A Simple Quantum Sketch With Applications to Graph Algorithms

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Sandia National Labs

Sampling

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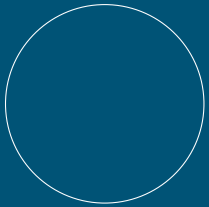

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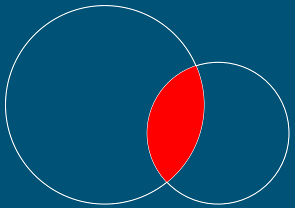

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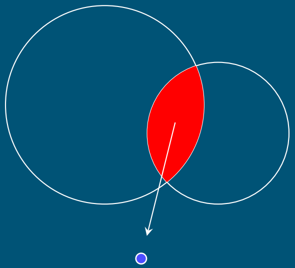

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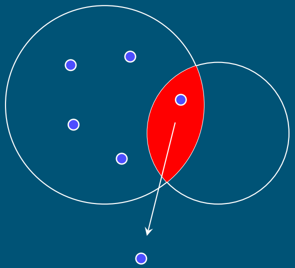

Simple solution: uniformly sample $O(1)$ elements from S, using $O(\log|U|)$ bits.

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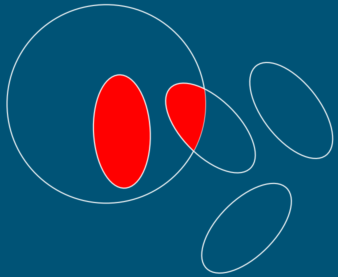

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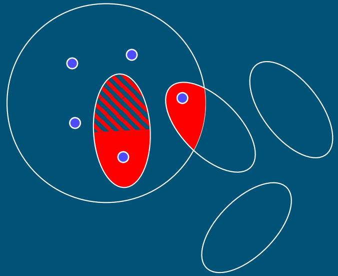

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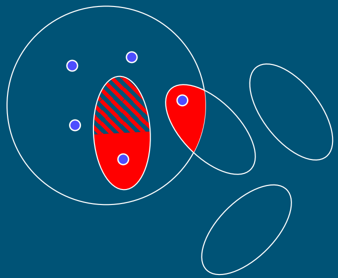

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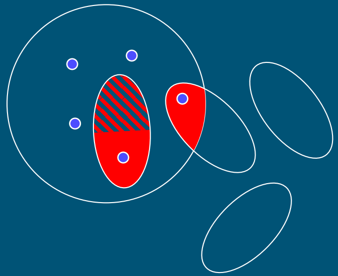

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- Need $\Theta\left(\sqrt{|S|}\right)$ samples even with $|\{p \in P : p \subseteq S\}| = \Omega(|P|)$.
- This is optimal up to a log factor by reduction to Boolean Hidden Matching [Gavinsky, Kempe, Kerenidis, Raz, de Wolf].

Quantum Sketch

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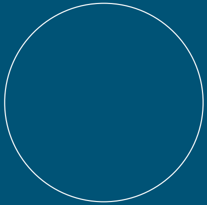

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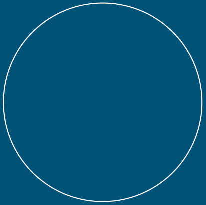

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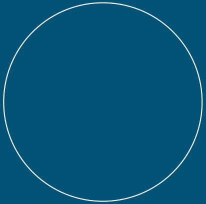

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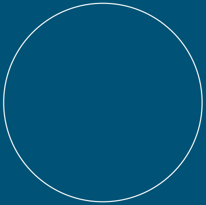

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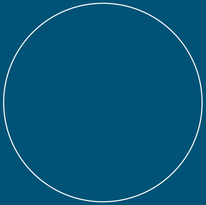

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- Otherwise: can return p s.t. $|p \cap S| = 1$, but with $\frac{1}{2}$ chance of -1 label.

Constructing the Quantum Sketch

Two Quantum Primitives

Superposition

A superposition $\sum_{x \in U} \alpha_x |x\rangle$ over U is a unit-length vector in $\mathbb{C}^U$.

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Projective Measurement

Using labeled projectors (Λ_i) s.t. $\sum_i \Lambda_i = I$, measure state ψ. With probability $\|\Lambda_i \psi\|_2^2$, get result i and transform ψ to $\Lambda_i \psi / \|\Lambda_i \psi\|_2$.

