Sampling in Query Evaluation

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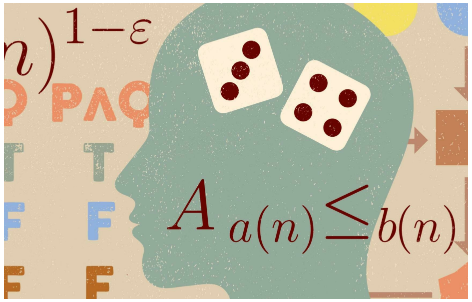


The goals of this tutorial

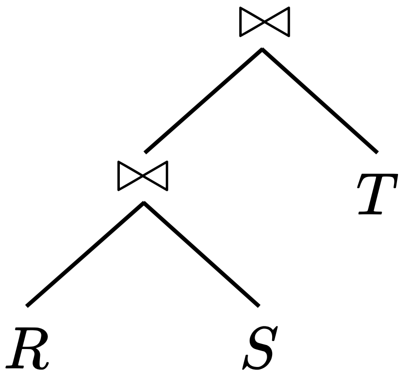
- Show some fundamental problems that motivate the use of sampling in databases
- Explain the difficulties behind these problems
- Show some tools that are used to do sampling in this context
- Explain how these tools can be used to provide (partial) solutions to these problems
- Convince the audience that there are interesting open problems in the area, and also that sampling tools could be very useful 🙂

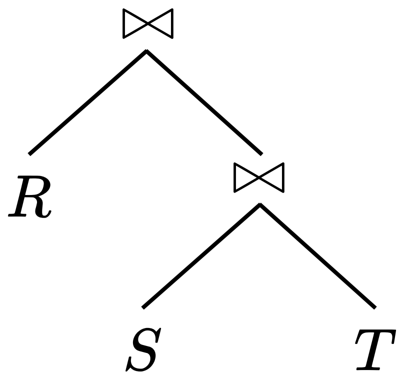
Motivation: Three related problems

Problem 1: query optimization

The task is to compute $R[A, B] \bowtie S[B, C] \bowtie T[C, D]$

$(R \bowtie S) \bowtie T$ $\qquad$ $R \bowtie (S \bowtie T)$ $\qquad$ $(R \bowtie T) \bowtie S$



