{pmatrix} \begin{pmatrix} \mathbf{I} & \mathbf{L}_{FF}^{-1}\mathbf{L}_{FC} \\ 0 & \mathbf{I} \end{pmatrix}.$$

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Schur complement of $\mathbf{L}$ onto C
is supported on C

Recursive Cholesky decomposition

- Recursively factor the symmetric matrix $\mathbf{L} = \mathbf{AWA}^\top$ based on separator tree
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Recursive partitions defined using separator tree

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Schur complement of $\mathbf{L}$ onto C is supported on C

Useful Schur complement properties (for efficient data structure updates)

Transitivity:

If $X \subseteq Y \subseteq V(G)$, then

$$\mathbf{Sc}(\mathbf{Sc}(\mathbf{L}, Y), X) = \mathbf{Sc}(\mathbf{L}, X)\,.$$

Decomposability:

If $\mathbf{L} = \mathbf{L}_1 + \ldots + \mathbf{L}_k$, and the $\mathbf{L}_i$'s supports intersect on C and are otherwise pairwise disjoint, then

$$\mathbf{Sc}(\mathbf{L}, C) = \mathbf{Sc}(\mathbf{L}_1, C) + \ldots + \mathbf{Sc}(\mathbf{L}_k, C)\,.$$

Decomposition of $\mathbf{P}_w$

$$\mathbf{P}_w = \mathbf{W}^{1/2}\mathbf{A}^\top(\mathbf{A}\mathbf{W}\mathbf{A}^\top)^{-1}\mathbf{A}\mathbf{W}^{1/2}$$

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\begin{aligned}
\mathbf{P}_w &= \mathbf{W}^{1/2}\mathbf{A}^\top(\mathbf{A}\mathbf{W}\mathbf{A}^\top)^{-1}\mathbf{A}\mathbf{W}^{1/2} \\
&= \mathbf{W}^{1/2}\mathbf{A}^\top\mathbf{\Pi}^{(\eta)\top}\ldots\mathbf{\Pi}^{(1)\top}\mathbf{\Gamma}\mathbf{\Pi}^{(1)}\ldots\mathbf{\Pi}^{(\eta)}\mathbf{A}\mathbf{W}^{1/2}
\end{aligned}
$$

where

$$
\mathbf{\Pi}^{(i)} = \boldsymbol{I} - \sum_{H \in \mathcal{T}(i)} \mathbf{L}^{(H)}_{\partial H, F_H}\left(\mathbf{L}^{(H)}_{F_H, F_H}\right)^{-1}.
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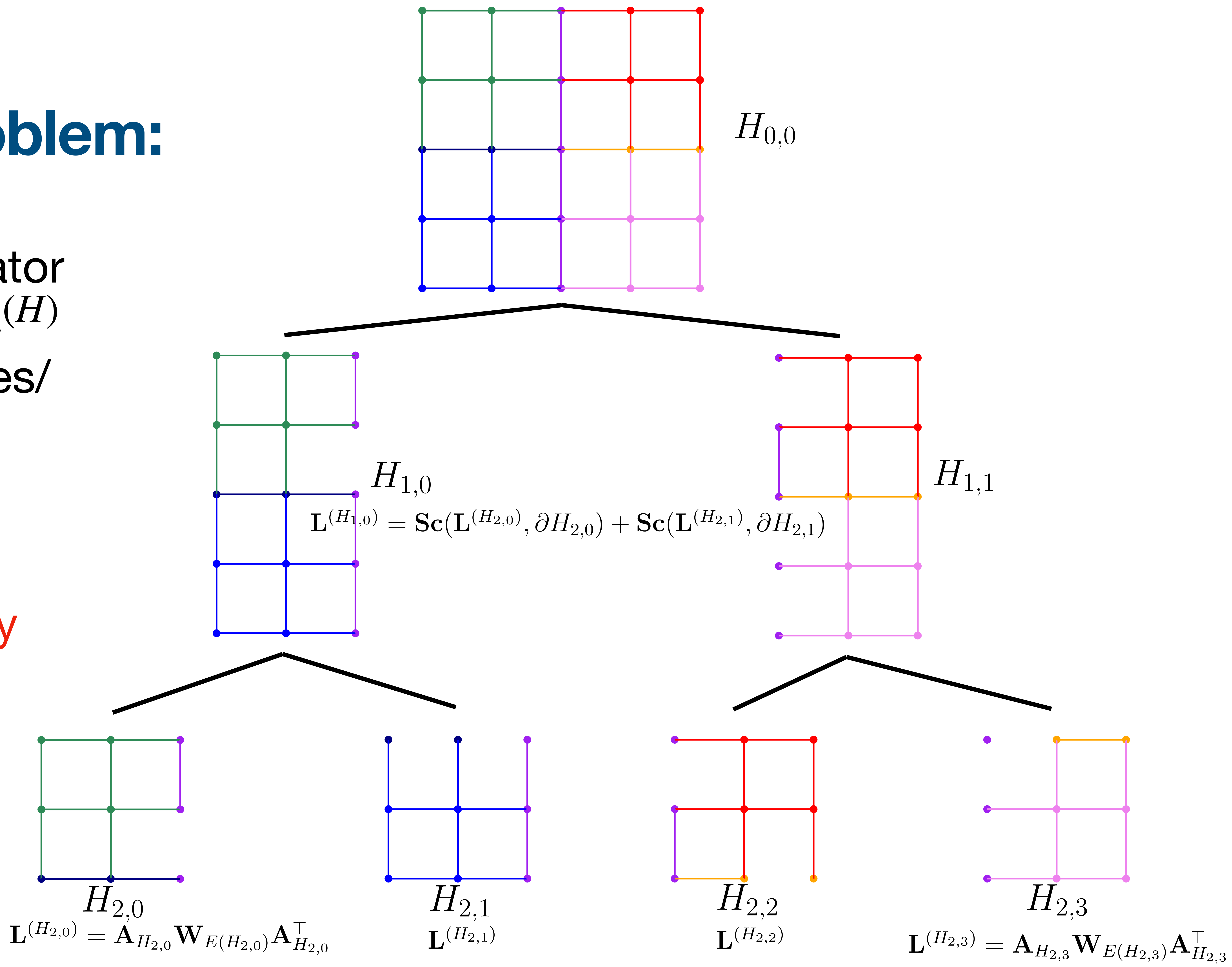
Can be further decomposed based on edges of separator tree.

Matrix/inverse maintenance problem:

Every node H in separator tree maintains matrix $\mathbf{L}^{(H)}$ and some other matrices/inverses

$\mathbf{L}^{(H)}$ is supported on separator and boundary of H.



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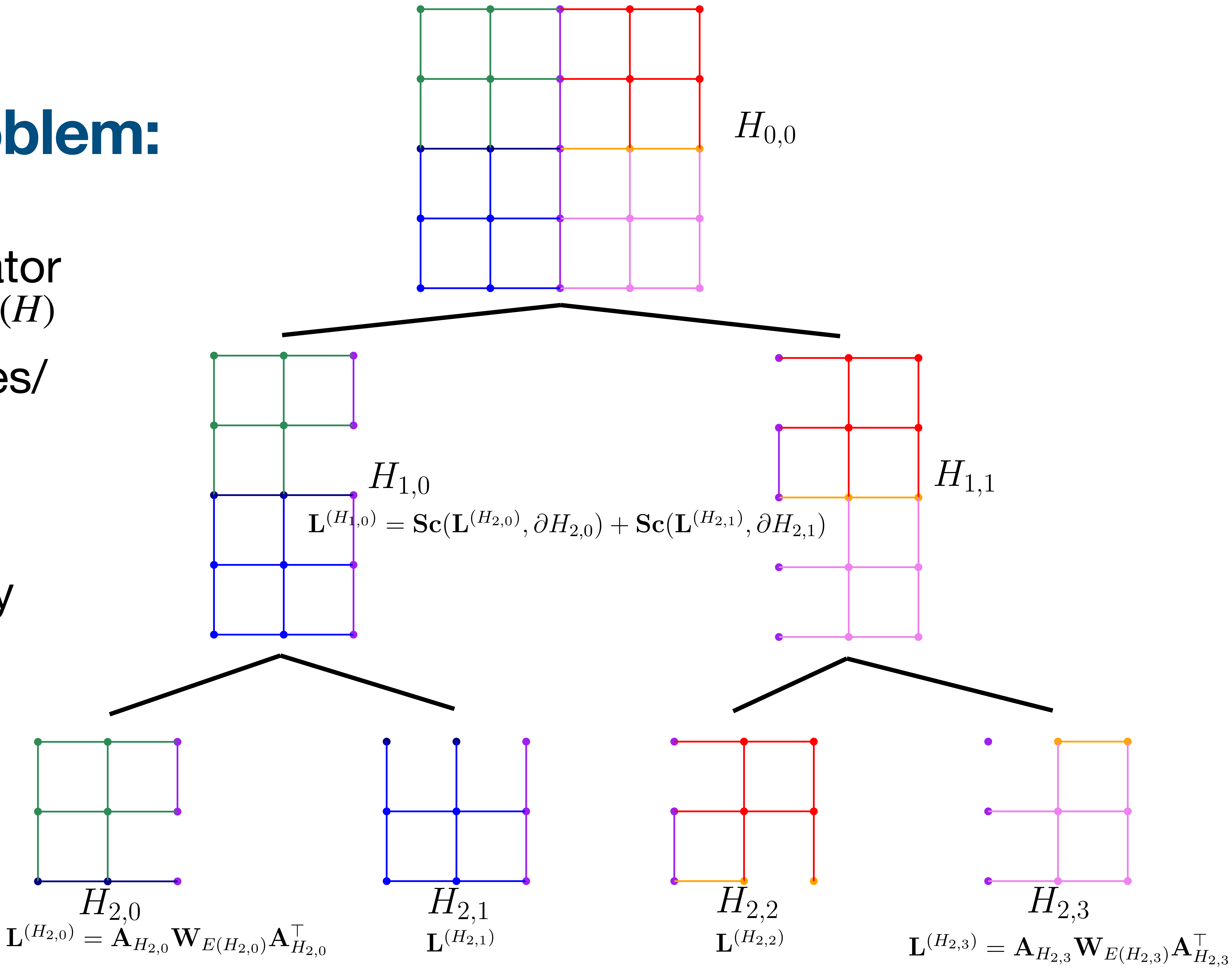
Theorem (DGLSY24):

Efficient algorithm for separable graphs.



Flow problems:
Use approximations to improve runtime

If the LP is a flow problem, then $\mathbf{AWA}^\top$ is a weighted Laplacian.

- [Spielman-Tang, 04] Laplacian solvers in nearly-linear time
- [Kyng-Sachdeva, 16], [Goranci-Henzinger-Peng, 18] sparse, approximate Schur complements in nearly-linear time

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Theorem (**D**GGPSY22): Nearly-linear time min-cost flow on planar graphs.