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- So when we return p with size-1 overlap, it comes with a random sign.

Application: Quantum Advantage for Counting Triangles in the Stream

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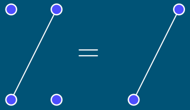

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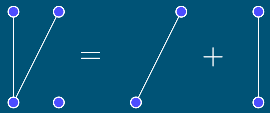

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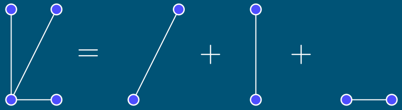

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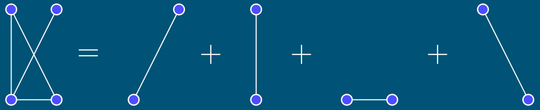

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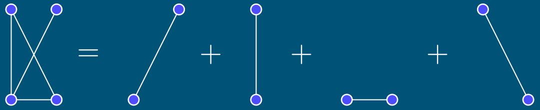

We will be concerned with *space complexity*.

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With this sketch we obtain quantum advantage for single-pass streaming problems of independent interest.

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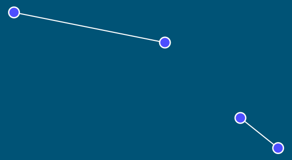

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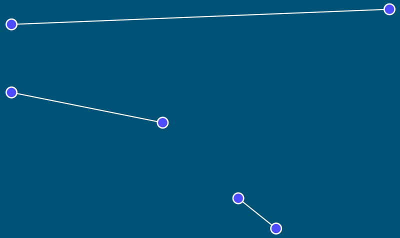

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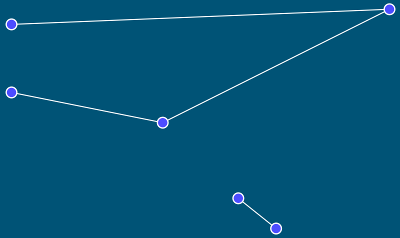

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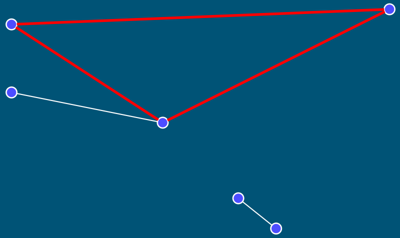

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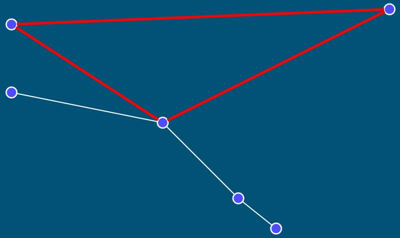

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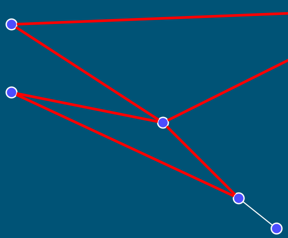

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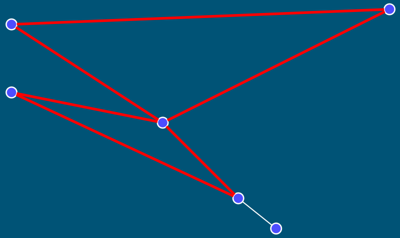

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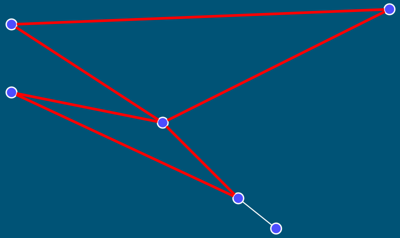

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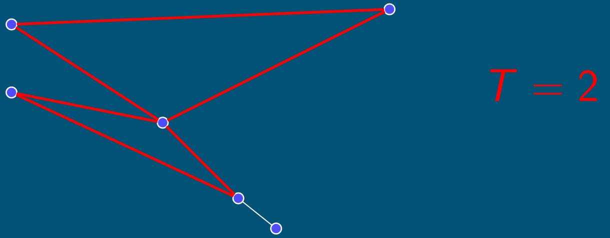

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- For our purposes: assume $\Omega(m)$ edge-disjoint triangles in a $\Theta(m)$-edge graph.