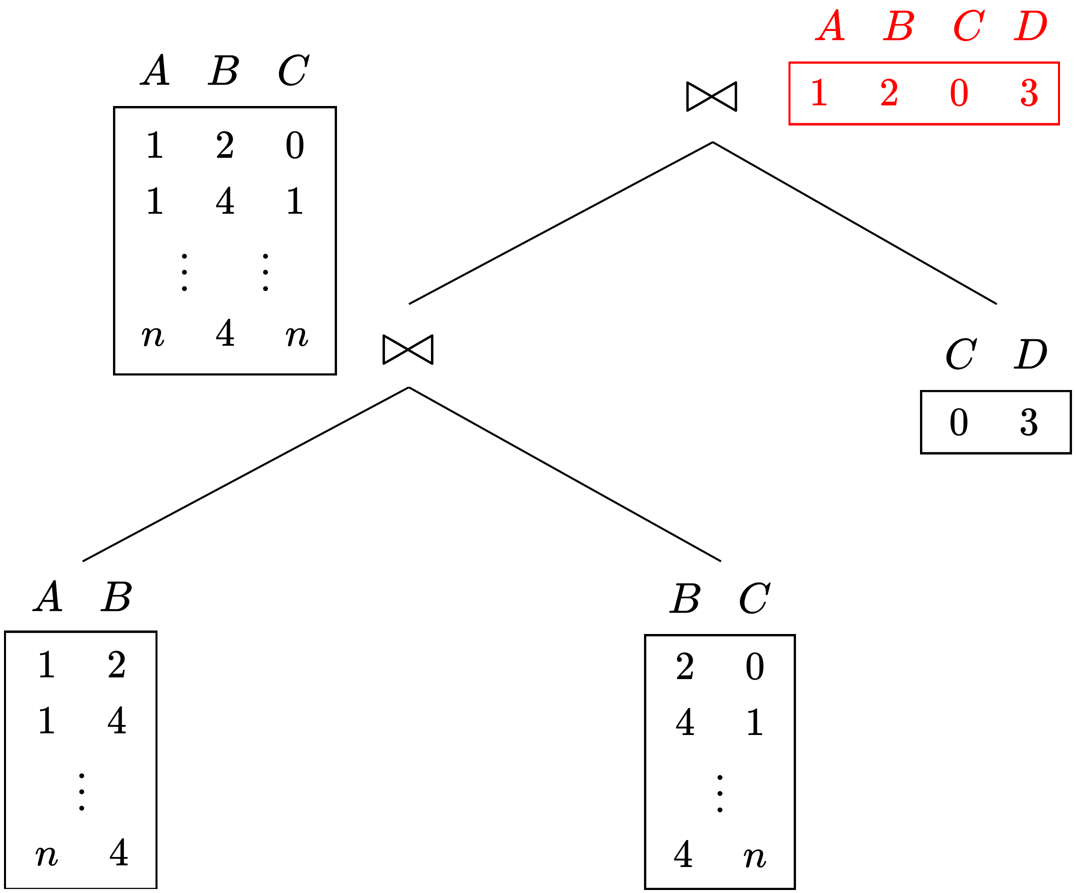


$$(R \bowtie S) \bowtie T$$

R	A	B
	1	2
	1	4
	$\vdots$	
	n	4

S	B	C
	2	0
	4	1
	$\vdots$	
	4	n

T	C	D
	0	3

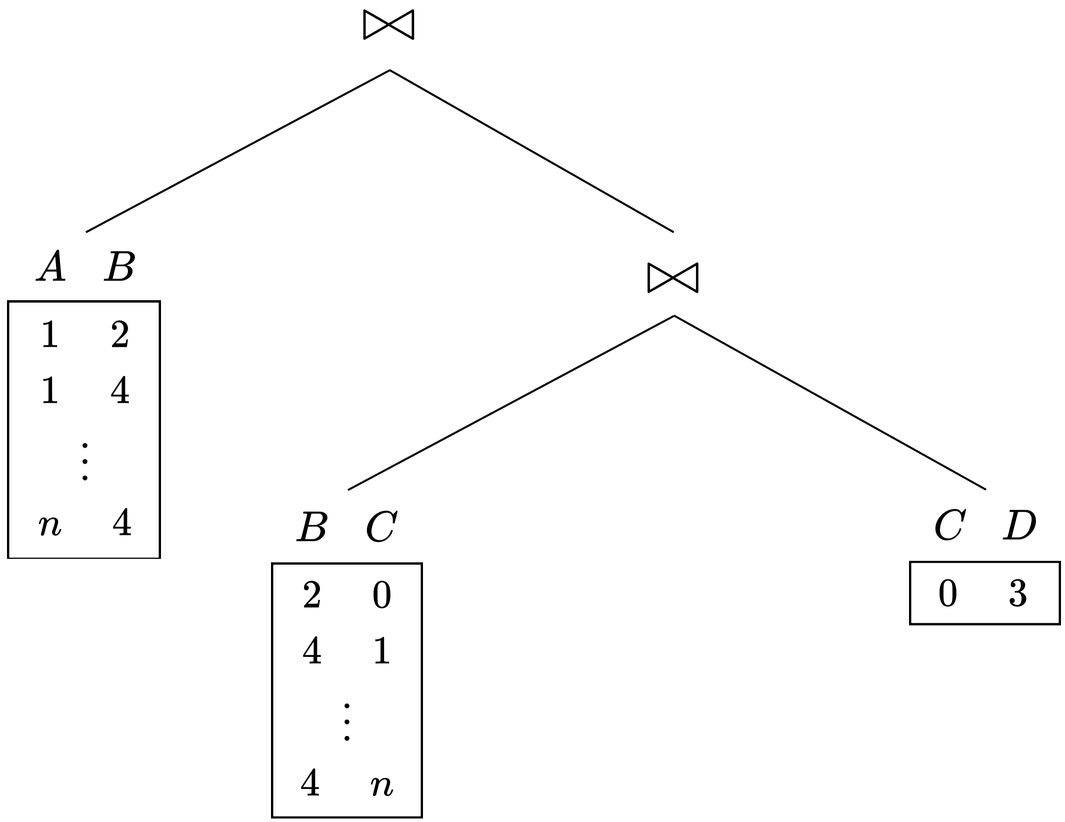


$$R \bowtie (S \bowtie T)$$

R	A	B
	1	2
	1	4
	$\vdots$	
	n	4

S	B	C
	2	0
	4	1
	$\vdots$	
	4	n

T	C	D
	0	3

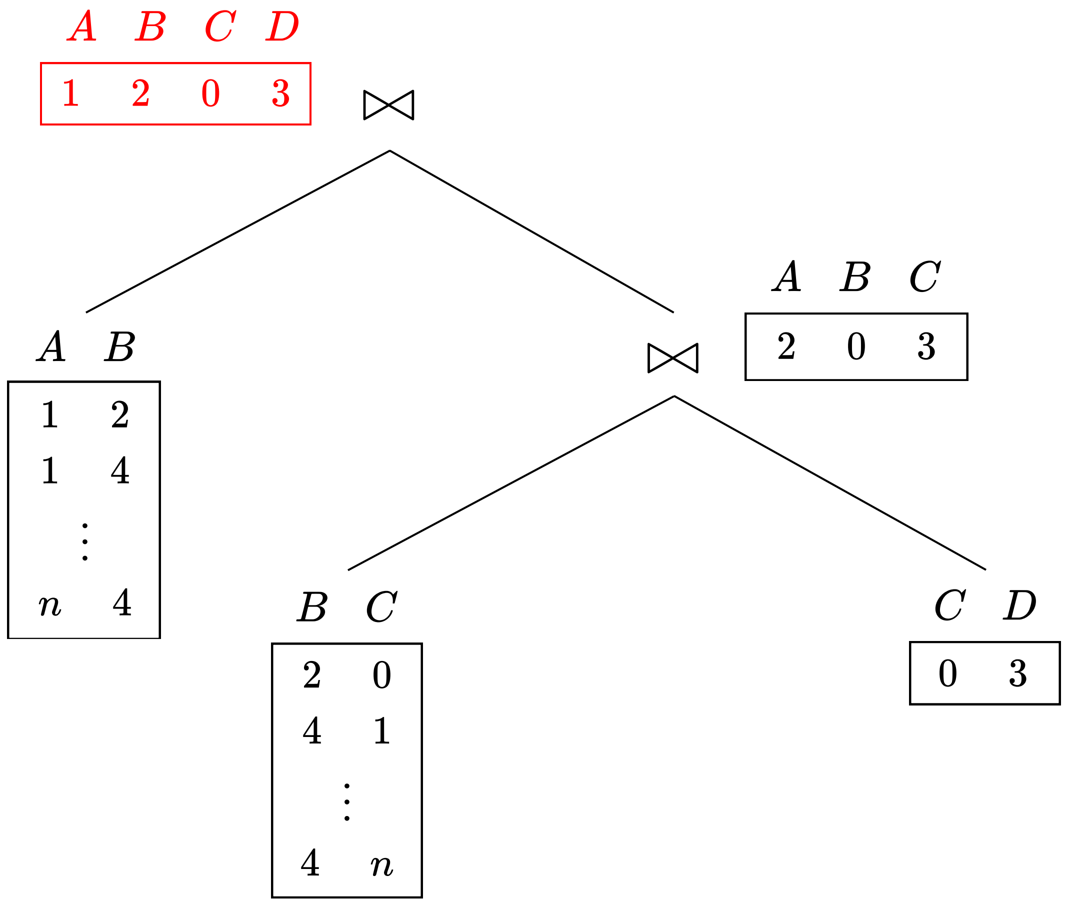


$$R \bowtie (S \bowtie T)$$

R	A	B
	1	2
	1	4
	$\vdots$	
	n	4

S	B	C
	2	0
	4	1
	$\vdots$	
	4	n

T	C	D
	0	3

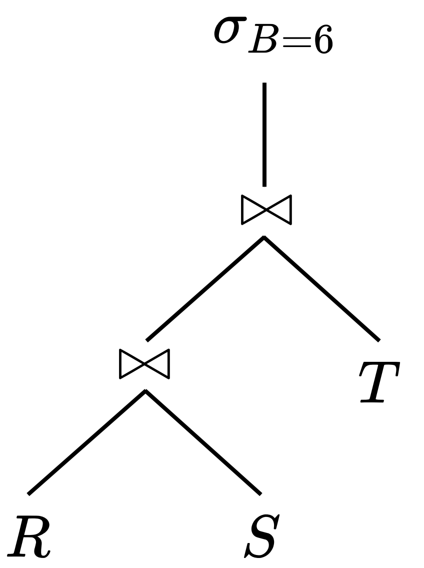


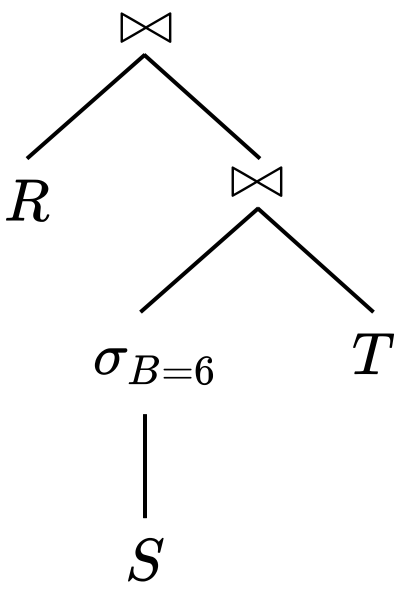
Query optimization

Now the task is to compute $\sigma_{B=4}(R[A,B] \bowtie S[B,C] \bowtie T[C,D])$

$\sigma_{B=4}((R \bowtie S) \bowtie T)$

$R \bowtie (\sigma_{B=4}(S) \bowtie T)$



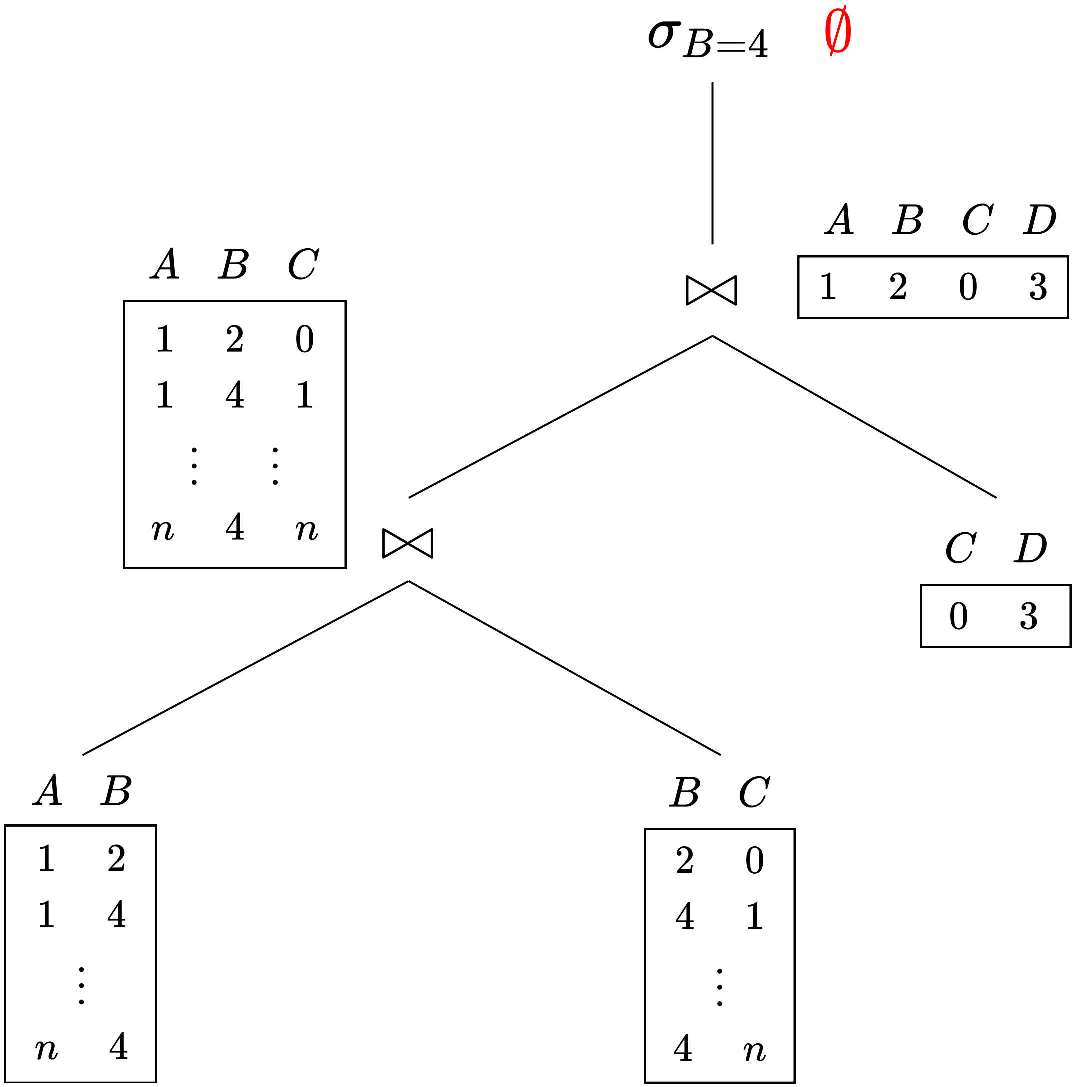


$$\sigma_{B=4}((R \bowtie S) \bowtie T)$$

R	A	B
	1	2
	1	4
	$\vdots$	
	n	4

S	B	C
	2	0
	4	1
	$\vdots$	
	4	n

T	C	D
	0	3

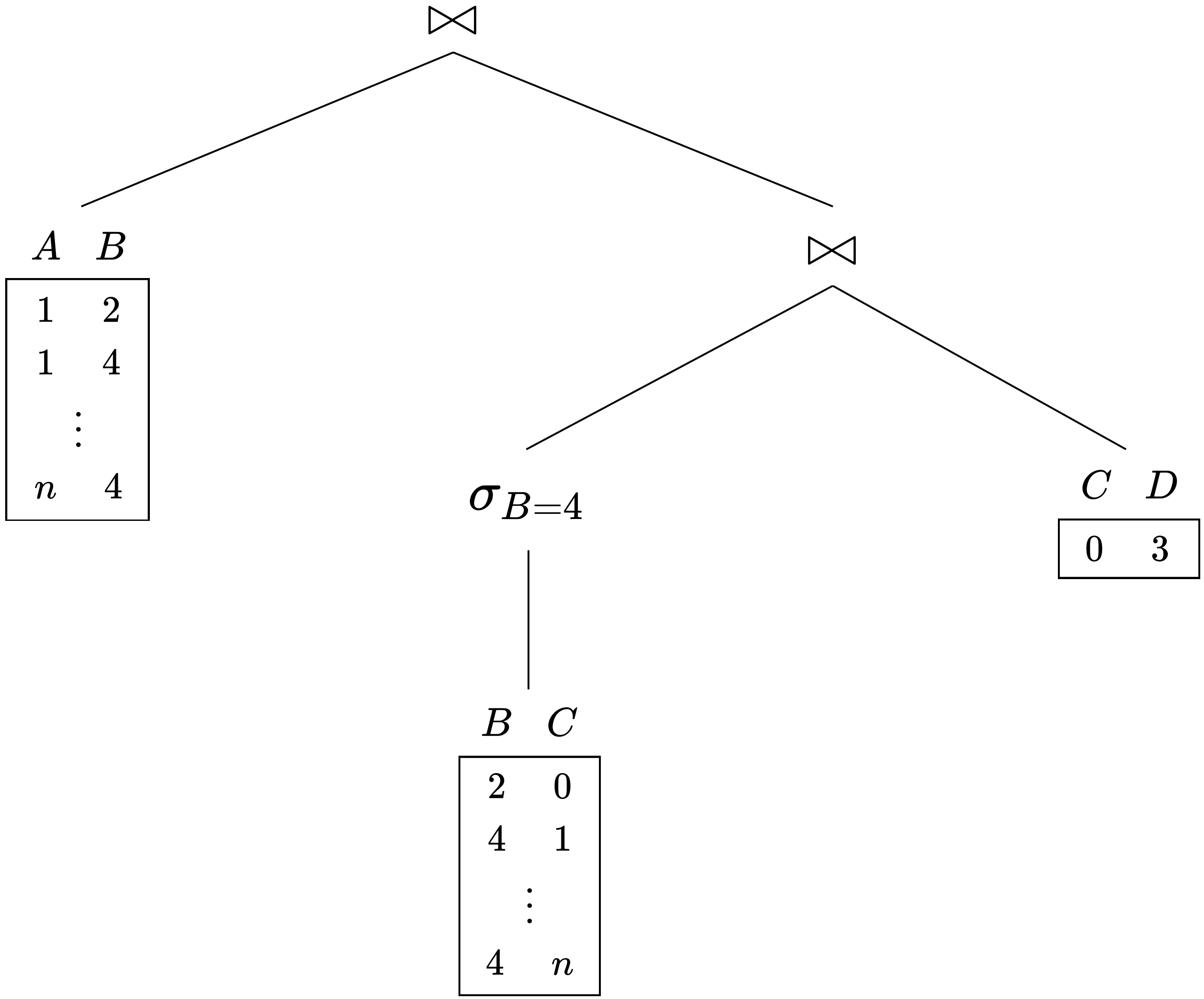


$$R \bowtie (\sigma_{B=4}(S) \bowtie T)$$

R	A	B
	1	2
	1	4
	$\vdots$	
	n	4

S	B	C
	2	0
	4	1
	$\vdots$	
	4	n

T	C	D
	0	3

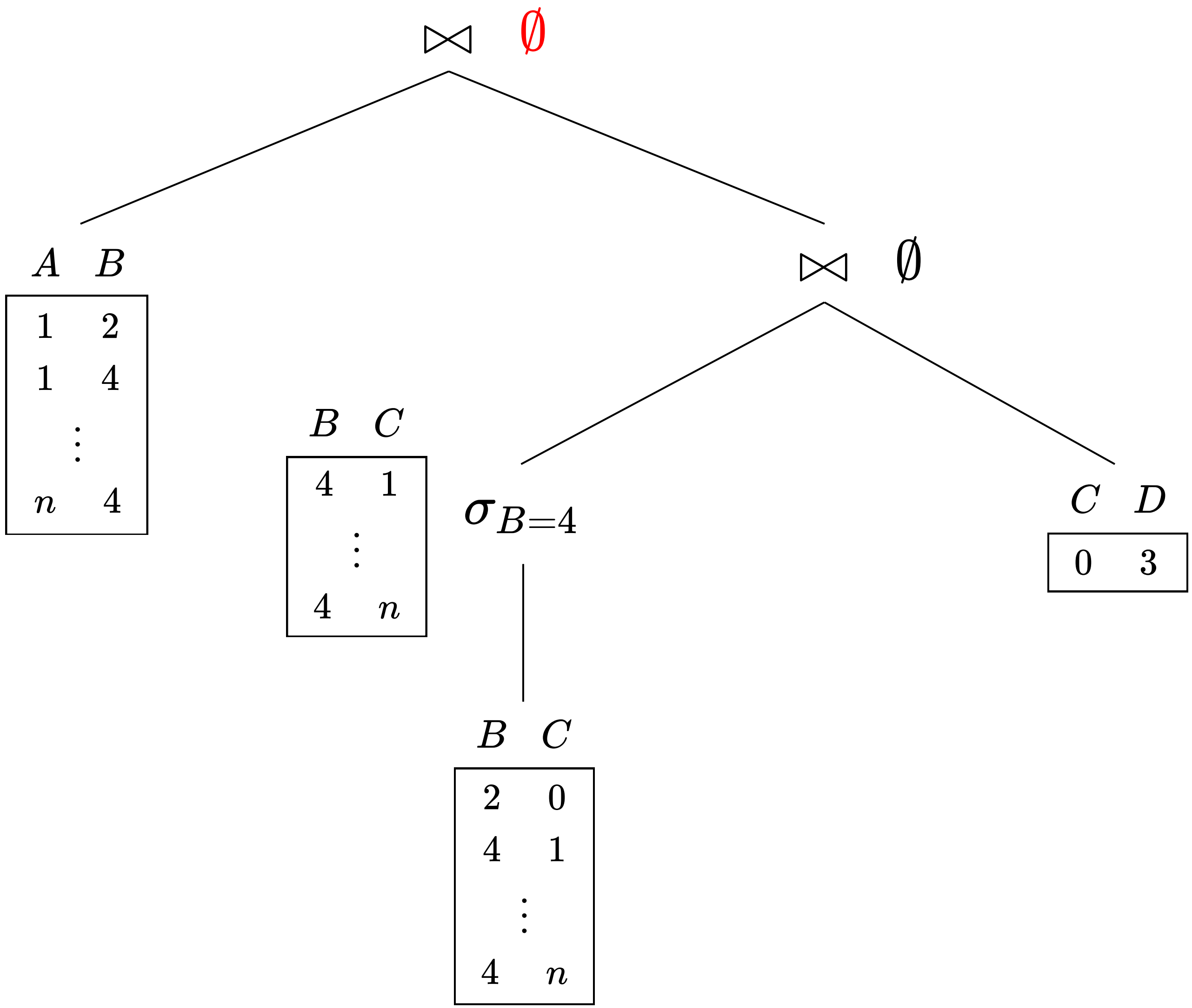


$$R \bowtie (\sigma_{B=4}(S) \bowtie T)$$

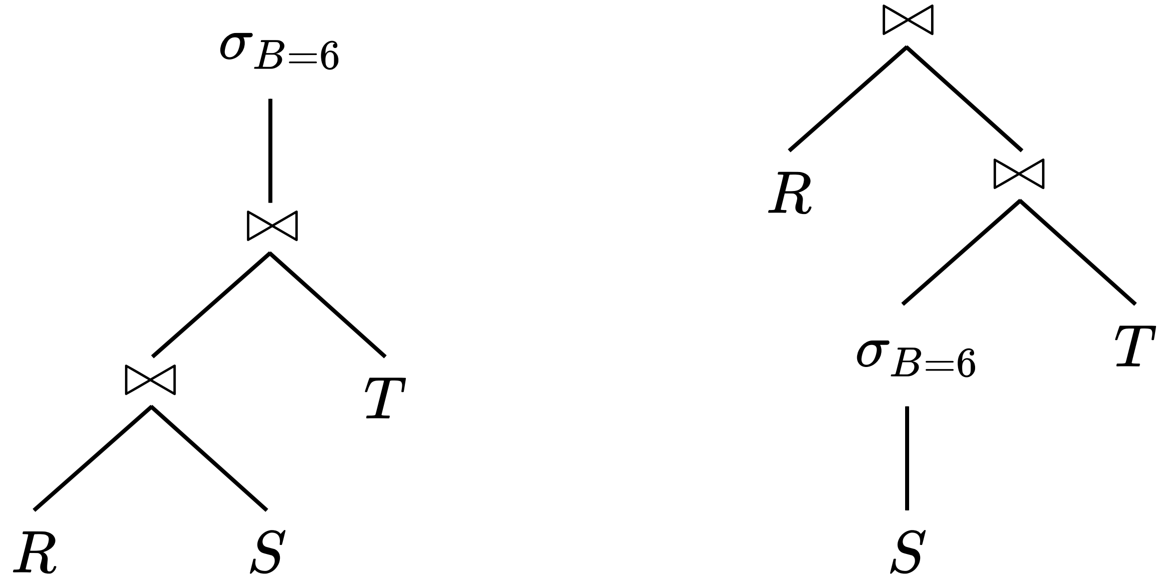
R	A	B
	1	2
	1	4
		$\vdots$
	n	4

S	B	C
	2	0
	4	1
		$\vdots$
	4	n

T	C	D
	0	3



Cardinality estimation



To compare query plans we need estimations of the cardinalities of the intermediate results

- Such estimations should be computed (very) efficiently

Problem 2: approximate query processing [HHW97,HH99]

The task is to compute the aggregate query $\mathrm{COUNT}(R[A,B] \bowtie S[B,C] \bowtie T[C,D])$

Not a good strategy to solve this task by first computing $R[A,B] \bowtie S[B,C] \bowtie T[C,D]$

- We can approximate the answer by doing a cardinality estimation

Problem 2: approximate query processing [HHW97,HH99]

Can we also approximate $\mathrm{SUM}_D(R[A,B] \bowtie S[B,C] \bowtie T[C,D])$ and $\mathrm{AVG}_A(R[A,B] \bowtie S[B,C] \bowtie T[C,D])$?

What kind of guarantees can be offered about the results of these approximations?

- How can such guarantees be obtained?

Problem 3: query exploration

The answer to a query can be very large

It can be more informative to:

- Return the number of answers
- Enumerate the answers with polynomial (constant) delay
- Generate an answer uniformly at random

Problem 3: query exploration

Returning the number of answers to a query can be solved again by using cardinality estimation