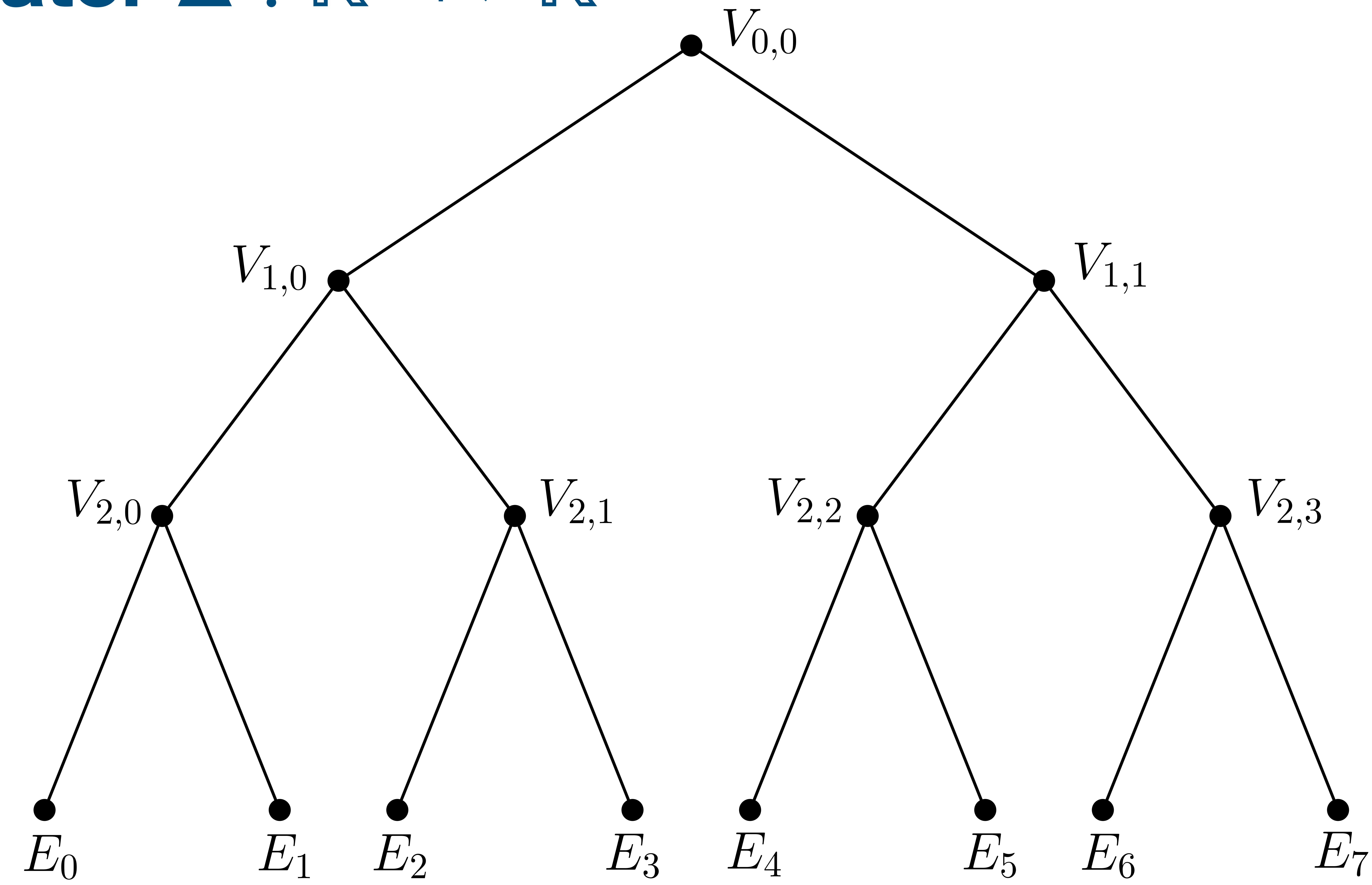
Theorem (**D**Y23+): $\tilde{O}(m\sqrt{t})$ time min-cost flow on treewidth t graphs.

Tree operator

Tree operator $\Delta : \mathbb{R}^V \mapsto \mathbb{R}^E$

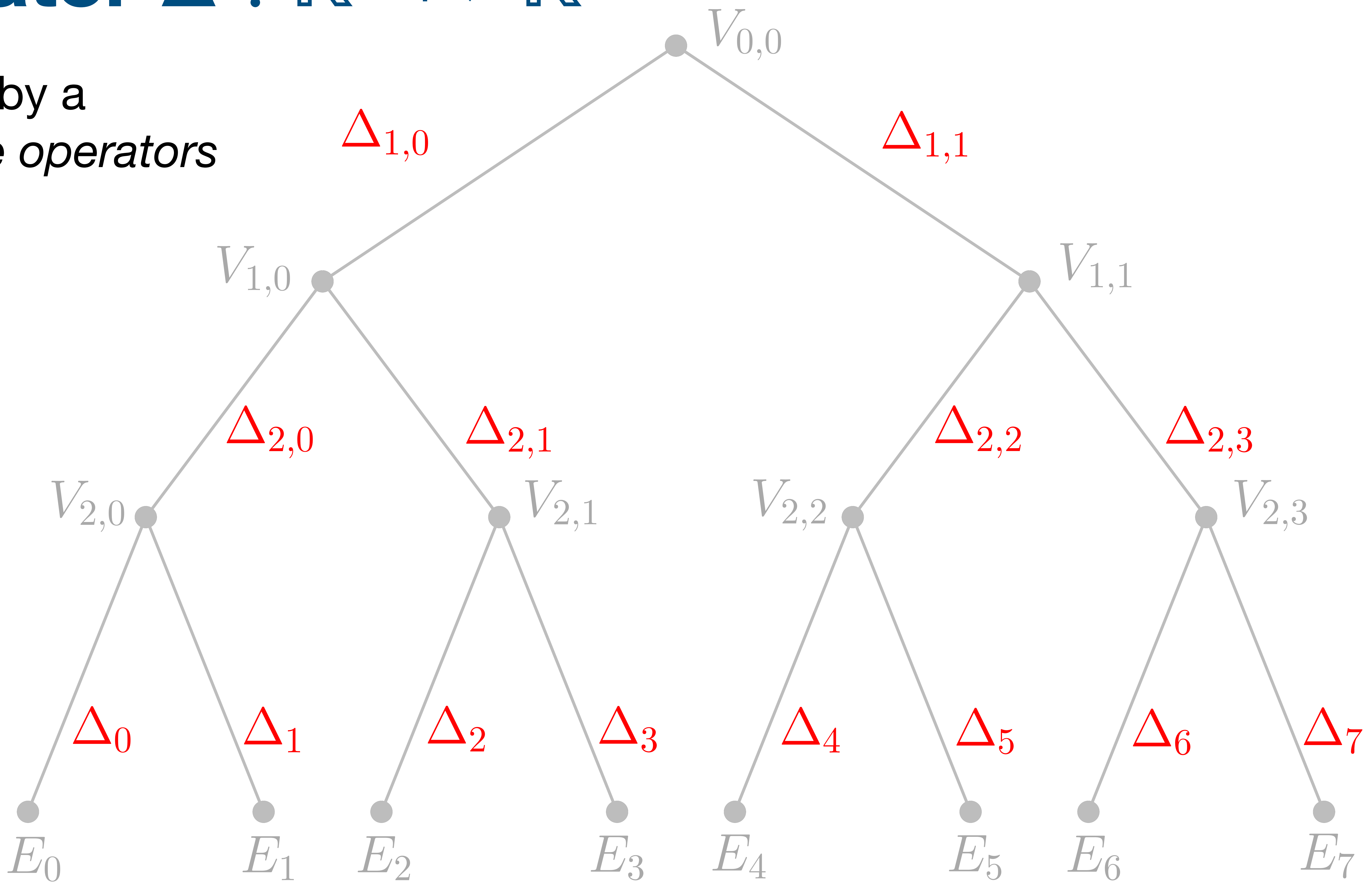
$$V := \dot{\bigcup} V_{i,j}$$

$$E := \dot{\bigcup} E_i$$



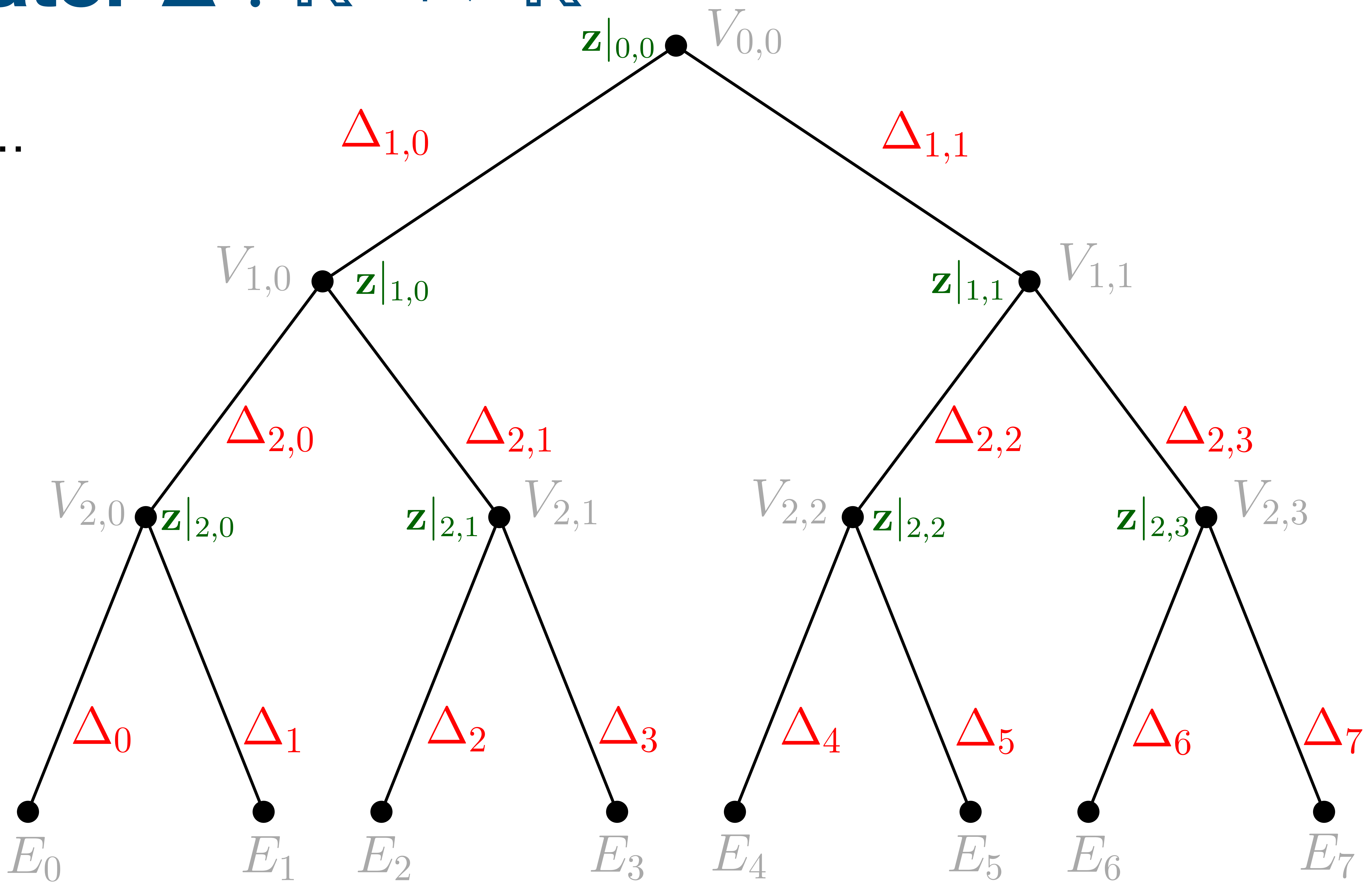
Tree operator $\Delta : \mathbb{R}^V \mapsto \mathbb{R}^E$

Δ is represented by a collection of *edge operators*



Tree operator $\Delta : \mathbb{R}^V \mapsto \mathbb{R}^E$

To compute $\Delta \mathbf{z}$...