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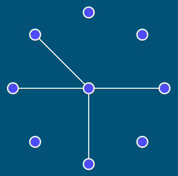

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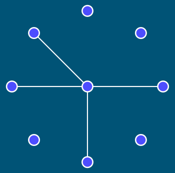

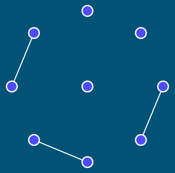

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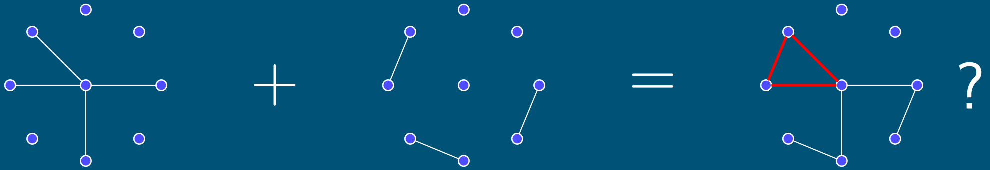

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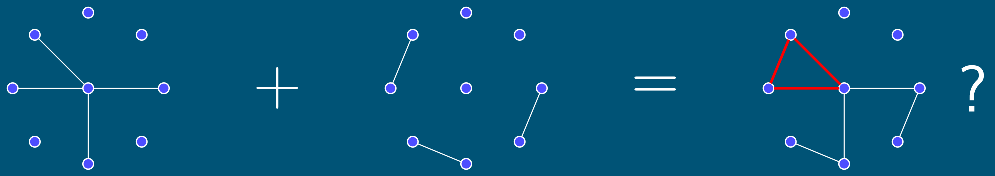

- Intuitively: if we keep k star edges, we have a $\sim T \times \left(\frac{k}{m}\right)^2 = \Theta\left(\frac{k^2}{m}\right)$ chance of getting both edges of at least one of $T = \Theta(m)$ triangles, so need $k = \Omega(\sqrt{m})$.

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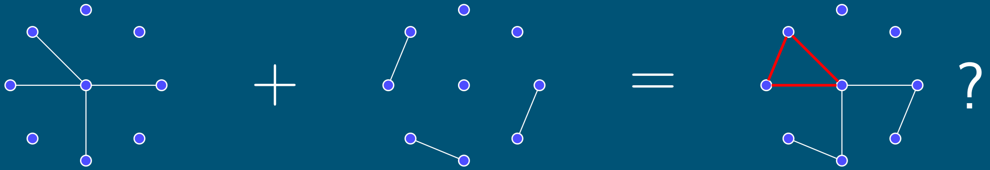

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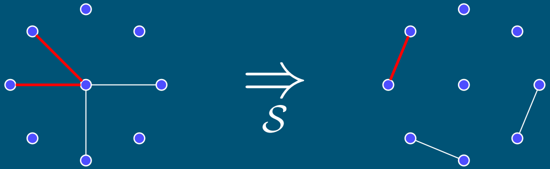

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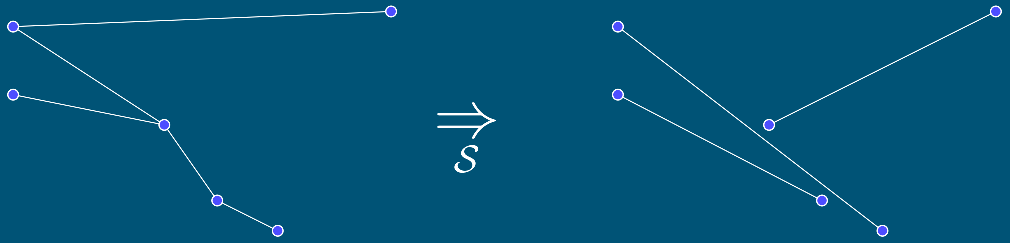
- Implies exponential quantum algorithm for the two-player version of this *specific* instance of triangle counting.

Sketching General Graphs

Generalizing to a general triangle-free graph (V, G) followed by a matching M is simple: make S the set of all edges in the first graph, and set $P = \{(uv, uw) : u \in V, vw \in M\}$.