Problem 3: query exploration

Returning the number of answers to a query can be solved again by using cardinality estimation

Cardinality estimation can also help to generate at random an answer to a query

- Can we sample with uniform distribution?
- Can sampling be used for cardinality estimation?

What do these problems have in common?

Sampling plays a central role in the development of solutions for these problems

The complexity of counting and uniform generation

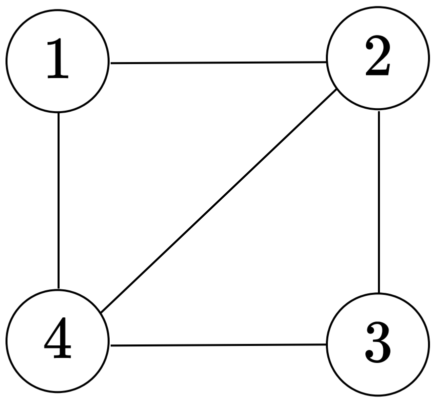
Hardness of counting

The problem of counting the number of answers to a join query is #P-complete

This can be easily shown by reducing from the problem of counting the number of 3-colorings of a graph

Hardness of counting

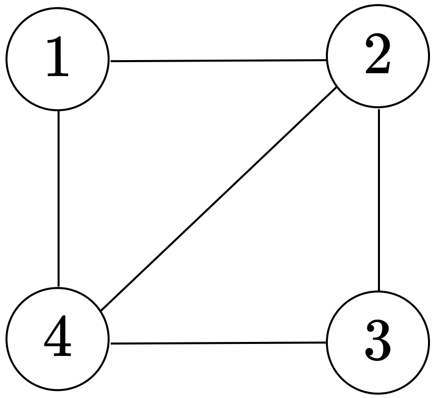
The problem of counting the number of answers to a join query is #P-complete



$$E(x_1, x_2) \wedge E(x_2, x_3) \wedge E(x_3, x_4) \wedge E(x_4, x_1) \wedge E(x_4, x_2)$$

Hardness of counting

The problem of counting the number of answers to a join query is #P-complete



$$Q(x_1, x_2, x_3, x_4) = E(x_1, x_2) \wedge E(x_2, x_3) \wedge E(x_3, x_4) \wedge E(x_4, x_1) \wedge E(x_4, x_2)$$

Hardness of counting

The problem of counting the number of answers to a join query is #P-complete

$$Q(x_1, x_2, x_3, x_4) = E(x_1, x_2) \wedge E(x_2, x_3)$$

$$\wedge E(x_3, x_4) \wedge E(x_4, x_1) \wedge$$

$$E(x_4, x_2)$$

E	
1	2
1	3
2	1
2	3
3	1
3	2

Number of 3-colorings: $|Q(E)|$

Hardness of uniform generation

There is no randomized polynomial-time algorithm for uniform generation of the answers to a join query (unless NP = RP)

If such an algorithm exists, then there exists an FPRAS for the problem of counting the number of answers to a join query (by Jerrum-Valiant-Vazirani)

Then there exists a BPP algorithm problem of verifying whether a join query has a non-empty set of answers

Hardness of uniform generation

There is no randomized polynomial-time algorithm for uniform generation of the answers to a join query (unless NP = RP)

Then there exists a BPP algorithm problem of verifying whether a join query has a non-empty set of answers

But the problem of verifying whether a join query has a non-empty set of answers is NP-complete

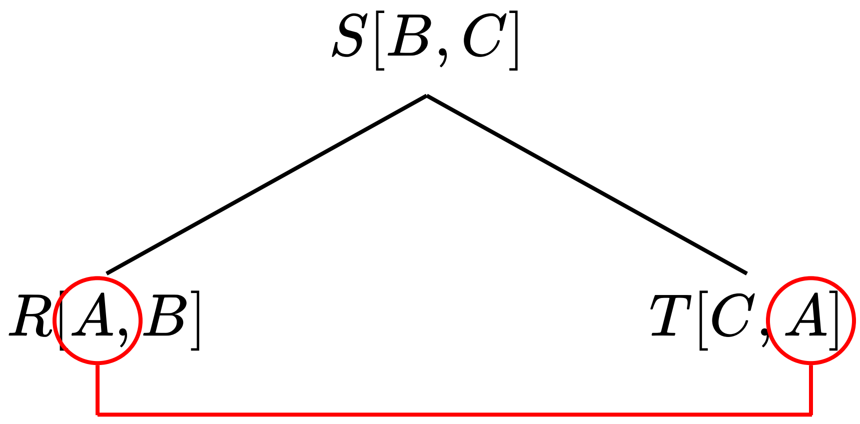
How can we get better complexity?

Consider acyclic queries

- Or a class of queries with a bounded degree of acyclicity, such as bounded treewidth or bounded hypertree width