Tree operator $\Delta : \mathbb{R}^V \mapsto \mathbb{R}^E$

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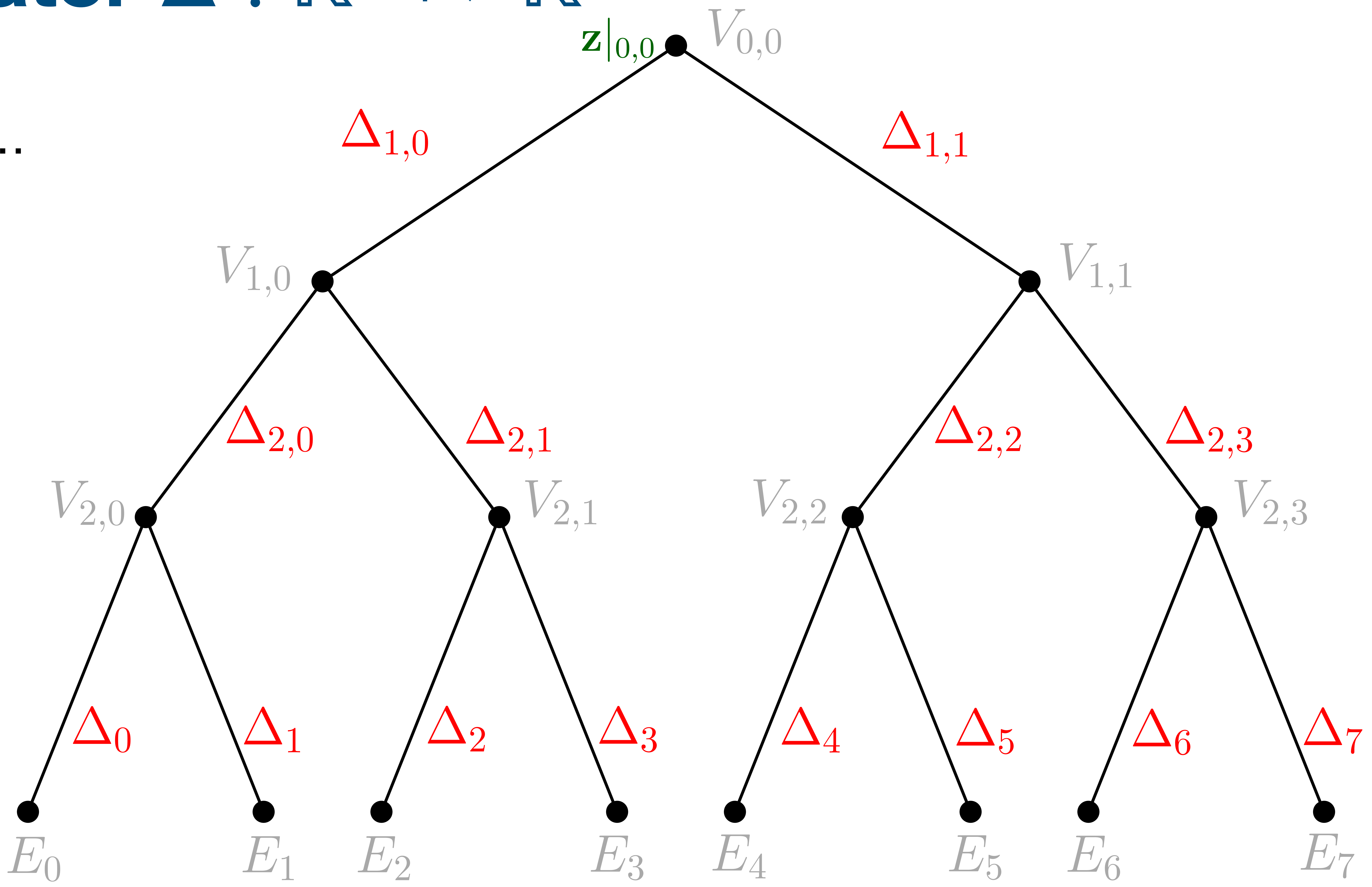

Tree operator $\Delta : \mathbb{R}^V \mapsto \mathbb{R}^E$

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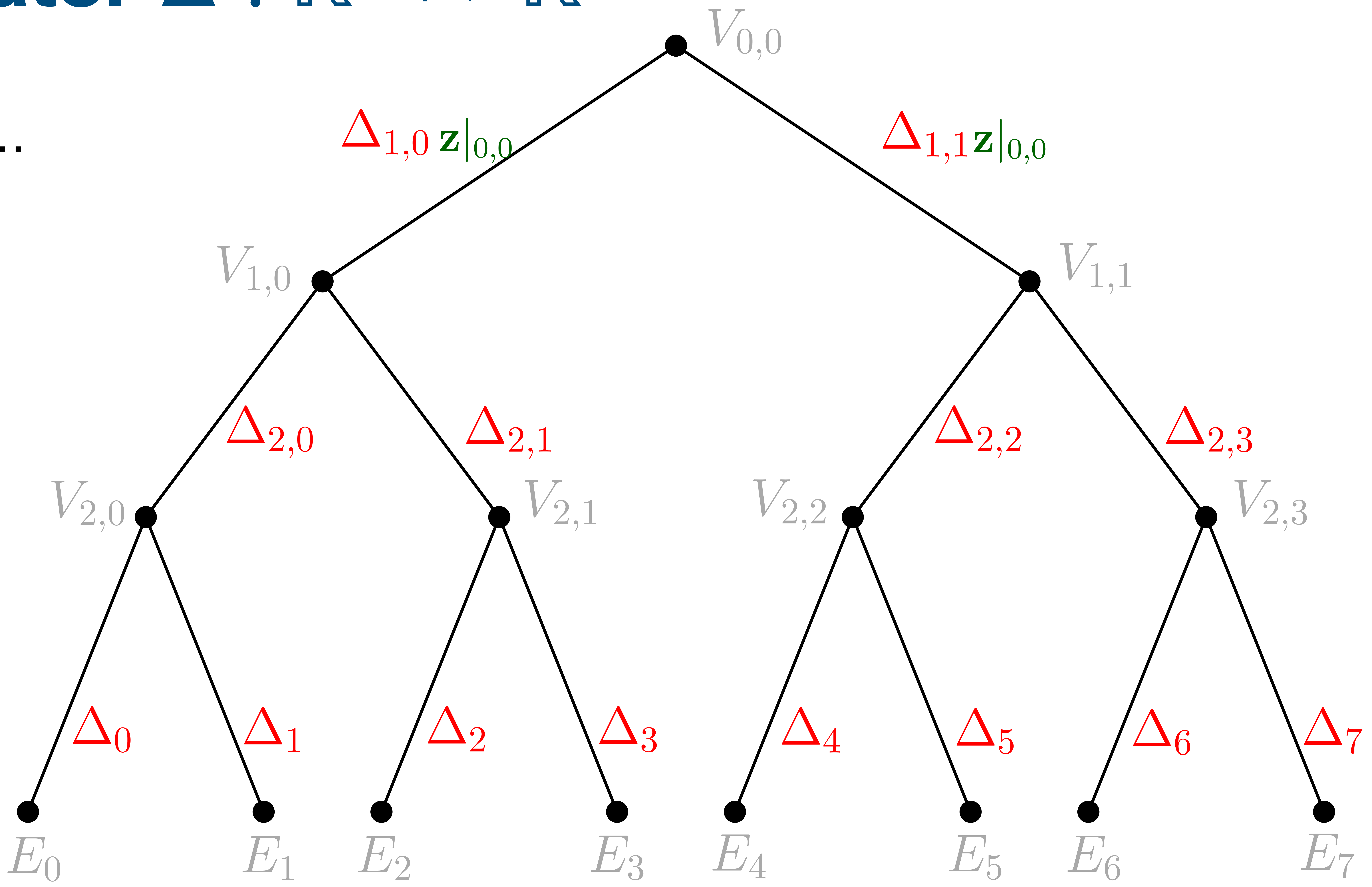
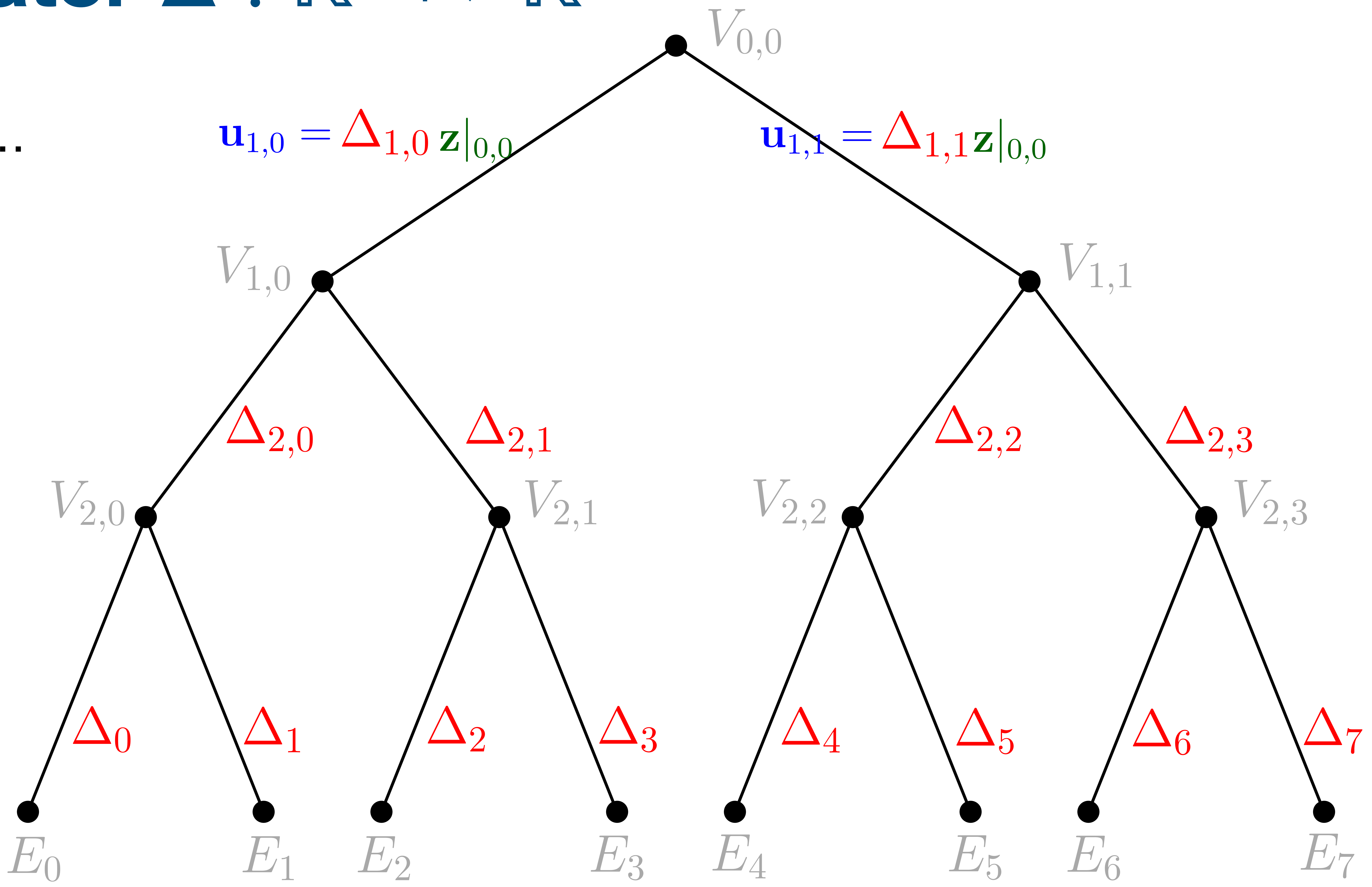