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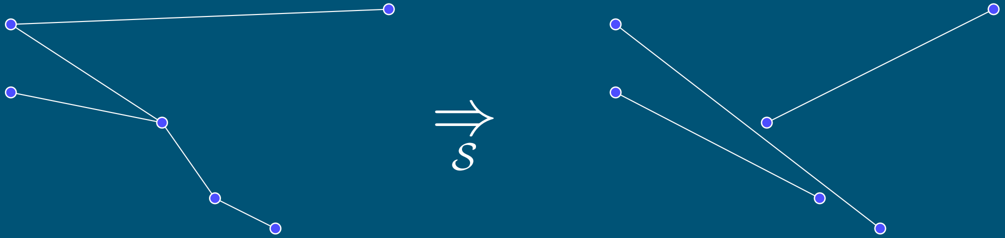

Problems arise when the triangle-completing edges don't form a matching.

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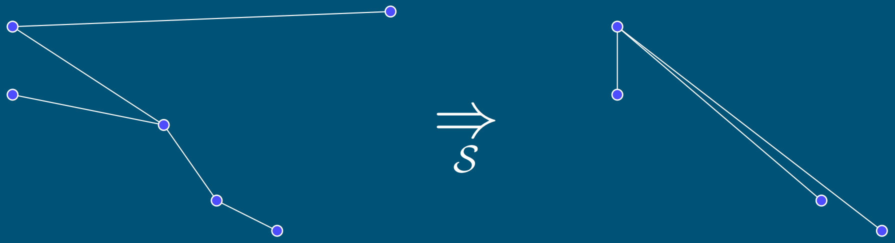
Our sketch doesn't work for non-matching edges because they don't give us disjoint pairs.



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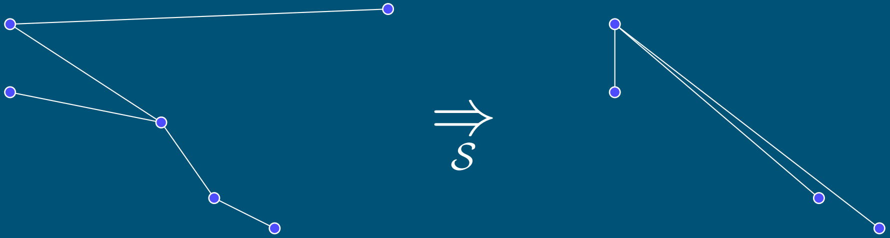

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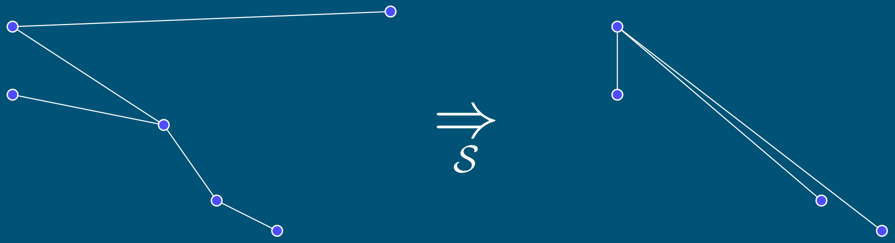

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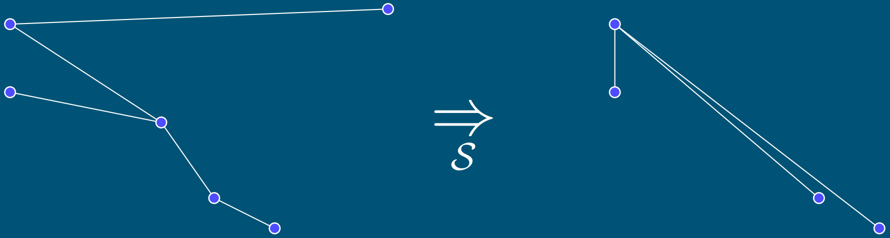

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- Worst-case instances turn out to be "classically easy".
- Interpolation between classical and quantum estimators then allows a $O\left(n^{2/5}\right)$ v. $\Omega(\sqrt{n})$ quantum space advantage.