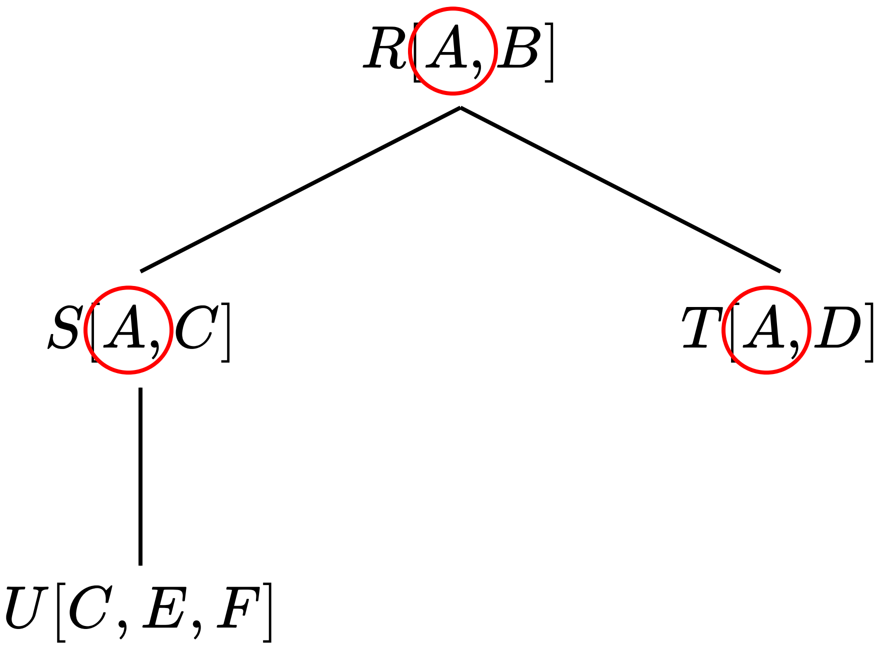
Counting in the acyclic case

$$R[A,B] \cancel{\bowtie S[B,C] \bowtie} T[C,A]$$



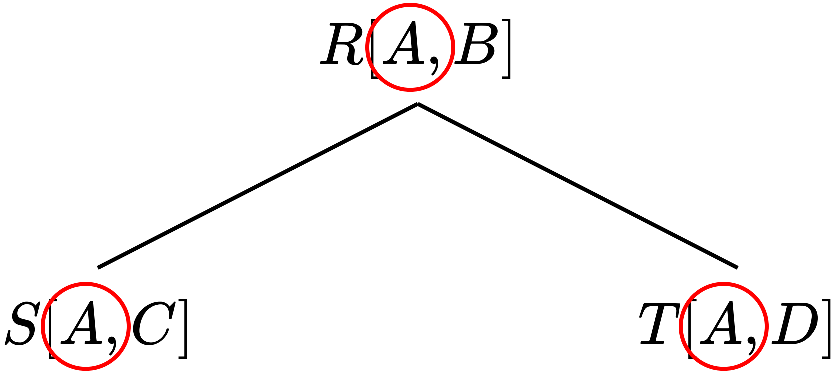
Counting in the acyclic case

$$R[A,B] \bowtie S[A,C] \bowtie T[A,D] \bowtie U[C,E,F]$$

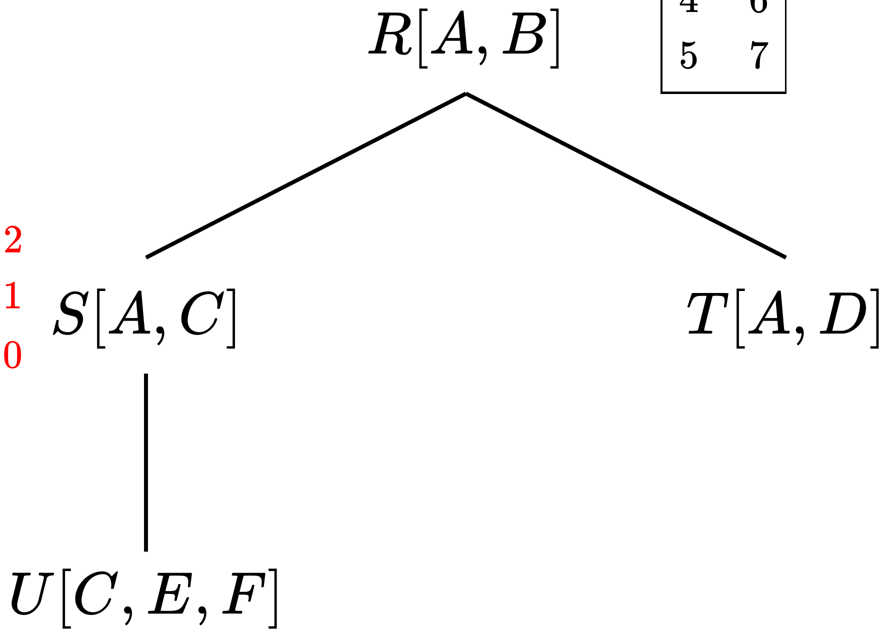


Counting in the acyclic case

$$R[A,B] \bowtie S[A,C] \bowtie T[A,D] \bowtie U[C,E,F]$$



Counting in the acyclic case



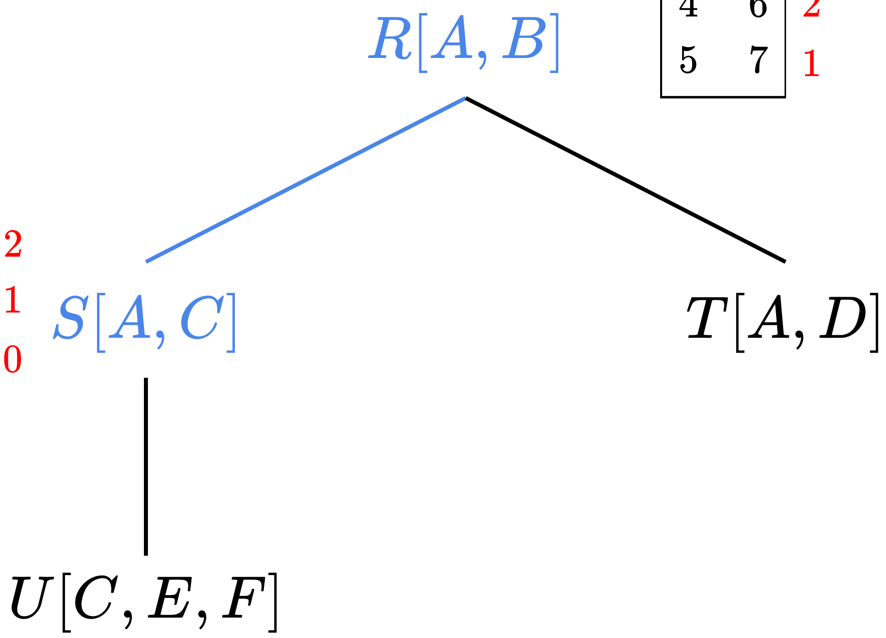
R	A	B
	4	6
	5	7

S	A	C	
	4	1	2
	5	2	1
	4	3	0

T	A	D	
	4	1	1
	4	2	1
	4	3	1
	5	4	1
	5	5	1

U	C	E	F	
	1	3	6	1
	1	4	7	1
	2	5	8	1

Counting in the acyclic case



R	A	B	
	4	6	2
	5	7	1

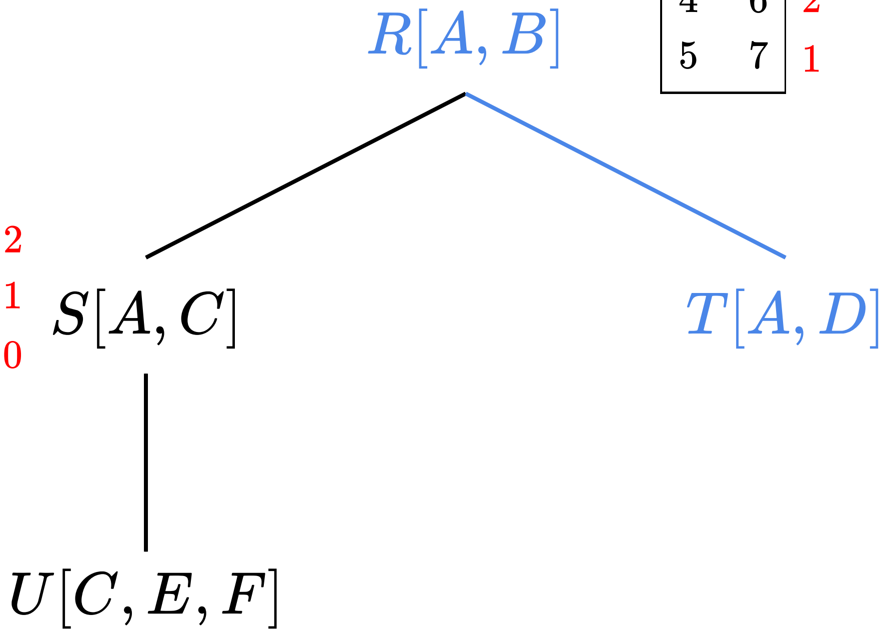
S	A	C	
	4	1	2
	5	2	1
	4	3	0

T	A	D	
	4	1	1
	4	2	1
	4	3	1
	5	4	1
	5	5	1

U	C	E	F	
	1	3	6	1
	1	4	7	1
	2	5	8	1

Counting in the acyclic case

R	A	B	
	4	6	2
	5	7	1



S	A	C	
	4	1	2
	5	2	1
	4	3	0

T	A	D	
	4	1	1
	4	2	1
	4	3	1
	5	4	1
	5	5	1

U	C	E	F	
	1	3	6	1
	1	4	7	1
	2	5	8	1

Counting in the acyclic case

$$\begin{tikzcd}
 & R[A,B] \arrow[dl, no head] \arrow[dr, no head] & \\
S[A,C] \arrow[d, no head] & & T[A,D] \\
U[C,E,F] & &
\end{tikzcd}$$

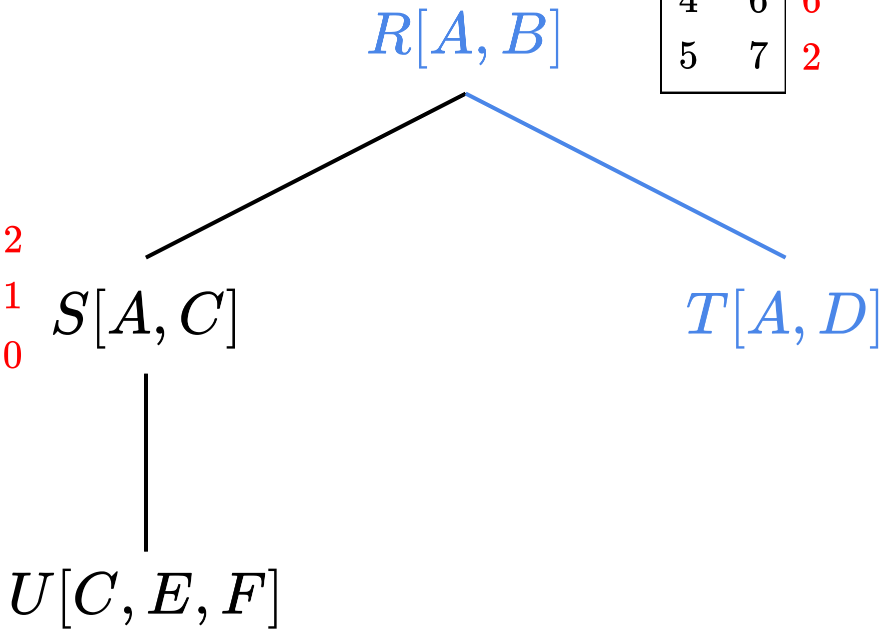
R	A	B	
	4	6	$2 \cdot 3$
	5	7	$1 \cdot 2$