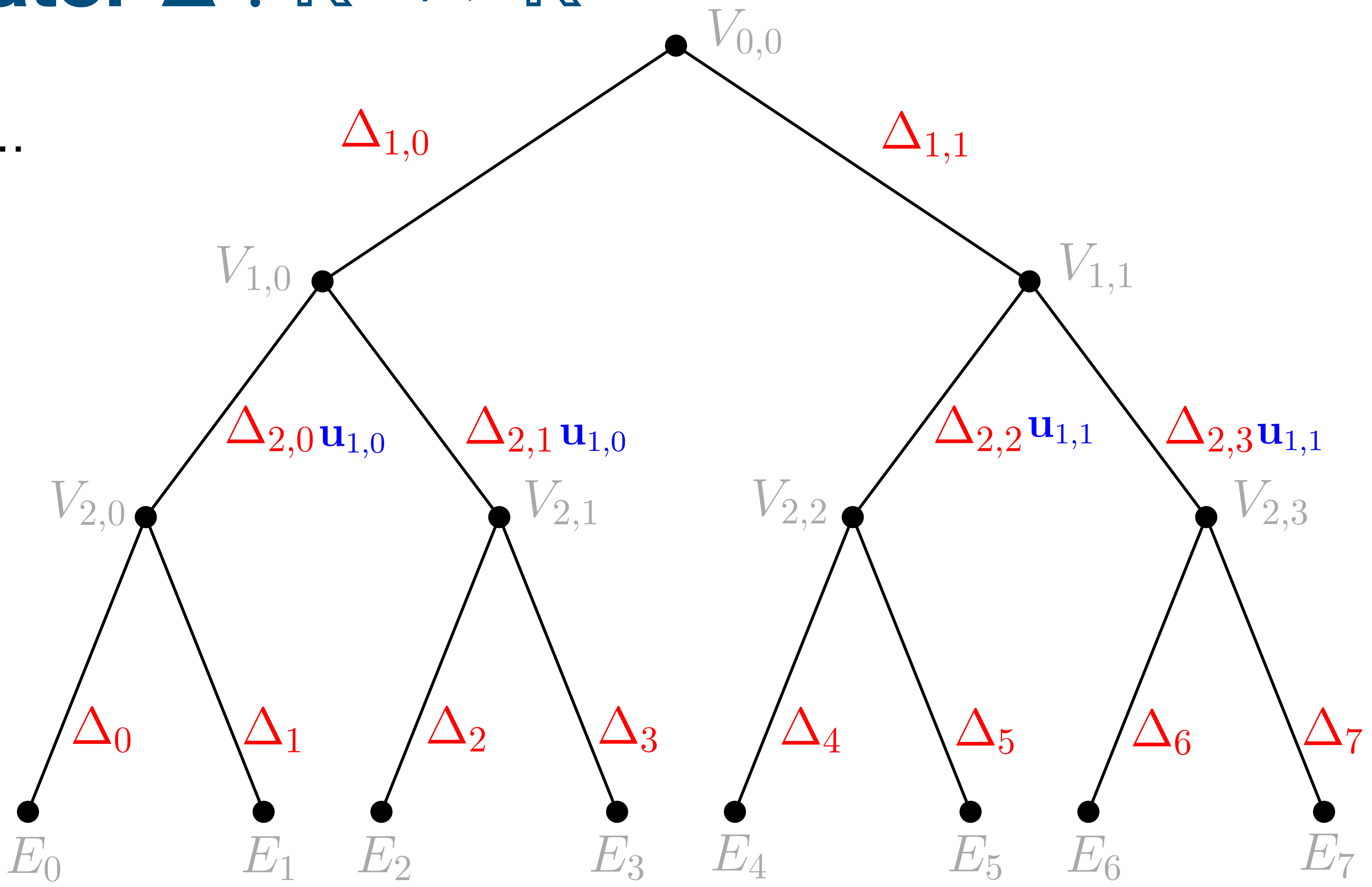
Tree operator $\Delta : \mathbb{R}^V \mapsto \mathbb{R}^E$

To compute $\Delta \mathbf{z}$...



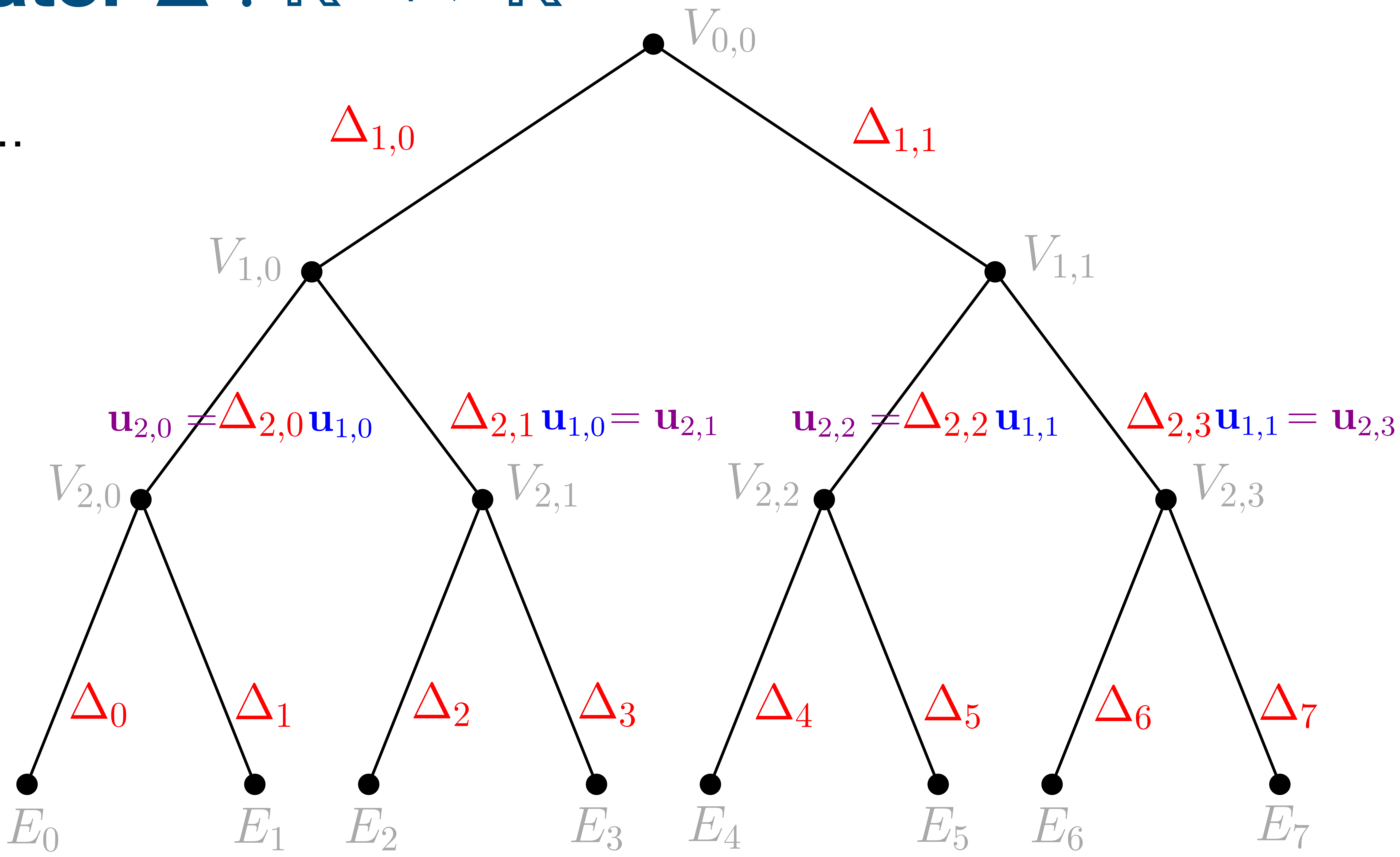
Tree operator $\Delta : \mathbb{R}^V \mapsto \mathbb{R}^E$

To compute $\Delta \mathbf{z}$...