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Excecuting a full query is then equivalent to doing a partial query for each $p \in P$ (in any order).

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This gives us quantum space advantage for counting triangles in the stream.

Theorem (Informal, K. '21)

There is a $\widetilde{O}\left(m^{2/5}\right)$-qubit streaming algorithm for counting triangles in the stream. ($\Omega(\sqrt{m})$ classically)

Interlude: Implementing the Streaming Properties of the Sketch

Swapping

To implement our swap operation, we need one more quantum primitive.

Unitary Evolution

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- One such transformation is swapping the basis elements $|x\rangle$ and $|y\rangle$.
- As our sketch is $\frac{1}{\sqrt{|S|}} \sum_{x \in S} |x\rangle$, this will do nothing if neither or both of x and y are in S, and swap them if exactly one is.

Partial Queries

We can implement our partial query operation using the measurement postulate we already have.

Projective Measurement

Using labeled projectors (Λ_i) s.t. $\sum_i \Lambda_i = I$, measure state ψ. With probability $\|\Lambda_i \psi\|_2^2$, get result i and transform ψ to $\Lambda_i \psi / \|\Lambda_i \psi\|_2$.

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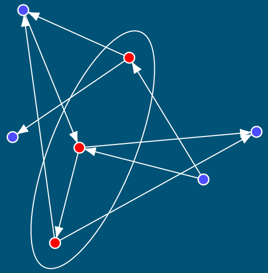
- If we do *not* return one of them, the superposition is projected onto the space orthogonal to $\mathrm{span}(|x\rangle, |y\rangle)$, i.e. x and y are deleted from S.

Application: Exponential Advantage for Maximum Directed Cut

Maximum Directed Cut

Given a *directed* graph, what is the maximum, over partitions $V_0 \sqcap V_1 = V$, number of edges $\overrightarrow{uv}$ such that $u \in V_0$ and $v \in V_1$?



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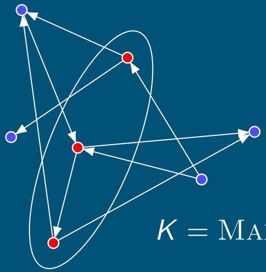

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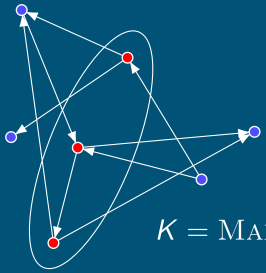

$K = \textsc{Max-DiCut}(G) = 5$

We want to output an α-approximation in the minimum amount of space, i.e. $K' \in [\alpha K, K]$, using as little space as possible.

Complexity of Approximating Max-DiCut

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- [**K.**, Parekh, Voronova] This can be achieved in polylog(n) space with our quantum sketch.

(First Order) Bias Histogram

The *bias* of a vertex v is $\frac{d_v^{\text{out}} - d_v^{\text{in}}}{d_v}$.

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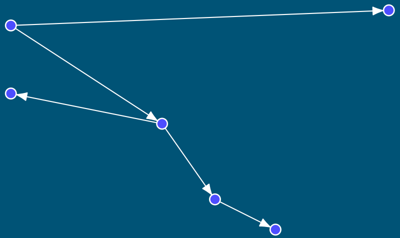
		Head Bias			
		$-1 \leq b < -0.5$	$-0.5 \leq b < 0$	$0 \leq b < 0.5$	$0.5 \leq b \leq 1$
Tail Bias	$-0.5 \leq b < 0$	1245	2333	2974	9309
	$0 \leq b < 0.5$	9361	8421	82	66
	$0 \leq b < 0.5$	955	2133	5369	621
	$0.5 \leq b \leq 1$	3530	5312	4789	8472

Given a partition of (B_i) of $[-1, 1]$ by thresholds, we want to know how many edges there are from B_i to B_j for each i, j.

Estimating the Bias Histogram

Return to the two-player setting: Alice and Bob both have graphs, and Alice wants to send Bob a message that will let him estimate their joint bias histogram.