S	A	C	
	4	1	2
	5	2	1
	4	3	0

T	A	D	
	4	1	1
	4	2	1
	4	3	1
	5	4	1
	5	5	1

U	C	E	F	
	1	3	6	1
	1	4	7	1
	2	5	8	1

Counting in the acyclic case



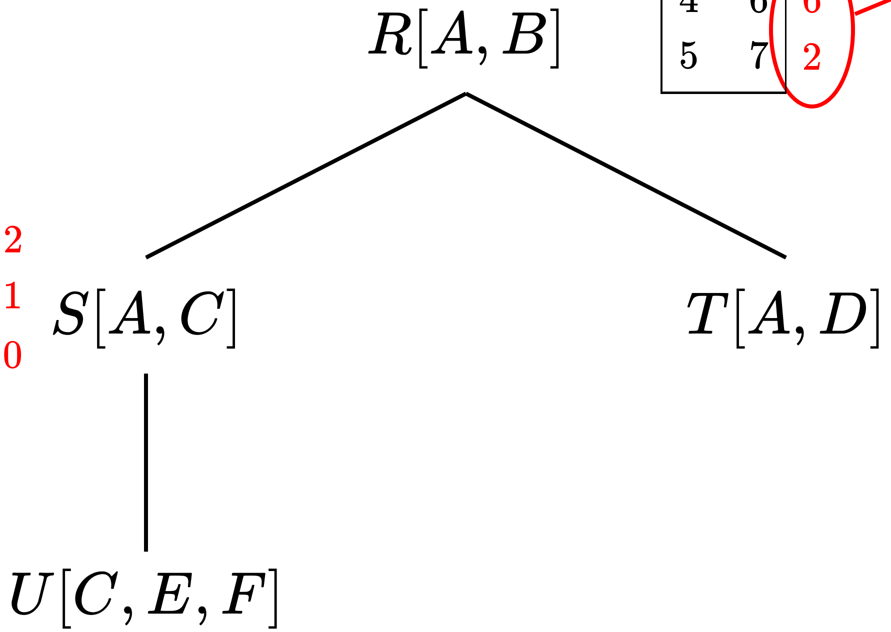
R	A	B	
	4	6	6
	5	7	2

S	A	C	
	4	1	2
	5	2	1
	4	3	0

T	A	D	
	4	1	1
	4	2	1
	4	3	1
	5	4	1
	5	5	1

U	C	E	F	
	1	3	6	1
	1	4	7	1
	2	5	8	1

Counting in the acyclic case



R	A	B	
	4	6	6
	5	7	2

8

S	A	C	
	4	1	2
	5	2	1
	4	3	0

T	A	D	
	4	1	1
	4	2	1
	4	3	1
	5	4	1
	5	5	1

U	C	E	F	
	1	3	6	1
	1	4	7	1
	2	5	8	1

Uniform generation in the acyclic case

$R[A, B]$

R	A	B	
	4	6	6
	5	7	2

$S[A, C]$

S	A	C	
	4	1	2
	5	2	1
	4	3	0

$T[A, D]$

T	A	D	
	4	1	1
	4	2	1
	4	3	1
	5	4	1
	5	5	1

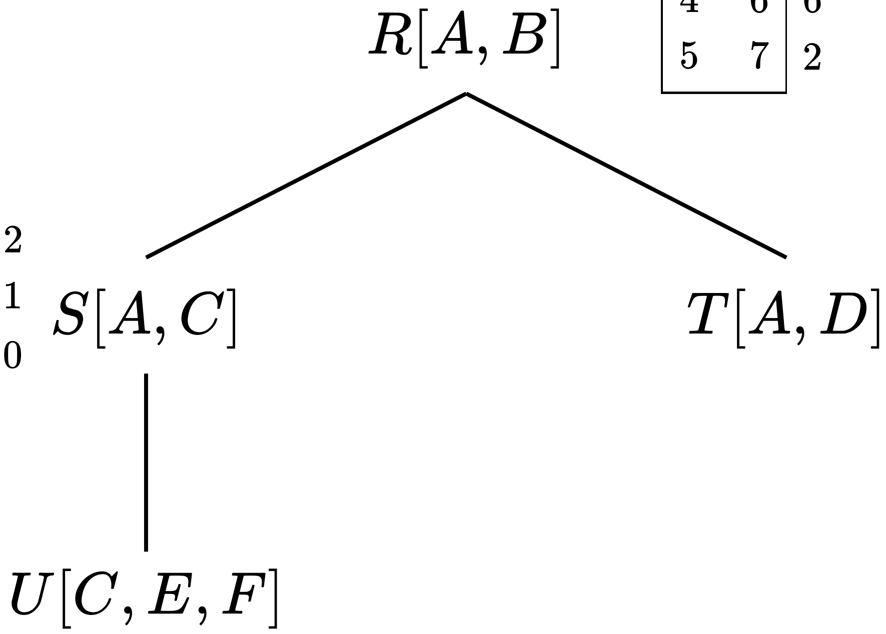
$U[C, E, F]$

U	C	E	F	
	1	3	6	1
	1	4	7	1
	2	5	8	1

Uniform generation in the acyclic case

A	B	C	D	E	F

R	A	B	
	4	6	6
	5	7	2



S	A	C	
	4	1	2
	5	2	1
	4	3	0

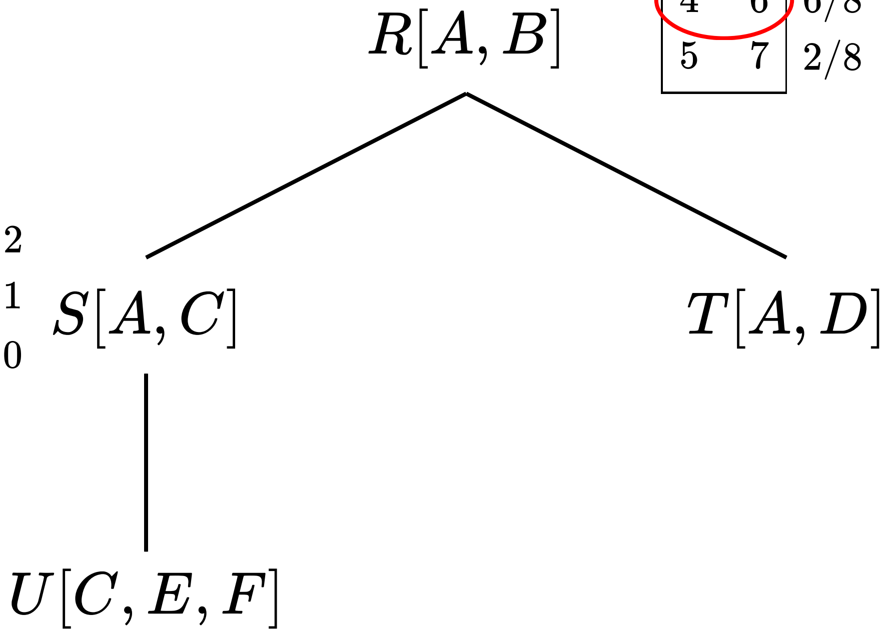
T	A	D	
	4	1	1
	4	2	1
	4	3	1
	5	4	1
	5	5	1