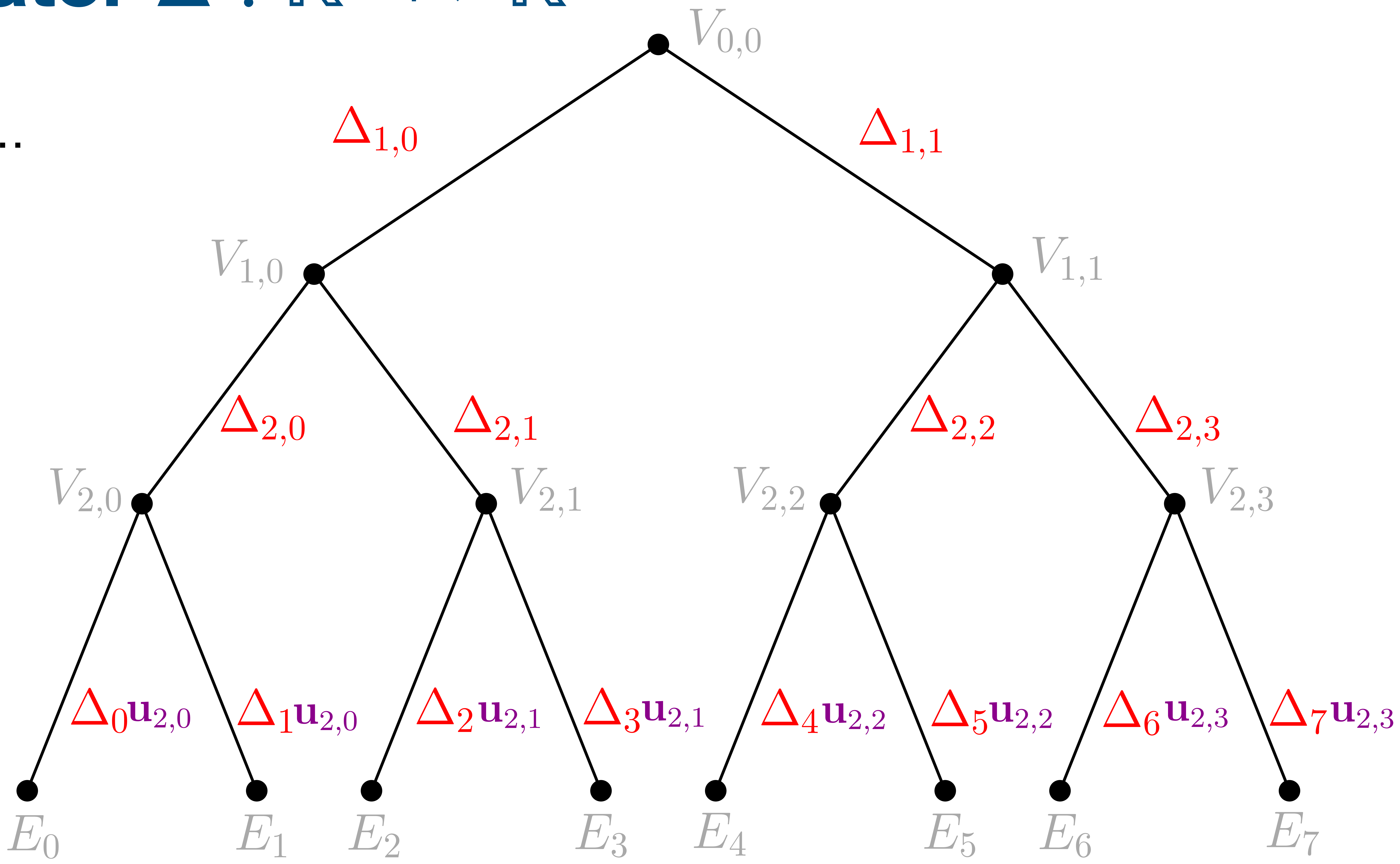
Tree operator $\Delta : \mathbb{R}^V \mapsto \mathbb{R}^E$

To compute $\Delta \mathbf{z}$...



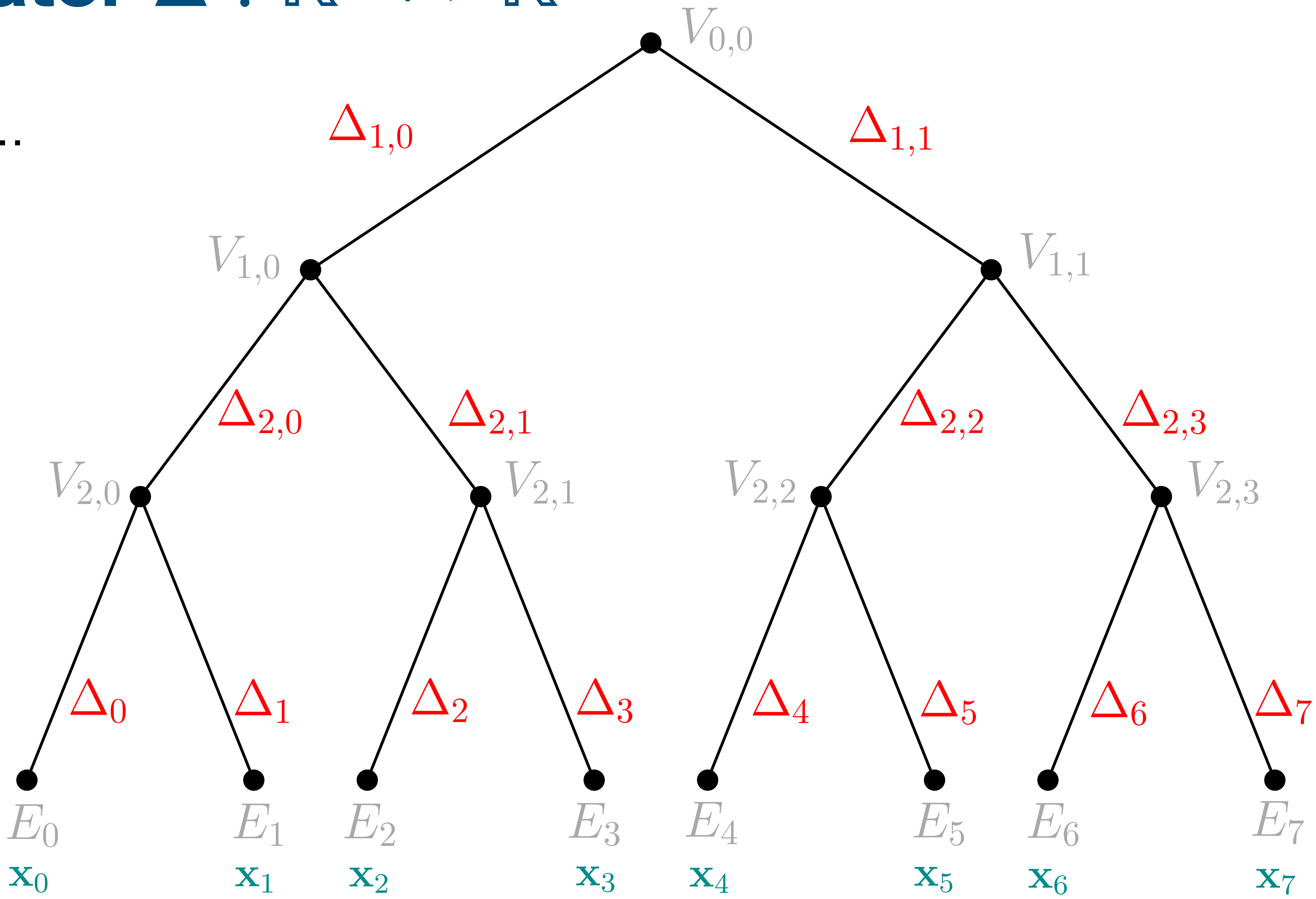
Tree operator $\Delta : \mathbb{R}^V \mapsto \mathbb{R}^E$

To compute $\Delta \mathbf{z}$...



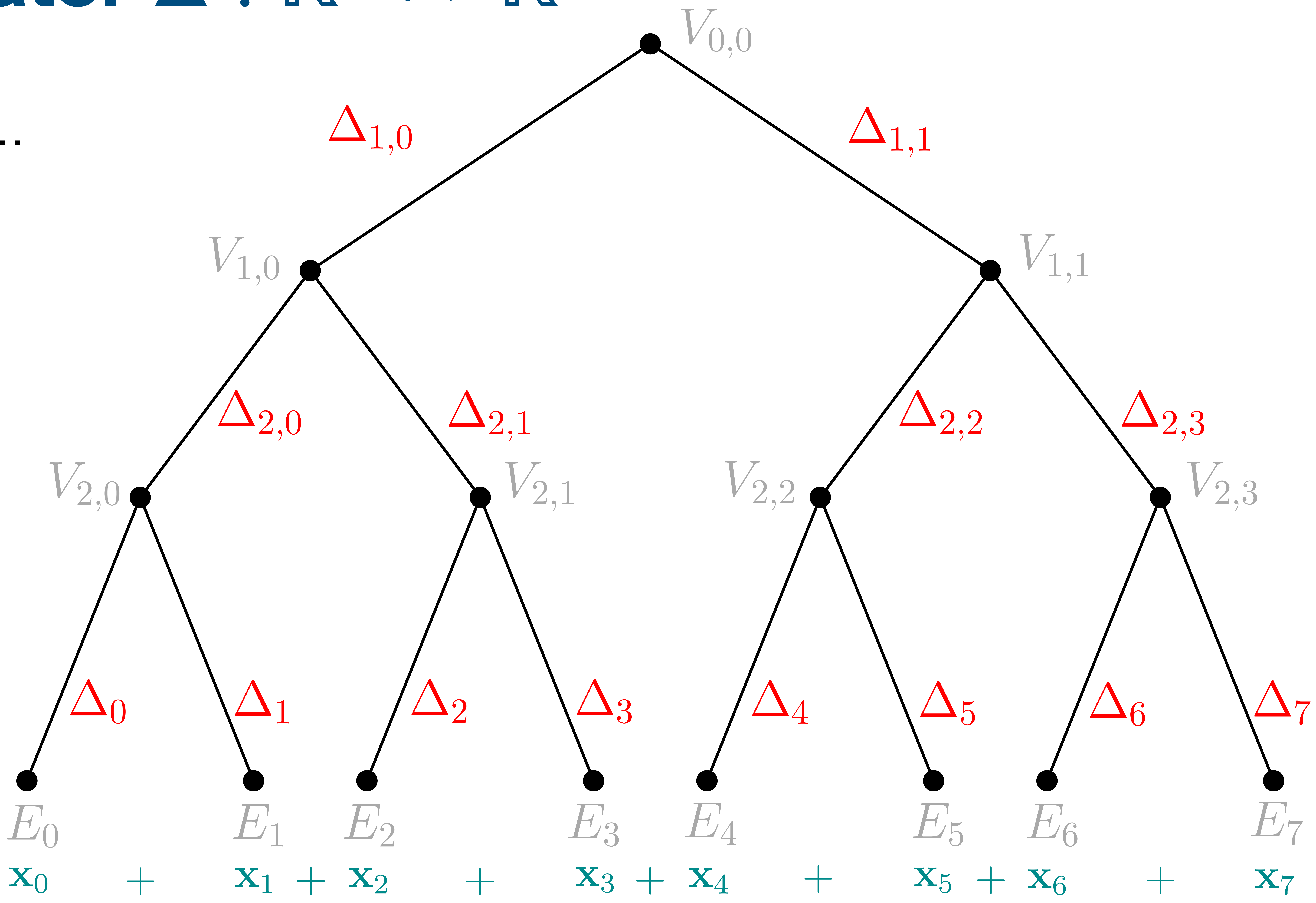
Tree operator $\Delta : \mathbb{R}^V \mapsto \mathbb{R}^E$

To compute $\Delta\mathbf{z}$...