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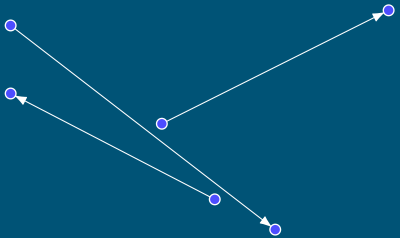

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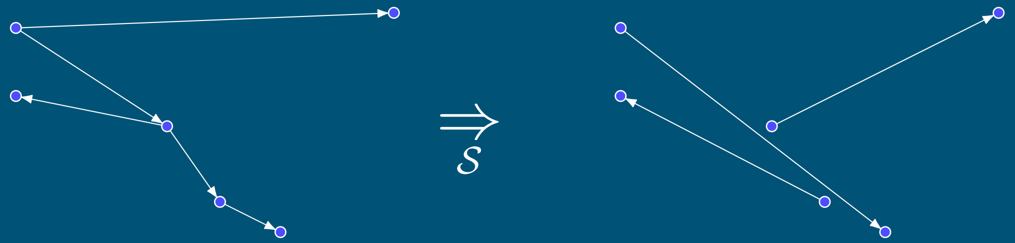

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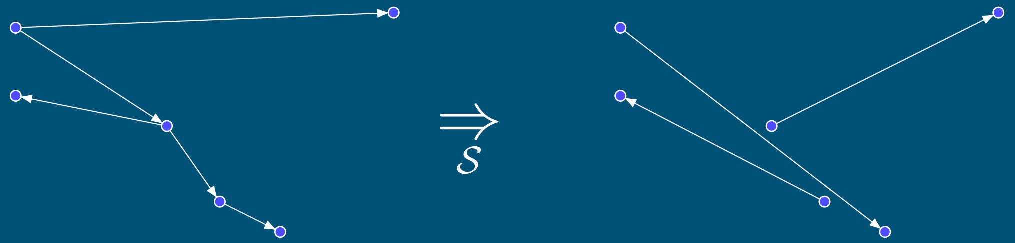
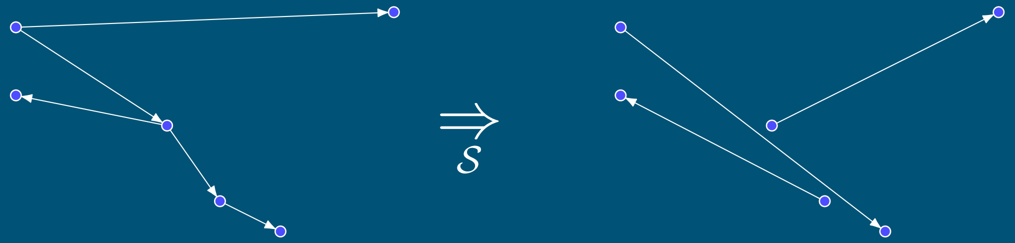


- Alice's edges are easy: she can sample edges and send them to Bob with their endpoint out- and in- degrees.
- So the challenge is sampling head-tail degree pairs from Alice's graph that correspond to edges in Bob's graph.

Sketching for the Bias Histogram

We can use our sketch to sample labeled pairs of Alice's vertices corresponding to Bob's edges, allowing him to calculate the biases of the endpoints of a sample of his edges with $O(\log(n))$ qubits of communication from Alice.



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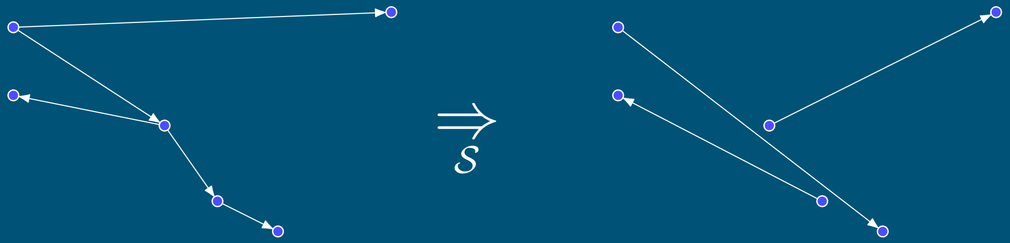