U	C	E	F	
	1	3	6	1
	1	4	7	1
	2	5	8	1

Uniform generation in the acyclic case

A	B	C	D	E	F

R	A	B	
	4	6	6/8
	5	7	2/8



S	A	C	
	4	1	2
	5	2	1
	4	3	0

T	A	D	
	4	1	1
	4	2	1
	4	3	1
	5	4	1
	5	5	1

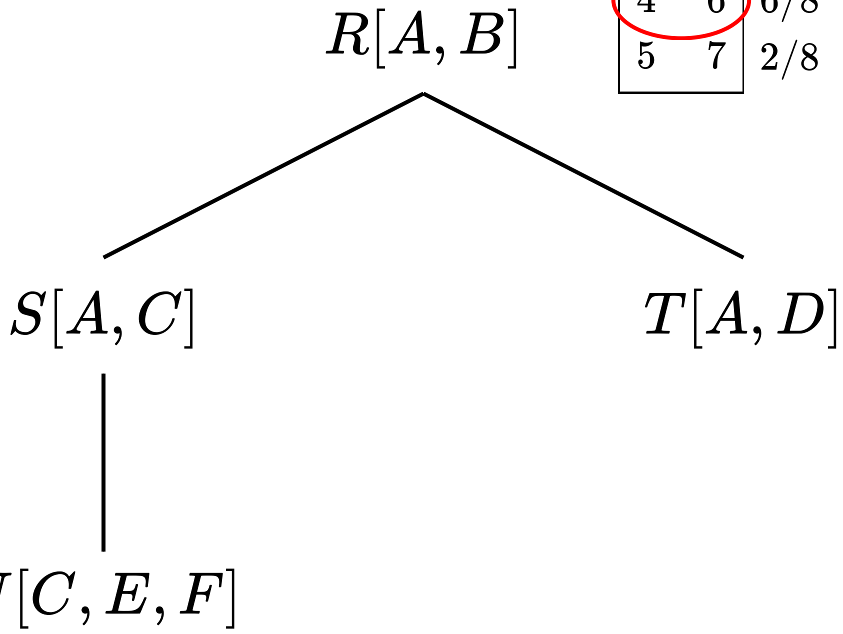
U	C	E	F	
	1	3	6	1
	1	4	7	1
	2	5	8	1

Uniform generation in the acyclic case

A	B	C	D	E	F
4	6				

$\frac{3}{4}\cdot$

R	A	B	
	4	6	6/8
	5	7	2/8



S	A	C	
	4	1	2
	5	2	1
	4	3	0

T	A	D	
	4	1	1
	4	2	1
	4	3	1
	5	4	1
	5	5	1

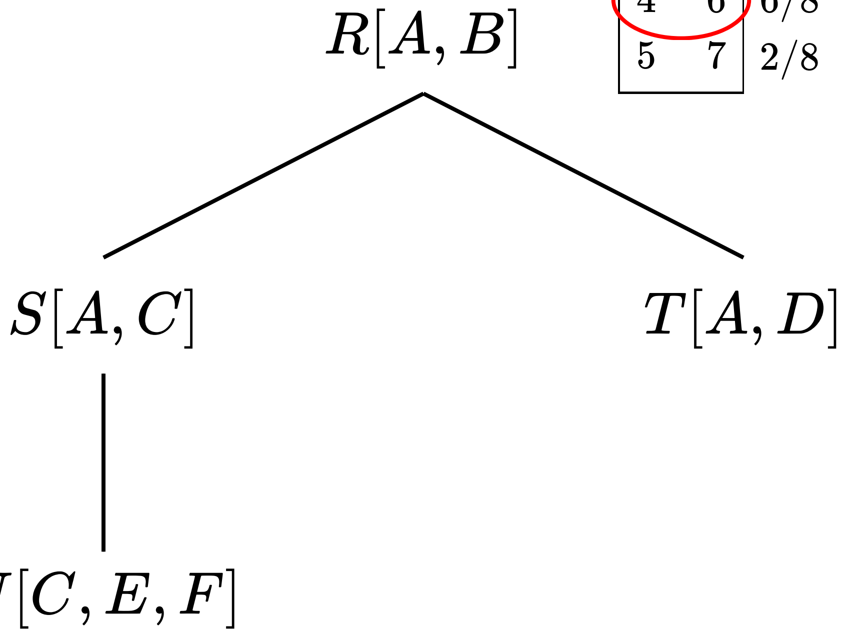
U	C	E	F	
	1	3	6	1
	1	4	7	1
	2	5	8	1

Uniform generation in the acyclic case

A	B	C	D	E	F
4	6				

$\frac{3}{4}\cdot$

R	A	B	
	4	6	6/8
	5	7	2/8



S	A	C	
	4	1	2/2
	4	3	0/2

T	A	D	
	4	1	1
	4	2	1
	4	3	1
	5	4	1
	5	5	1

U	C	E	F	
	1	3	6	1
	1	4	7	1
	2	5	8	1

Uniform generation in the acyclic case

A	B	C	D	E	F
4	6	1			

$\frac{3}{4} \cdot 1 \cdot$

R	A	B	
	4	6	6/8
	5	7	2/8

$R[A, B]$

S	A	C	
	4	1	2/2
	4	3	0/2

$S[A, C]$

$T[A, D]$

T	A	D	
	4	1	1
	4	2	1
	4	3	1
	5	4	1
	5	5	1

U	C	E	F	
	1	3	6	1
	1	4	7	1
	2	5	8	1

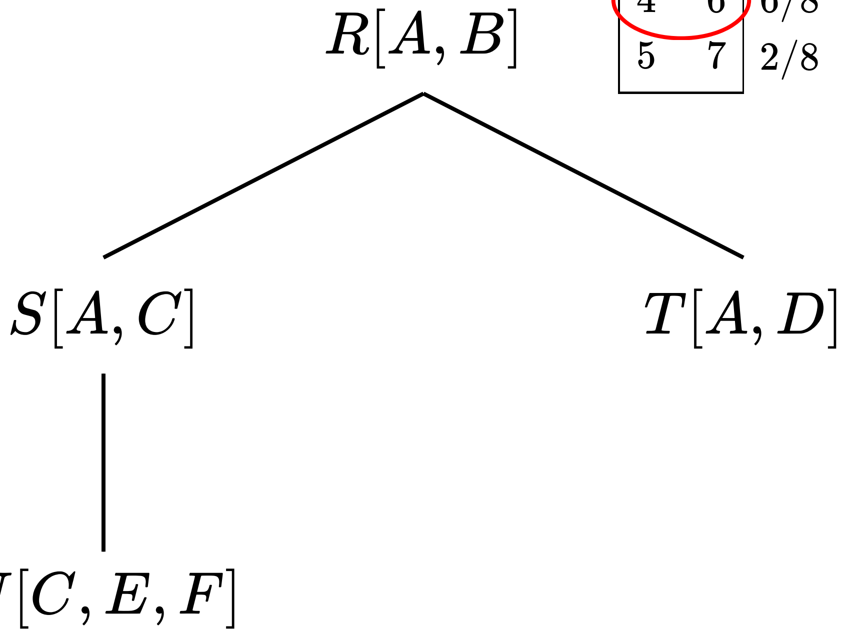
$U[C, E, F]$

Uniform generation in the acyclic case

A	B	C	D	E	F
4	6	1			

$\frac{3}{4} \cdot 1 \cdot$

R	A	B	
	4	6	$6/8$
	5	7	$2/8$



S	A	C	
	4	1	$2/2$
	4	3	$0/2$

T	A	D	
	4	1	1
	4	2	1
	4	3	1
	5	4	1
	5	5	1

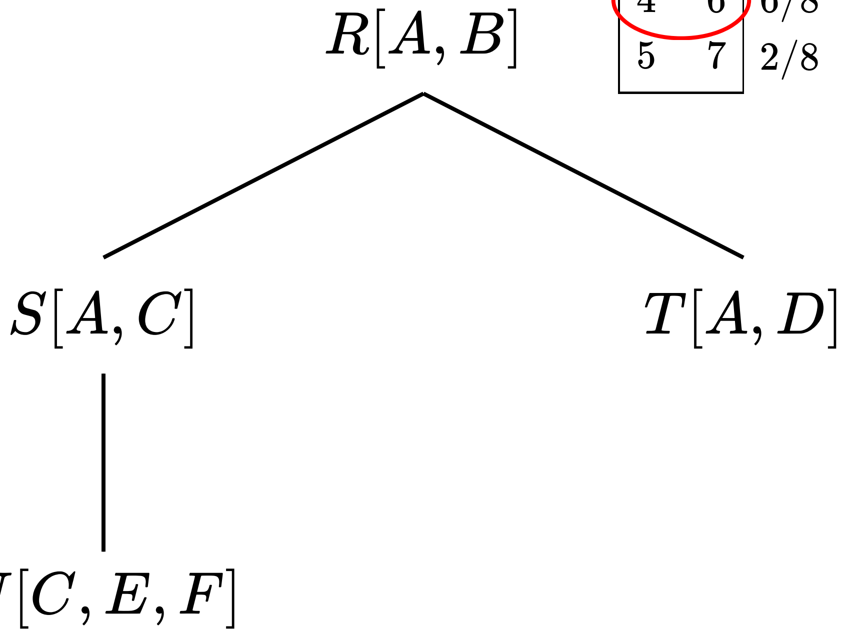
U	C	E	F	
	1	3	6	$1/2$
	1	4	7	$1/2$

Uniform generation in the acyclic case

A	B	C	D	E	F
4	6	1		4	7

$$\frac{3}{4} \cdot 1 \cdot \frac{1}{2} \cdot$$

R	A	B	
	4	6	$6/8$
	5	7	$2/8$



S	A	C	
	4	1	$2/2$
	4	3	$0/2$

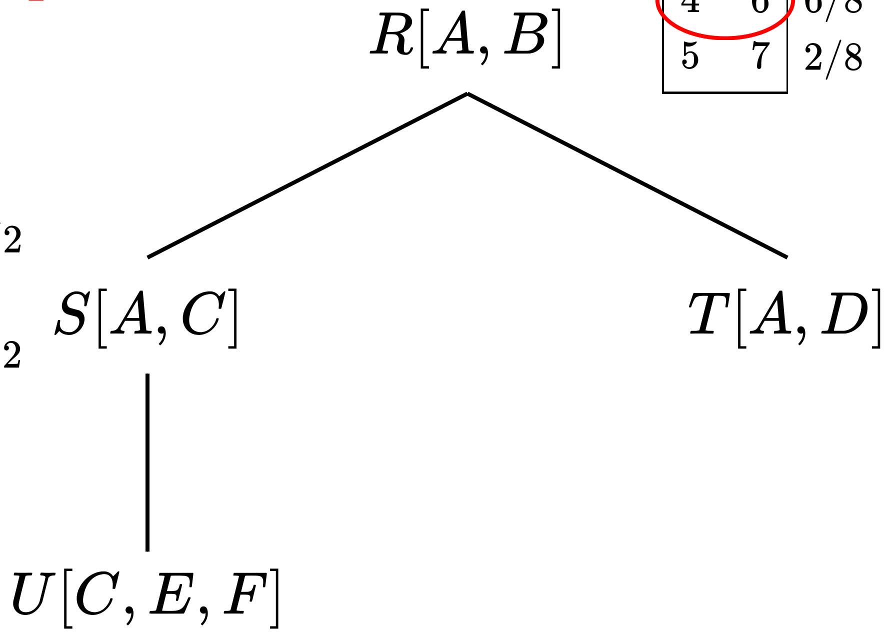
T	A	D	
	4	1	1
	4	2	1
	4	3	1
	5	4	1
	5	5	1

U	C	E	F	
	1	3	6	$1/2$
	1	4	7	$1/2$

Uniform generation in the acyclic case

A	B	C	D	E	F
4	6	1		4	7

$\frac{3}{4} \cdot 1 \cdot \frac{1}{2} \cdot$



R	A	B	
	4	6	6/8
	5	7	2/8

S	A	C	
	4	1	2/2
	4	3	0/2

T	A	D	
	4	1	1/3
	4	2	1/3
	4	3	1/3

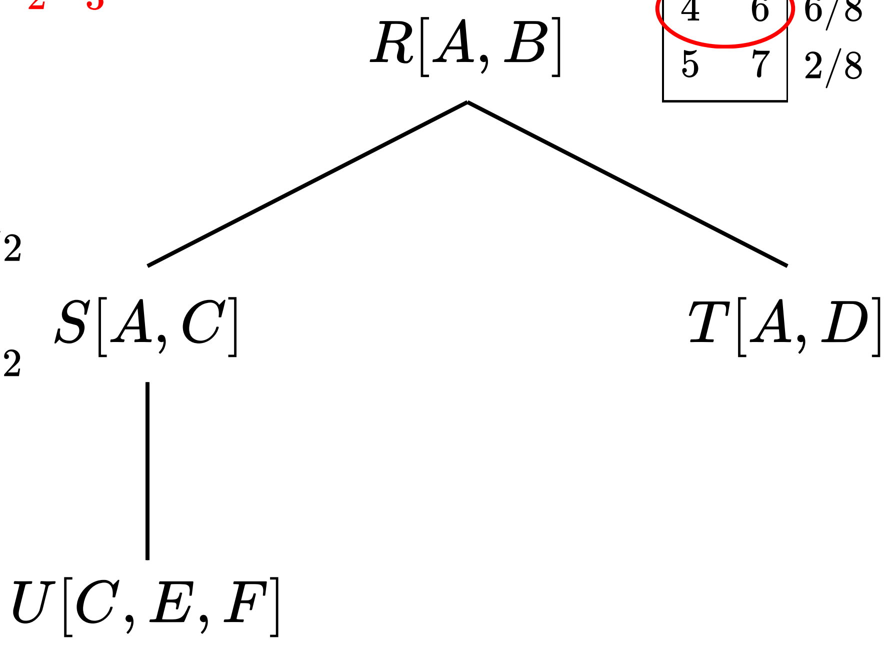
U	C	E	F	
	1	3	6	1/2
	1	4	7	1/2

Uniform generation in the acyclic case

A	B	C	D	E	F
4	6	1	2	4	7

$$\frac{3}{4} \cdot 1 \cdot \frac{1}{2} \cdot \frac{1}{3}$$

R	A	B	
	4	6	$6/8$
	5	7	$2/8$



S	A	C	
	4	1	$2/2$
	4	3	$0/2$

T	A	D	
	4	1	$1/3$
	4	2	$1/3$
	4	3	$1/3$

U	C	E	F	
	1	3	6	$1/2$
	1	4	7	$1/2$

Uniform generation in the acyclic case

A	B	C	D	E	F
4	6	1	2	4	7

$$\frac{3}{4} \cdot 1 \cdot \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{8}$$

$R[A,B]$

R	A	B	
	4	6	$6/8$
	5	7	$2/8$

$S[A,C]$

S	A	C	
	4	1	$2/2$
	4	3	$0/2$

$T[A,D]$

T	A	D	
	4	1	$1/3$
	4	2	$1/3$
	4	3	$1/3$

$U[C,E,F]$

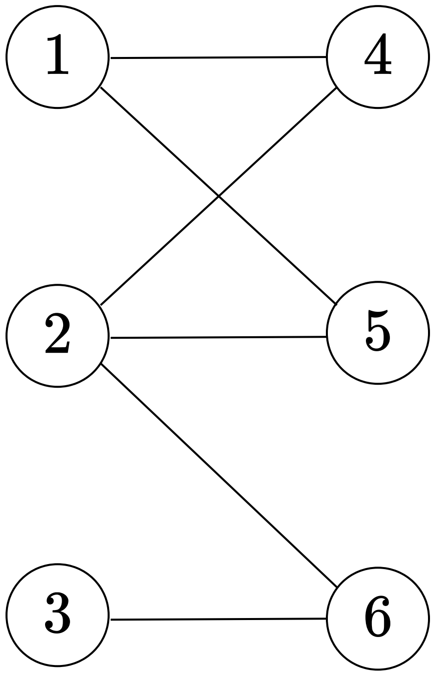
U	C	E	F	
	1	3	6	$1/2$
	1	4	7	$1/2$

Does this work with other operators?

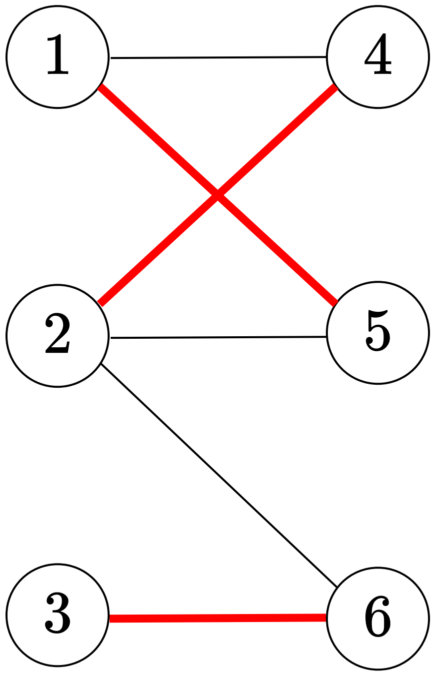
The previous approach for acyclic queries can be extended to consider the selection operator σ

But it does not work if the projection operator π is included

Hardness of counting with projection [PS13]

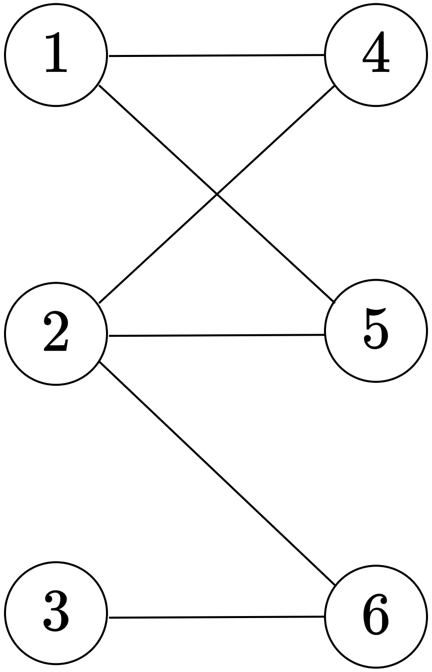


Hardness of counting with projection [PS13]



The problem of counting the number of perfect matchings in a bipartite graph is #P-complete

Hardness of counting with projection [PS13]



I_1

4
5

I_2

4
5
6

I_3

6

D

4	5
4	6
5	4
5	6
6	4
6	5

Hardness of counting with projection [PS13]

I_1
4
5

I_2
4
5
6

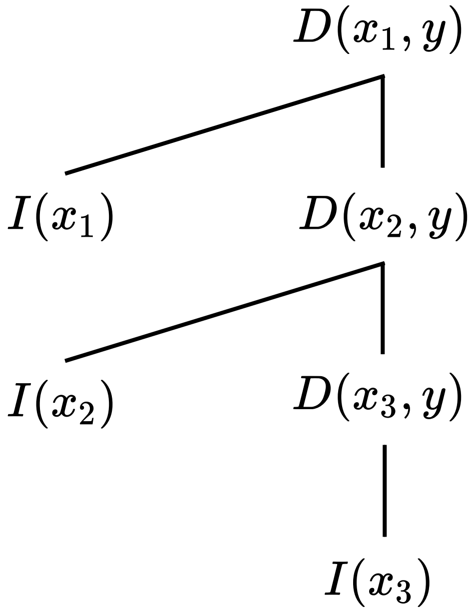
I_3
6

D	
4	5
4	6
5	4
5	6
6	4
6	5

$$F(x_1, x_2, x_3) = I_1(x_1) \wedge I_2(x_2) \wedge I_3(x_3)$$

$$W(x_1, x_2, x_3) = I_1(x_1) \wedge I_2(x_2) \wedge I_3(x_3) \wedge \exists y\, (D(x_1, y) \wedge D(x_2, y) \wedge D(x_3, y))$$

Hardness of counting with projection [PS13]



$$W(x_1, x_2, x_3) = I_1(x_1) \wedge I_2(x_2) \wedge I_3(x_3) \wedge \exists y\, (D(x_1, y) \wedge D(x_2, y) \wedge D(x_3, y))$$

Hardness of counting with projection [PS13]

I_1
4
5

I_2
4
5
6

I_3
6

D	
4	5
4	6
5	4
5	6
6	4
6	5

Number of perfect matchings:
$|F(I_1, I_2, I_3)| - |W(I_1, I_2, I_3, D)|$

$$F(x_1, x_2, x_3) = I_1(x_1) \wedge I_2(x_2) \wedge I_3(x_3)$$

$$W(x_1, x_2, x_3) = I_1(x_1) \wedge I_2(x_2) \wedge I_3(x_3) \wedge \exists y \, (D(x_1, y) \wedge D(x_2, y) \wedge D(x_3, y))$$

Does this rule out efficient uniform generation?