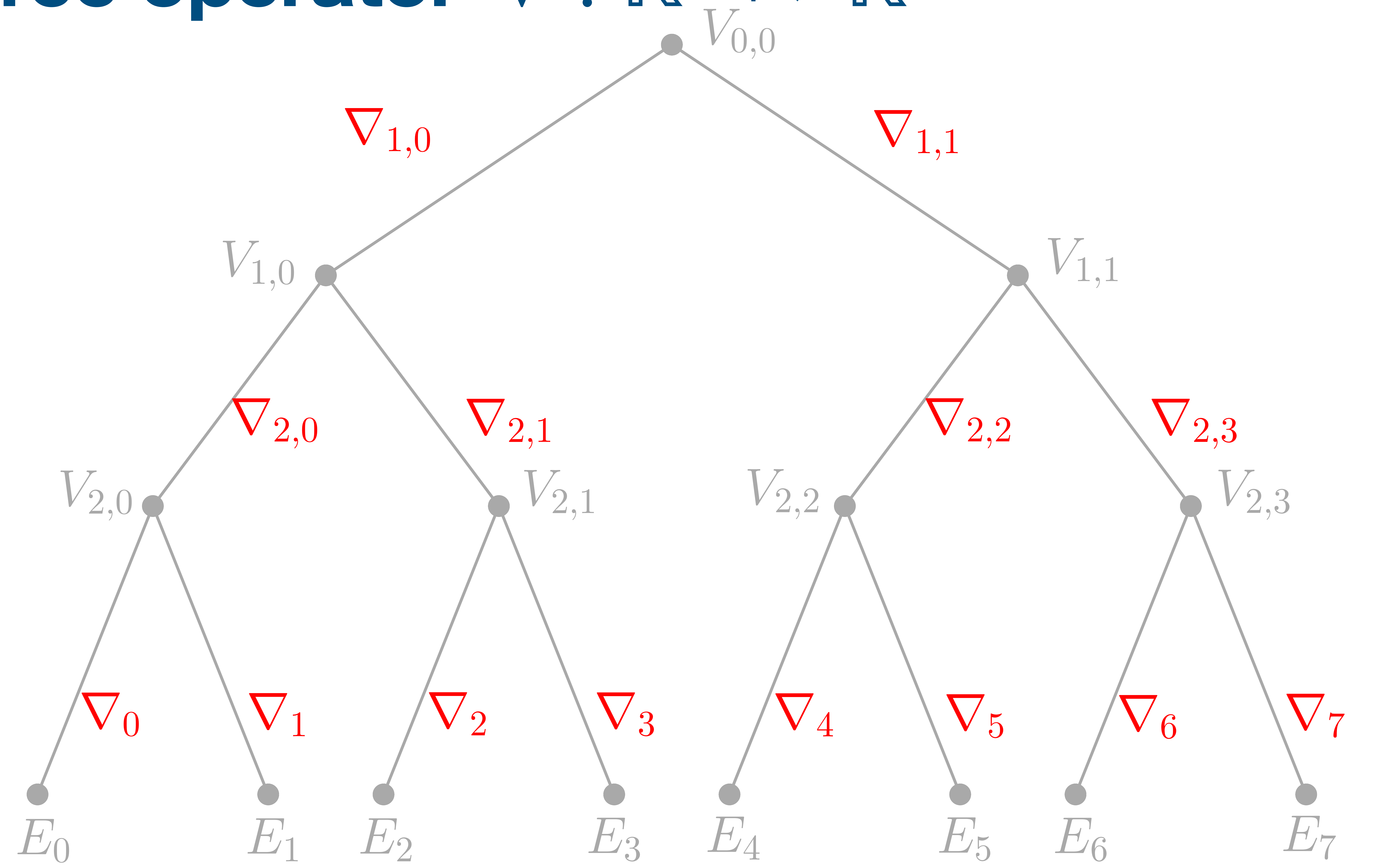


Tree operator $\Delta : \mathbb{R}^V \mapsto \mathbb{R}^E$

To compute $\Delta \mathbf{z}$...

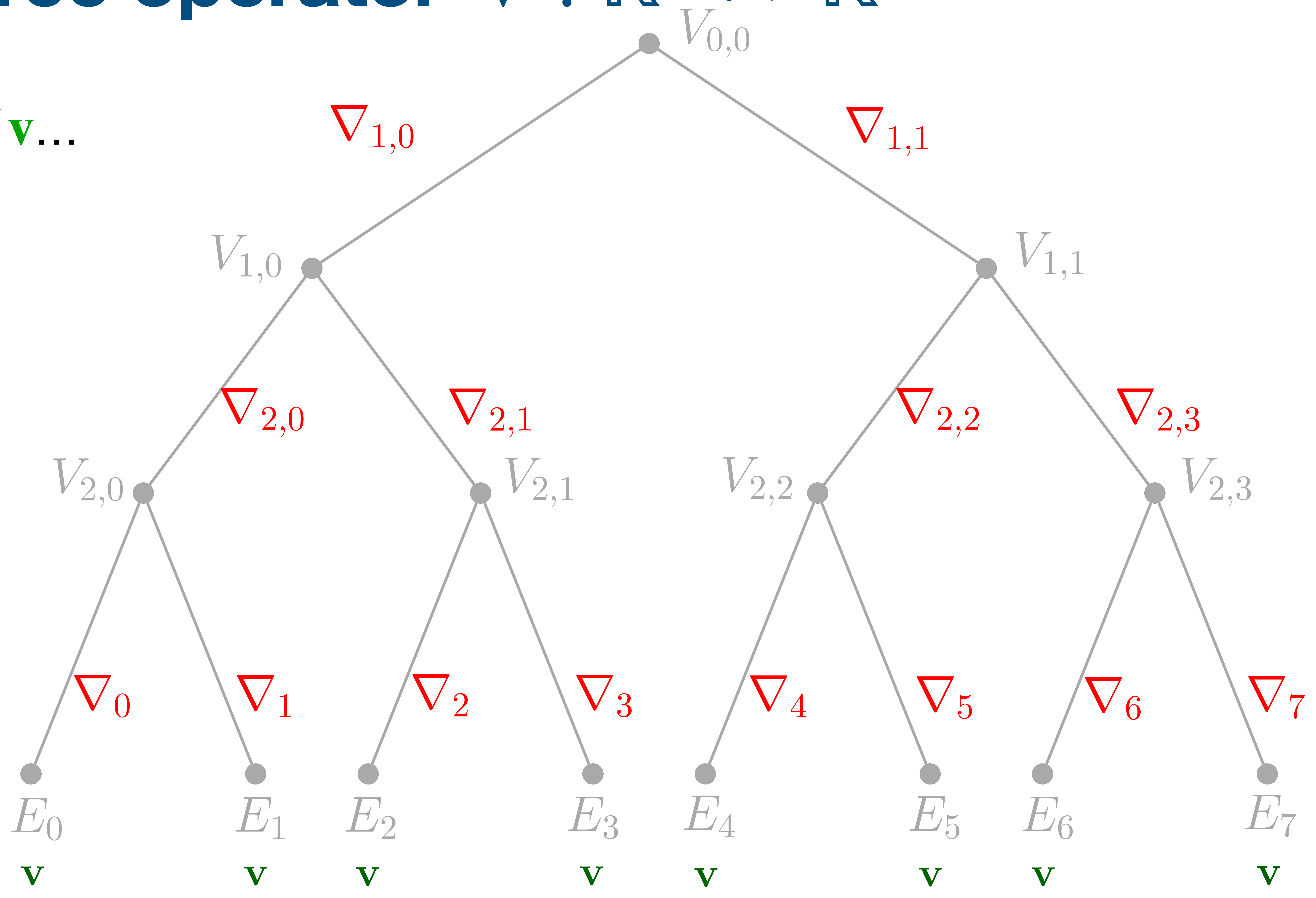


Inverse tree operator $\nabla : \mathbb{R}^E \mapsto \mathbb{R}^V$



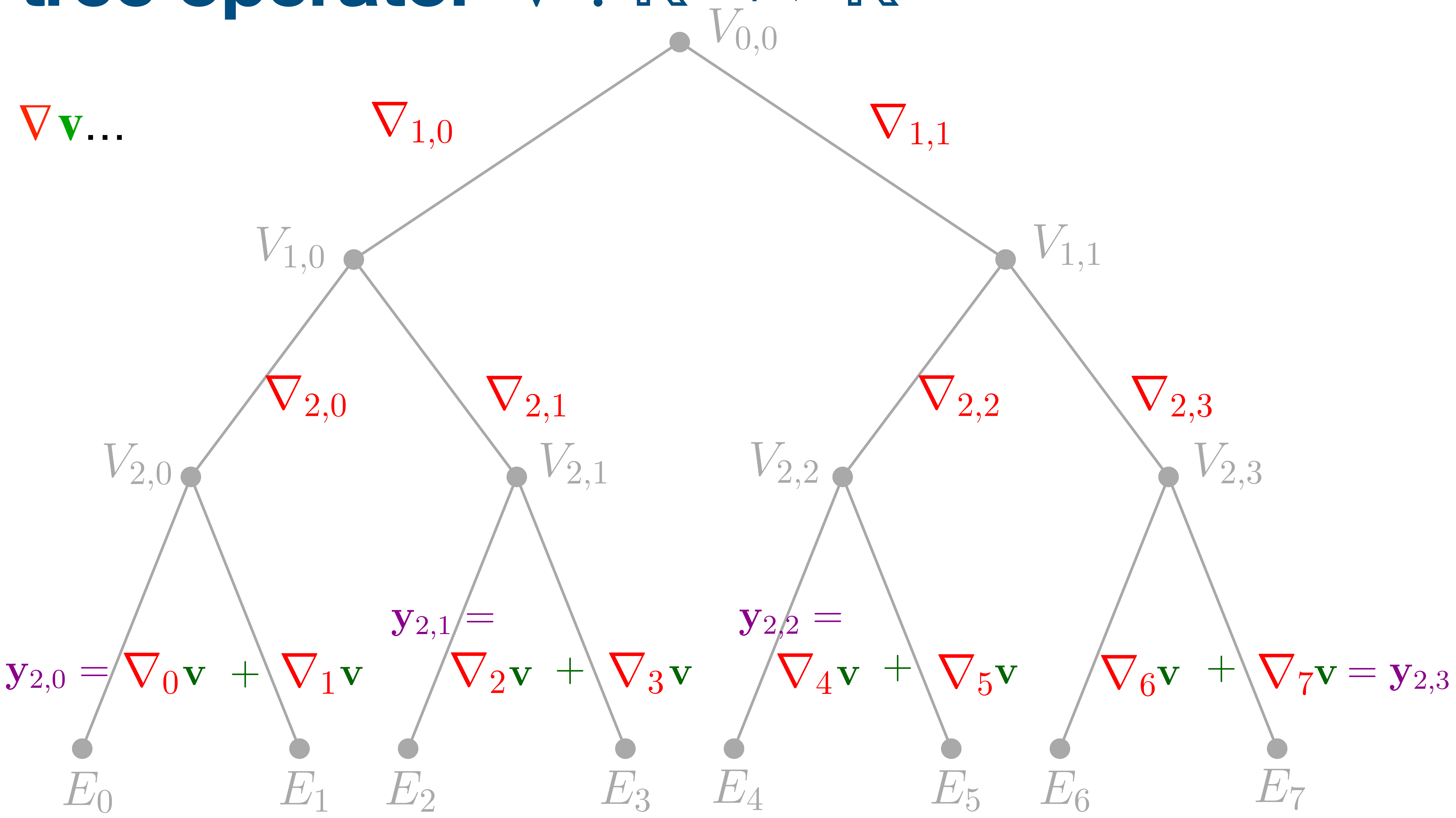
Inverse tree operator $\nabla : \mathbb{R}^E \mapsto \mathbb{R}^V$