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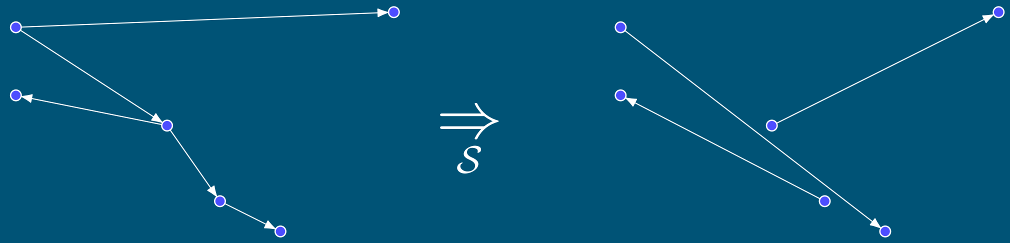

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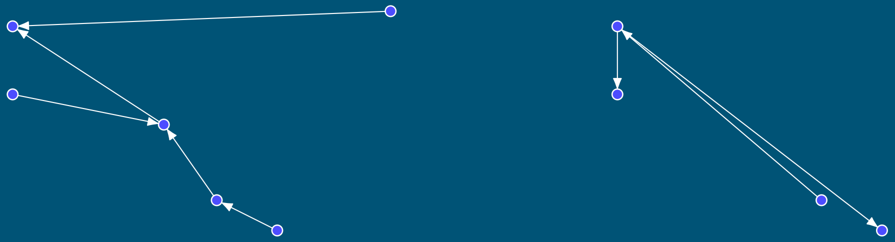
Works if he has a matching. But what if he doesn't?

Copying

Suppose Alice knew how many edges Bob had incident to each of her vertices.



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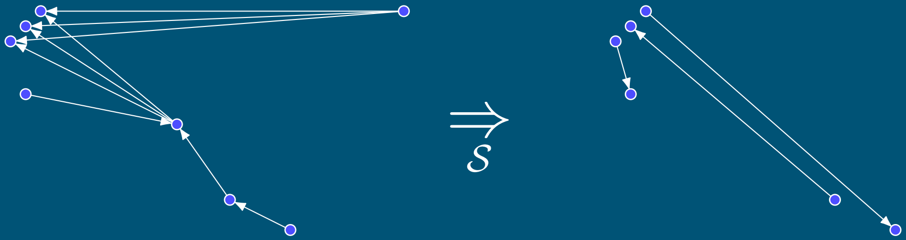

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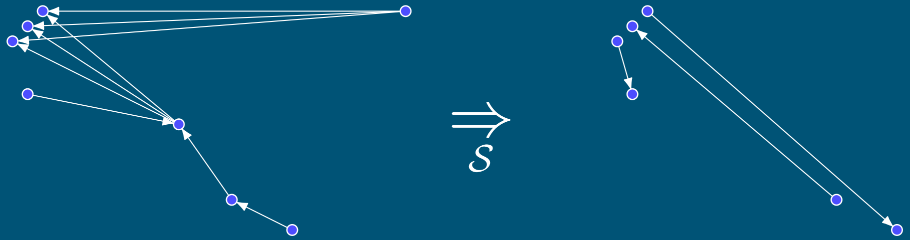

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- She will still put at most $m = |E_{\text{Bob}}|$ elements total in her sketch set S, and so the needed property of $|p \in P : p \subseteq S| = \Omega(|S|)$ is preserved.

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- We can therefore combine classical and quantum sampling to estimate the bias histogram in polylog(n) space.

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- Use classical sampling to estimate edge counts for edges $\overrightarrow{uv}$ such that b_u or b_v are dominated by edges that arrive *after* $\overrightarrow{uv}$.

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- Now S contains $(v, 0) \ldots (v, d_v)$, and we can then query $((u, d_1), (v, d_2))$ if we want to sample restricted to edges with endpoint degrees at least (d_1, d_2).
- Actual algorithm is complicated by need to track in- and out-degrees separately.

Exponential Quantum Advantage for Maximum Directed Cut

Theorem (Informal, K., Parekh, Voronova)

There is a $\operatorname{polylog}(n)$ *space quantum streaming algorithm that* 0.4835*-approximates the* $\textsc{Max-DiCut}$ *value of a graph.*

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Contrasts with the undirected problem, where no quantum advantage is possible for any approximation ratio [Kapralov, Krachun '19], [**K.**, Parekh '22].

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- Can we characterize which CSPs admit quantum space advantage in the stream?