No, the argument for join queries does not apply here

- The problem of verifying whether an acyclic conjunctive query has a non-empty set of answers can be solved in polynomial time

For practical applications

- We need to consider both acyclic and cyclic queries
- We need to include all relational algebra operators
- We need to consider aggregation

Part I: join, selection and aggregation

A bit of notation

- $\mathrm{dom}(A)$: domain of attribute A
- Given a tuple r and an attribute A, $r[A]$ is the value of r in the attribute A
- $r \sim s$: r and s have the same values in their common attributes
- $R \ltimes S = \{r \in R \mid \exists s \in S : r \sim s\}$
  - If X is the set of attributes of R, then $R \ltimes S = \pi_X(R \bowtie S)$

Uniform generation

Sampling with uniform distribution [093,CMN99]

We would like to generate uniformly at random a tuple in $R[A, B] \bowtie S[B, C]$

Ideally, the probability of choosing a tuple $t \in R \bowtie S$ should be

$$\frac{1}{|R \bowtie S|}$$

Sampling with uniform distribution: first attempt

To produce a sample do the following:

1. Generate uniformly at random $r \in R$
2. Generate uniformly at random $s \in S$
3. If $r \sim s$, then return (r, s)

Sampling with uniform distribution: first attempt

Tuples in the join are generated uniformly. If $r \sim s$:

$$\Pr((r, s) \text{ is generated}) = \frac{1}{|R||S|}$$

The probability that a tuple is generated is

$$\frac{|R \bowtie S|}{|R||S|}$$

If $|R \bowtie S| \ll |R||S|$, then this probability can be very small

Sampling with uniform distribution: second attempt

To produce a sample do the following:

1. Generate uniformly at random $r \in R$
2. Generate uniformly at random $s \in \sigma_{B=r[B]}(S)$
3. Return (r, s)

Sampling with uniform distribution: second attempt

But in this cases the tuples in the join are **not** generated uniformly.

Assuming $r \sim s$:

$$\begin{aligned}
&\Pr((r, s) \text{ is generated}) \\
&\qquad = \Pr(r \text{ is generated}) \Pr(s \text{ is generated} \mid r \text{ is generated}) \\
&\qquad = \frac{1}{|R|} \frac{1}{|S \ltimes \{r\}|}
\end{aligned}$$

Sampling with uniform distribution: second attempt

R	A	B
	1	2
	3	4

$$\frac{1}{2}$$

S	B	C
	2	1
	4	1
	$\vdots$	
	4	N

Sampling with uniform distribution: second attempt

R	A	B
	1	2
	3	4

$$\frac{1}{2N}$$

S	B	C
	2	1
	4	1
	$\vdots$	
	4	N

How do we solve this problem?

Sampling with uniform distribution: third attempt [093]

Let $M_B(S) = \max\limits_{v \in \mathrm{dom}(B)} |\sigma_{B=v}(S)|$

To produce a sample do the following:

1. Generate uniformly at random $r \in R$
2. Reject with probability
$$1 - \frac{|S \ltimes \{r\}|}{M_B(S)}$$
3. Generate uniformly at random $s \in \sigma_{B=r[B]}(S)$
4. Return (r, s)

Sampling with uniform distribution: third attempt [093]

The tuples in the join are generated uniformly.

Assuming $r \sim s$:

$$\Pr((r, s) \text{ is generated})$$

$$= \Pr(r \text{ is generated}) \Pr(s \text{ is generated} \mid r \text{ is generated})$$

$$= \frac{1}{|R|} \frac{|S \ltimes \{r\}|}{M_B(S)} \frac{1}{|S \ltimes \{r\}|} = \frac{1}{|R| M_B(S)}$$

Upper bound for $|R \bowtie S|$

A general framework for sampling [ZCLHY18]

Consider the join query $R_1[A_1, A_2] \bowtie R_2[A_2, A_3] \bowtie \cdots \bowtie R_n[A_n, A_{n+1}]$

Given $t \in R_i$, define

$$w(t) = |\{t\} \bowtie R_{i+1} \bowtie \cdots \bowtie R_n|$$

Besides, let

$$w(R) = \sum_{t \in R} w(t)$$

A general framework for sampling [ZCLHY18]

For each $t \in R_i$, we have that $w(t) = \; w(R_{i+1} \ltimes \{t\})$

$$
\begin{aligned}
w(t) &= |\{t\} \bowtie R_{i+1} \bowtie r_{i+1} \bowtie \;\; \cdots \bowtie R_n| \\
&= \sum_{t' \in R_{i+1}} |\{t\} \bowtie \{t'\} \bowtie R_{i+2} \cdots \bowtie R_n| \\
&= \sum_{t' \in R_{i+1} \,:\, t \sim t'} |\{t\} \bowtie \{t'\} \bowtie R_{i+2} \cdots \bowtie R_n| \\
&= \sum_{t' \in R_{i+1} \,:\, t \sim t'} |\{t'\} \bowtie R_{i+2} \cdots \bowtie R_n| \\
&= \sum_{t' \in R_{i+1} \ltimes \{t\}} w(t') = w(R_{i+1} \ltimes \{t\})
\end{aligned}
$$

A general framework for sampling [ZCLHY18]

We do not have access to the values $w(t)$ when sampling, but instead we have some approximations of them

Assume given an approximation W of w that satisfies the following properties

1. $W(t) \geq w(t)$
2. $W(t) = w(t) = 1$ for each $t \in R_n$
3. $W(t) \geq W(R_{i+1} \ltimes \{t\})$ for each $t \in R_i$

A general framework for sampling [ZCLHY18]

To produce a sample, do the following:

1. Generate $r_1 \in R_1$ with probability $\frac{W(r_1)}{W(R_1)}$
2. For $i = 2$ to n:
   - 2.1. Reject with probability $1 - \frac{W(R_i \ltimes \{r_{i-1}\})}{W(r_{i-1})}$
   - 2.2. Generate $r_i \in R_i \ltimes \{r_{i-1}\}$ with probability $\frac{W(r_i)}{W(R_i \ltimes \{r_{i-1}\})}$
3. Return $(r_1, r_2, \ldots, r_n)$

A general framework for sampling [ZCLHY18]

The tuples in the join are generated uniformly

$$\Pr((r_1, r_2) \text{ is generated})$$

$$= \Pr(r_1 \text{ is generated}) \Pr(r_2 \text{ is generated} \mid r_1 \text{ is generated})$$

A general framework for sampling [ZCLHY18]

The tuples in the join are generated uniformly

$$\begin{aligned}
&\Pr((r_1, r_2) \text{ is generated}) \\
&\qquad = \Pr(r_1 \text{ is generated}) \Pr(r_2 \text{ is generated} \mid r_1 \text{ is generated}) \\
&\qquad = \frac{W(r_1)}{W(R_1)} \cdot
\end{aligned}$$

A general framework for sampling [ZCLHY18]

The tuples in the join are generated uniformly

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$$= \Pr(r_1 \text{ is generated}) \Pr(r_2 \text{ is generated} \mid r_1 \text{ is generated})$$

$$= \frac{W(r_1)}{W(R_1)} \cdot$$

1. Generate $r_1 \in R_1$ with probability $\frac{W(r_1)}{W(R_1)}$

A general framework for sampling [ZCLHY18]

The tuples in the join are generated uniformly

$$
\begin{aligned}
&\Pr((r_1, r_2) \text{ is generated}) \\
&\qquad = \Pr(r_1 \text{ is generated}) \Pr(r_2 \text{ is generated} \mid r_1 \text{ is generated}) \\
&\qquad = \frac{W(r_1)}{W(R_1)} \cdot \frac{W(R_2 \ltimes \{r_1\})}{W(r_1)} \cdot
\end{aligned}
$$

2.1. Reject with probability $1 - \frac{W(R_i \ltimes \{r_{i-1}\})}{W(r_{i-1})}$

A general framework for sampling [ZCLHY18]

The tuples in the join are generated uniformly

$$\Pr((r_1, r_2) \text{ is generated})$$

$$= \Pr(r_1 \text{ is generated}) \Pr(r_2 \text{ is generated} \mid r_1 \text{ is generated})$$

$$= \frac{W(r_1)}{W(R_1)} \cdot \frac{W(R_2 \ltimes \{r_1\})}{W(r_1)} \cdot \frac{W(r_2)}{W(R_2 \ltimes \{r_1\})}$$

2.2. Generate $r_i \in R_i \ltimes \{r_{i-1}\}$ with probability $\frac{W(r_i)}{W(R_i \ltimes \{r_{i-1}\})}$

A general framework for sampling [ZCLHY18]

The tuples in the join are generated uniformly

$$\Pr((r_1, r_2) \text{ is generated})$$

$$= \Pr(r_1 \text{ is generated}) \Pr(r_2 \text{ is generated} \mid r_1 \text{ is generated})$$

$$= \frac{W(r_1)}{W(R_1)} \cdot \frac{W(R_2 \ltimes \{r_1\})}{W(r_1)} \cdot \frac{W(r_2)}{W(R_2 \ltimes \{r_1\})} = \frac{W(r_2)}{W(R_1)}$$

A general framework for sampling [ZCLHY18]

The tuples in the join are generated uniformly

$$\Pr((r_1, r_2, \ldots, r_n) \text{ is generated}) = \frac{W(r_n)}{W(R_1)}$$

A general framework for sampling [ZCLHY18]

The tuples in the join are generated uniformly

$$\Pr((r_1, r_2, \ldots, r_n) \text{ is generated}) = \frac{W(r_n)}{W(R_1)} = \frac{1}{W(R_1)}$$

A generalization of the idea of [093]

Assume that:

$W(r_1) = M_{A_2}(R_2)$ for each $r_1 \in R_1$

$W(r_2) = 1$ for each $r_2 \in R_2$

A generalization of the idea of [093]

Then: $$W(R_1) = \sum_{t \in R_1} W(t) = \sum_{t \in R_1} M_{A_2}(R_2) = |R_1| M_{A_2}(R_2)$$

$$W(R_2) = \sum_{t \in R_2} W(t) = \sum_{t \in R_2} 1 = |R_2|$$

Therefore:

$$\Pr((r_1, r_2) \text{ is generated}) = \frac{W(r_1)}{W(R_1)} \cdot \frac{W(R_2 \ltimes \{r_1\})}{W(r_1)} \cdot \frac{W(r_2)}{W(R_2 \ltimes \{r_1\})}$$

A generalization of the idea of [093]

Then: $$W(R_1) = \sum_{t \in R_1} W(t) = \sum_{t \in R_1} M_{A_2}(R_2) = |R_1| M_{A_2}(R_2)$$

$$W(R_2) = \sum_{t \in R_2} W(t) = \sum_{t \in R_2} 1 = |R_2|$$

Therefore:

$$\Pr((r_1, r_2) \text{ is generated}) = \frac{M_{A_2}(R_2)}{W(R_1)} \cdot \frac{W(R_2 \ltimes \{r_1\})}{M_{A_2}(R_2)} \cdot \frac{W(r_2)}{W(R_2 \ltimes \{r_1\})}$$

A generalization of the idea of [093]

Then: $$W(R_1) = \sum_{t \in R_1} W(t) = \sum_{t \in R_1} M_{A_2}(R_2) = |R_1| M_{A_2}(R_2)$$

$$W(R_2) = \sum_{t \in R_2} W(t) = \sum_{t \in R_2} 1 = |R_2|$$

Therefore:

$$\Pr((r_1, r_2) \text{ is generated}) = \frac{M_{A_2}(R_2)}{|R_1| M_{A_2}(R_2)} \cdot \frac{W(R_2 \ltimes \{r_1\})}{M_{A_2}(R_2)} \cdot \frac{W(r_2)}{W(R_2 \ltimes \{r_1\}}$$