To compute $\nabla \mathbf{v}$…



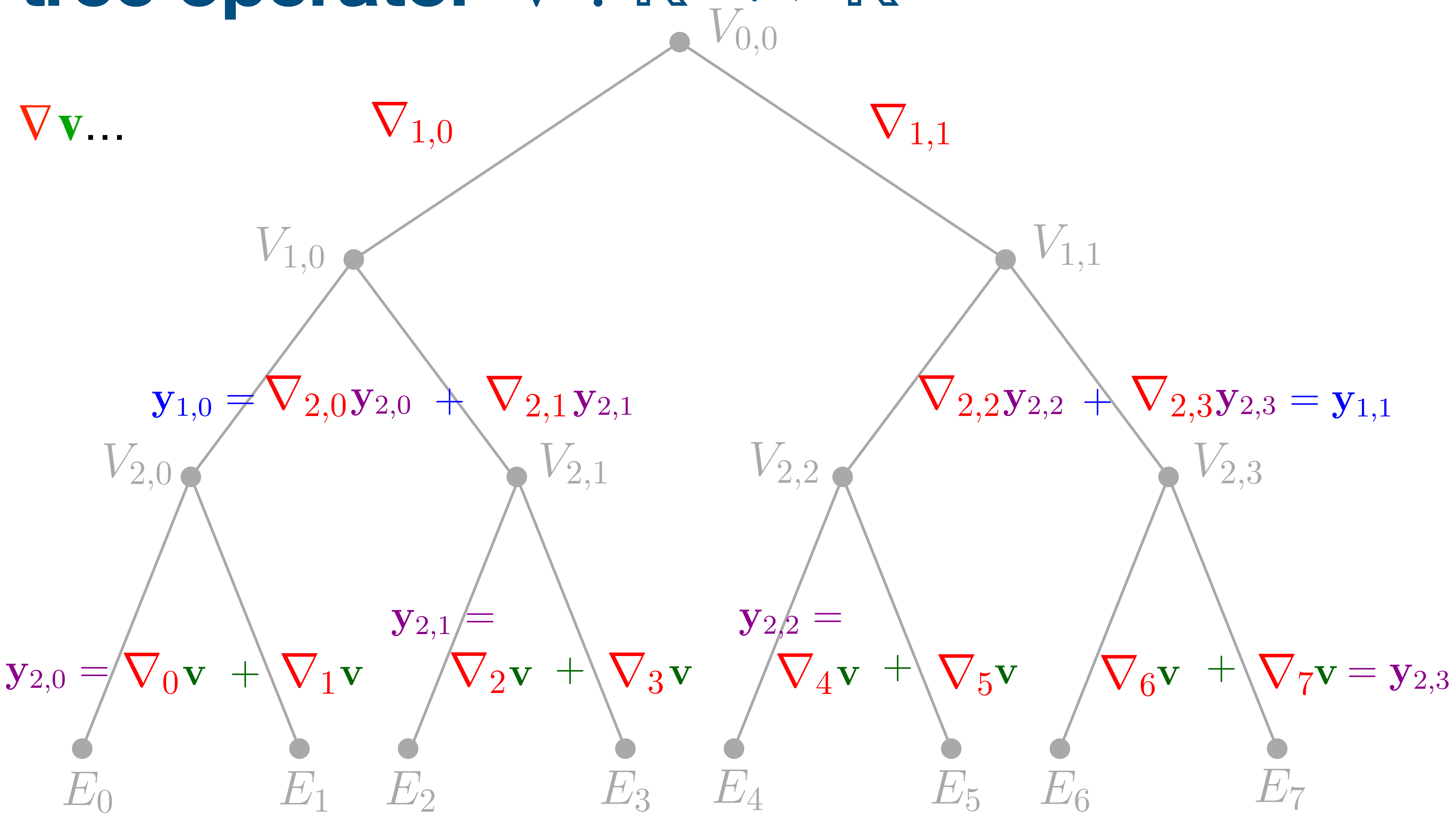
Inverse tree operator $\nabla : \mathbb{R}^E \mapsto \mathbb{R}^V$

To compute $\nabla \mathbf{v}$...



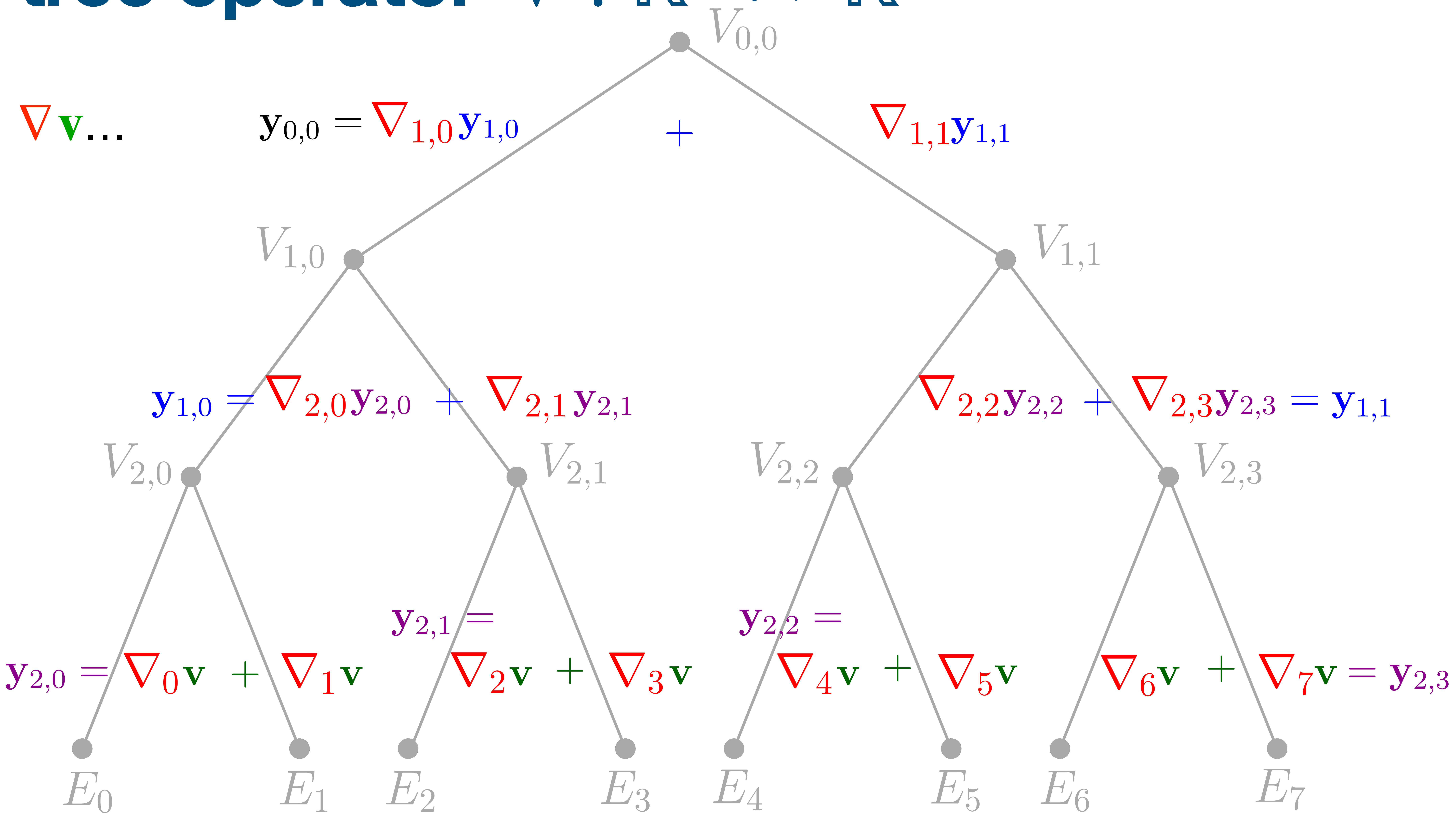
Inverse tree operator $\nabla : \mathbb{R}^E \mapsto \mathbb{R}^V$

To compute $\nabla \mathbf{v}$...



Inverse tree operator $\nabla : \mathbb{R}^E \mapsto \mathbb{R}^V$

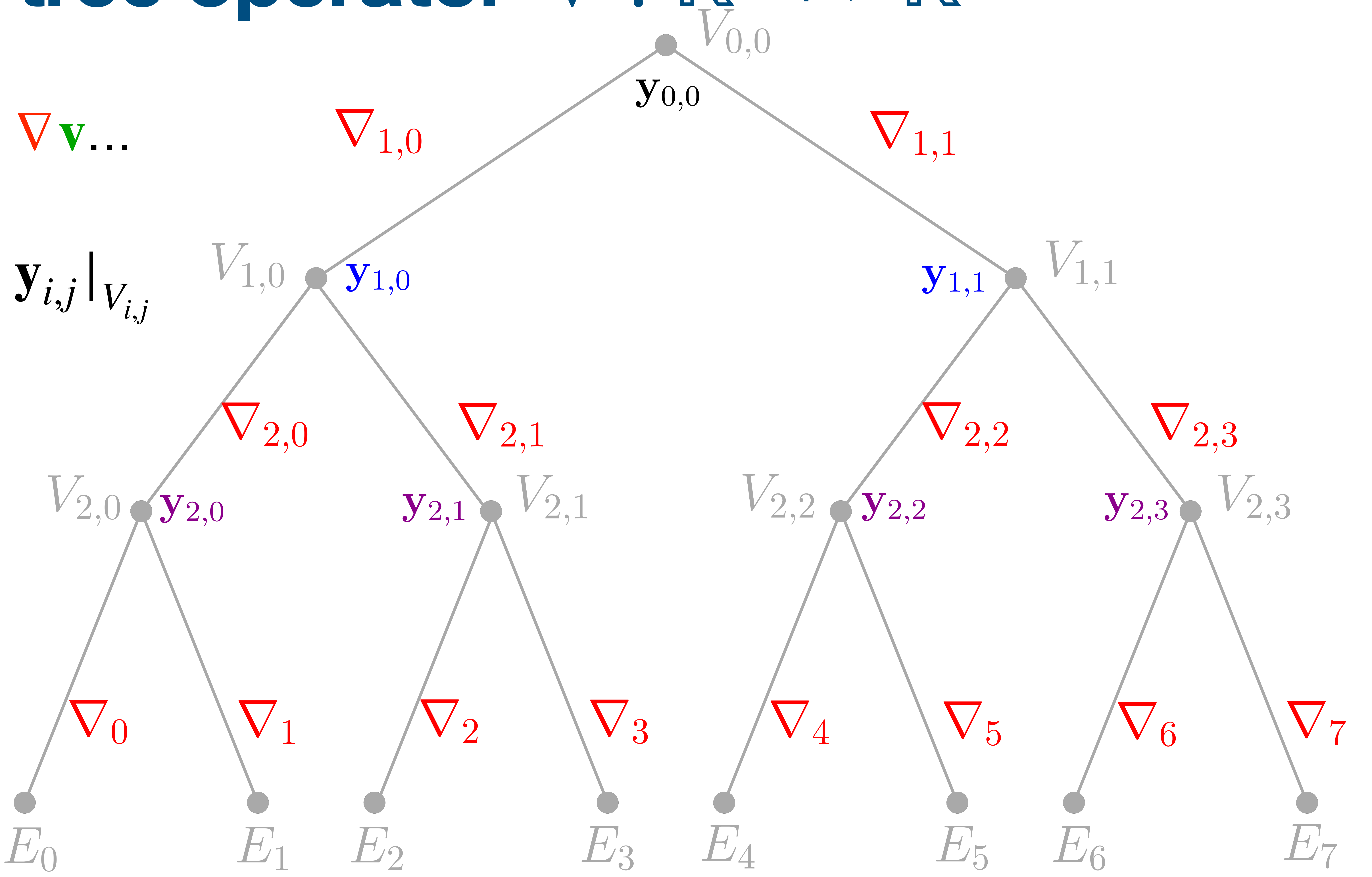
To compute $\nabla \mathbf{v}$...



Inverse tree operator $\nabla : \mathbb{R}^E \mapsto \mathbb{R}^V$

To compute $\nabla \mathbf{v}$...

$$(\nabla \mathbf{v})|_{V_{i,j}} = \mathbf{y}_{i,j}|_{V_{i,j}}$$



Fun lemma:

Δ is a tree operator if and only if $\Delta^\top$ is an inverse tree operator.

Proof:

Take the transpose of all edge operators.

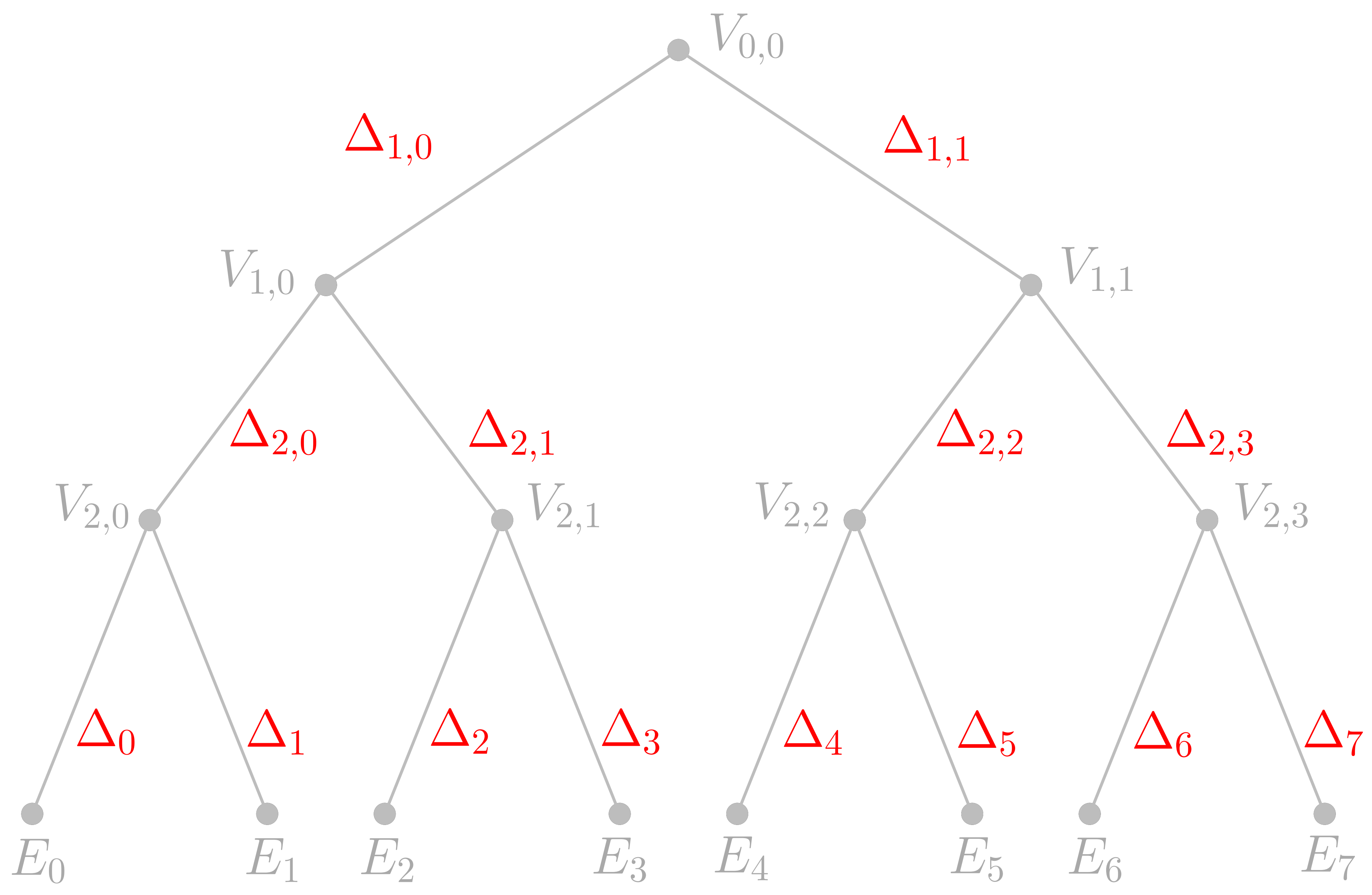
Complexity of Δ (and ∇)

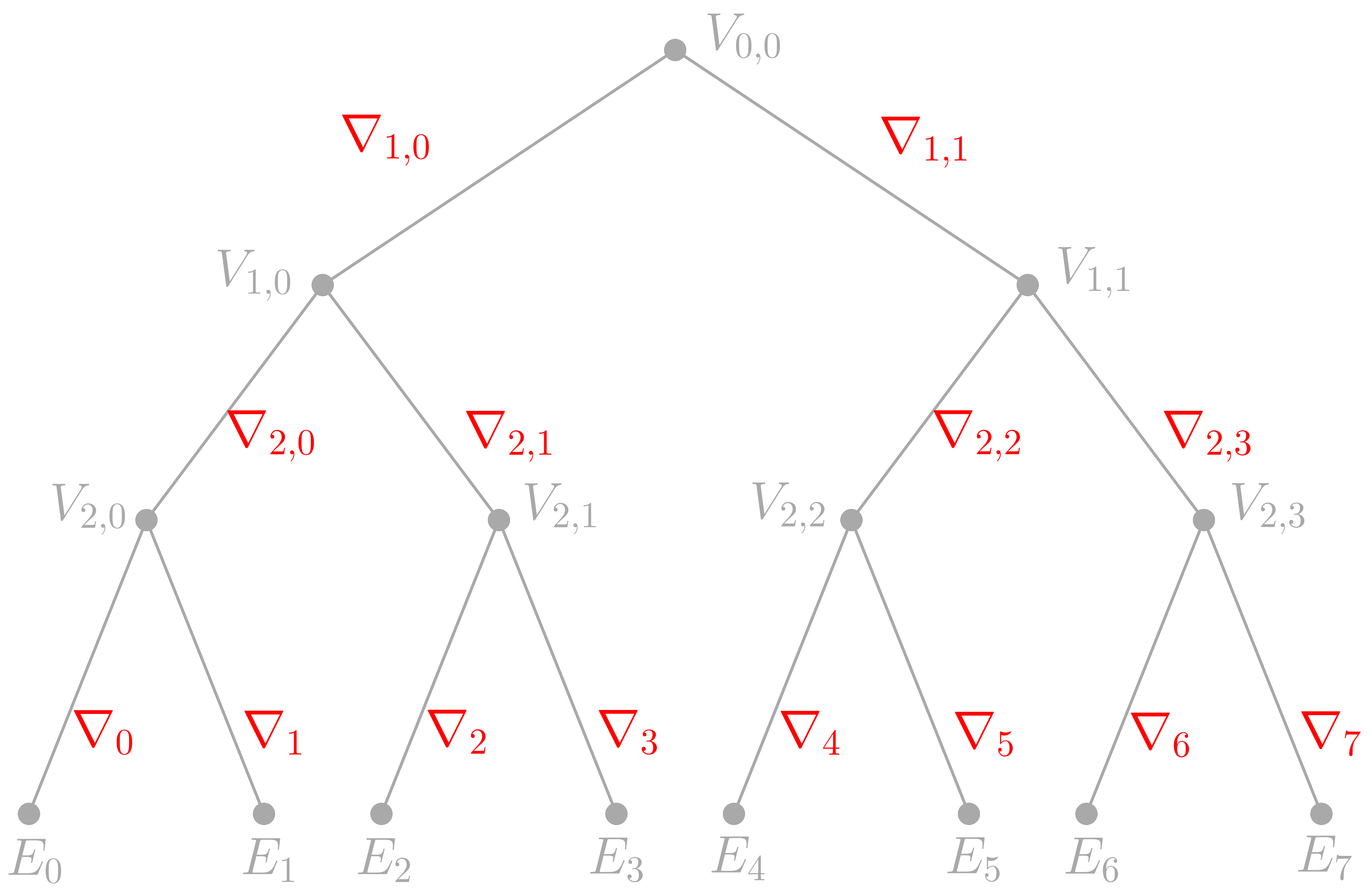
Say Δ has query complexity Q if the max time to apply k edge operators to k arbitrary vectors is at most $Q(k)$.

Recall Δ is a function of $\boldsymbol{w}$.

Say Δ has update complexity U if, when $\boldsymbol{w}$ changes in k coordinates, Δ can be updated in at most $U(k)$ time.

Easier to apply inverse tree operator





Implicit representation and heavy hitter detection both based on the tree structure

For a nice reference on this line of work, keep an eye out for my thesis

Thank you