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Then: $$W(R_1) = \sum_{t \in R_1} W(t) = \sum_{t \in R_1} M_{A_2}(R_2) = |R_1| M_{A_2}(R_2)$$

$$W(R_2) = \sum_{t \in R_2} W(t) = \sum_{t \in R_2} 1 = |R_2|$$

Therefore:

$$\Pr((r_1, r_2) \text{ is generated}) = \frac{M_{A_2}(R_2)}{|R_1| M_{A_2}(R_2)} \cdot \frac{|R_2 \ltimes \{r_1\}|}{M_{A_2}(R_2)} \cdot \frac{W(r_2)}{|R_2 \ltimes \{r_1\}|}$$

A generalization of the idea of [093]

Then:
$$W(R_1) = \sum_{t \in R_1} W(t) = \sum_{t \in R_1} M_{A_2}(R_2) = |R_1| M_{A_2}(R_2)$$

$$W(R_2) = \sum_{t \in R_2} W(t) = \sum_{t \in R_2} 1 = |R_2|$$

Therefore:

$$\Pr((r_1, r_2) \text{ is generated}) = \frac{M_{A_2}(R_2)}{|R_1| M_{A_2}(R_2)} \cdot \frac{|R_2 \ltimes \{r_1\}|}{M_{A_2}(R_2)} \cdot \frac{\color{red}{1}}{|R_2 \ltimes \{r_1\}|}$$

A generalization of the idea of [093]

Then: $$W(R_1) = \sum_{t \in R_1} W(t) = \sum_{t \in R_1} M_{A_2}(R_2) = |R_1| M_{A_2}(R_2)$$

$$W(R_2) = \sum_{t \in R_2} W(t) = \sum_{t \in R_2} 1 = |R_2|$$

Therefore:

$$\Pr((r_1, r_2) \text{ is generated}) = \frac{1}{|R_1|} \cdot \frac{|R_2 \ltimes \{r_1\}|}{M_{A_2}(R_2)} \cdot \frac{1}{|R_2 \ltimes \{r_1\}|}$$

A generalization of the idea of [093]

Then:

$$W(R_1) = \sum_{t \in R_1} W(t) = \sum_{t \in R_1} M_{A_2}(R_2) = |R_1| M_{A_2}(R_2)$$

$$W(R_2) = \sum_{t \in R_2} W(t) = \sum_{t \in R_2} 1 = |R_2|$$

Therefore:

$$\Pr((r_1, r_2) \text{ is generated}) = \frac{1}{|R_1|} \cdot \frac{|R_2 \ltimes \{r_1\}|}{M_{A_2}(R_2)} \cdot \frac{1}{|R_2 \ltimes \{r_1\}|} = \frac{1}{|R_1| M_{A_2}(R}$$

We can use better bounds

Define W as:

- $W(t) = \mathrm{AGM}(R_{i+1} \bowtie \cdots \bowtie R_n)$ for every $t \in R_i$ with $1 \leq i < n$
- $W(t) = 1$ for every $t \in R_n$

W satisfies the three properties

Sampling in the acyclic case

Consider an acyclic join query $R_1 \bowtie R_2 \bowtie \cdots \bowtie R_n$

Fix a join tree for this query

- $R_i \prec R_j$ indicates that R_i is an ancestor of R_j in this tree

Sampling in the acyclic case

Given $t \in R_i$, define

$$w(t) = \left| \{t\} \bowtie \left(\bigotimes_{R_j \,:\, R_i \prec R_j} R_j \right) \right|$$

Besides, if R_j is a child of R_i:

$$w(t, R_j) = \left| \{t\} \bowtie R_j \bowtie \left(\bigotimes_{R_k \,:\, R_j \prec R_k} R_k \right) \right|$$

Sampling in the acyclic case

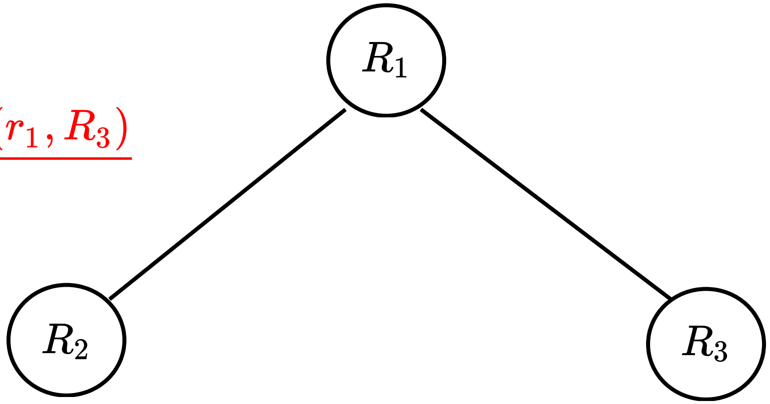
Assume given an approximation W of w that satisfies the following properties

1. $W(t) \geq w(t)$
2. $W(t, R_j) \geq w(t, R_j)$ if $t \in R_i$ and R_j is a child of R_i
3. $W(t) = w(t) = 1$ if $t \in R_i$ and R_i is a leaf
4. $W(t) \geq W(t, R_{k_1}) \cdot W(t, R_{k_2}) \cdot \ldots \cdot W(t, R_{k_\ell})$ if $t \in R_i$ and the children of R_i are $R_{k_1}, R_{k_2}, \ldots, R_{k_\ell}$
5. $W(t, R_j) \geq W(R_j \ltimes \{t\})$ if $t \in R_i$ and R_j is a child of R_i

Sampling in the acyclic case

Sample with probability: $\dfrac{W(r_1)}{W(R_1)}$

Reject with probability: $1 - \dfrac{W(r_1, R_2)W(r_1, R_3)}{W(r_1)}$



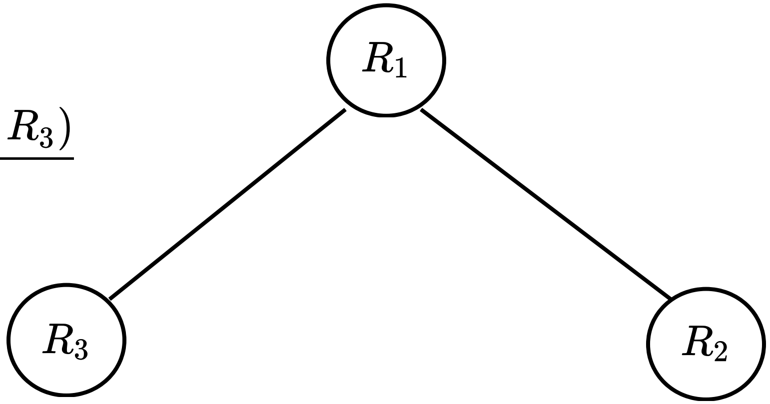
Reject with probability: $1 - \dfrac{W(R_2 \ltimes \{r_1\})}{W(r_1, R_2)}$ $\quad$ $1 - \dfrac{W(R_3 \ltimes \{r_1\})}{W(r_1, R_3)}$

Sample with probability: $\dfrac{W(r_2)}{W(R_2 \ltimes \{r_1\})}$ $\quad$ $\dfrac{W(r_3)}{W(R_3 \ltimes \{r_1\})}$

Sampling in the acyclic case

Sample with probability: $\dfrac{W(r_1)}{W(R_1)}$

Accept with probability: $\dfrac{W(r_1, R_2)W(r_1, R_3)}{W(r_1)}$



Accept with probability: $\dfrac{W(R_2 \ltimes \{r_1\})}{W(r_1, R_2)}$ $\dfrac{W(R_3 \ltimes \{r_1\})}{W(r_1, R_3)}$

Sample with probability: $\dfrac{W(r_2)}{W(R_2 \ltimes \{r_1\})}$ $\dfrac{W(r_3)}{W(R_3 \ltimes \{r_1\})}$

Sampling in the acyclic case

$$\Pr((r_1, r_2, r_3) \text{ is generated}) =$$

$$= \frac{W(r_1)}{W(R_1)} \cdot \frac{W(r_1, R_2) W(r_1, R_3)}{W(r_1)} \cdot \frac{W(R_2 \ltimes \{r_1\})}{W(r_1, R_2)} \cdot \frac{W(r_2)}{W(R_2 \ltimes \{r_1\})} \cdot \frac{W(R_3 \ltimes \{r_1\})}{W(r_1, R_3)} \cdot \frac{W(r_3)}{W(R_3 \ltimes \{r_1\})}$$

$$= \frac{W(r_2) W(r_3)}{W(R_1)}$$

$$= \frac{1}{W(R_1)}$$

Sampling in the cyclic case

Consider the join query $Q = R_1 \bowtie R_2 \bowtie \cdots \bowtie R_n$

Split Q into join queries Q_{acyclic} and Q_{rest} such that $Q = Q_{\text{acyclic}} \bowtie Q_{\text{rest}}$

- Assume that $\{A_1, \ldots, A_k\}$ is the set of attributes that queries Q_{acyclic} and Q_{rest} have in common

Sampling in the cyclic case

Let

$$M_{\mathrm{rest}} = \max_{(v_1,\ldots,v_k)\in\mathrm{dom}(A_1)\times\cdots\times\mathrm{dom}(A_k)} |\{t \in Q_{\mathrm{rest}} \mid \forall i \in \{1,\ldots,k\} : t[A_i] = v_i\}|$$

To produce a sample do the following:

1. Use the sample algorithm for the acyclic case to generate a tuple $t \in Q_{\mathrm{acyclic}}$
2. Reject with probability
$$1 - \frac{|Q_{\mathrm{rest}} \ltimes \{t\}|}{M_{\mathrm{rest}}}$$
3. Generate uniformly at random $t' \in Q_{\mathrm{rest}}$
4. Return (t, t')

Sampling in the cyclic case

The tuples in the join are generated uniformly

$$\Pr((t, t') \text{ is generated})$$

$$= \Pr(t \text{ is generated}) \Pr(t' \text{ is generated} \mid t \text{ is generated})$$

$$= \frac{1}{W(R_1)} \cdot \frac{|Q_{\text{rest}} \ltimes \{t\}|}{M_{\text{rest}}} \cdot \frac{1}{|Q_{\text{rest}} \ltimes \{t\}|} = \frac{1}{W(R_1) M_{\text{rest}}}$$

Estimation of cardinality and aggregates

Properties of estimators

Bias of an estimator $\hat{\theta}$ relative to θ is defined as

$\mathrm{Bias}(\hat{\theta}, \theta) = E[\hat{\theta}] - \theta$

- $\hat{\theta}$ is unbiased if $\mathrm{Bias}(\hat{\theta}, \theta) = 0$

$\hat{\theta}_n$ is consistent if $\hat{\theta}_n \xrightarrow{p} \theta$

- For every $\varepsilon > 0$: $\lim_{n \to \infty} \Pr(|\hat{\theta}_n - \theta| > \varepsilon) = 0$

We would like $\hat{\theta}_n$ to be computable in polynomial time in n

Confidence intervals

We would like to provide the following guarantee:

$$\Pr\left(\theta \in [f(\hat{\theta}), g(\hat{\theta})]\right) \geq 1 - \delta$$

Which is usually translated into the following:

$$\Pr\left(\theta \in [\hat{\theta}_n - \varepsilon(n), \hat{\theta}_n + \varepsilon(n)]\right) \geq 1 - \delta$$

Confidence intervals

Two fundamental tools to construct confidence intervals:

1. Central Limit Theorem
    - The confidence interval depends on the convergence rate, so it would be an approximation if we consider a fixed value n
    - A way to deal with this is to use the Berry–Esseen theorem, which gives a precise bound on the difference with the standard normal distribution

Confidence intervals

Two fundamental tools to construct confidence intervals:

2. Concentration inequalities: Chebyshev, Hoeffding, ...
    - The bounds produced are not approximations, but they are looser

In both cases it is convenient to have a *small* variance

Confidence intervals

Chebyshev inequality:

$$\Pr(|X - E[X]| \geq \varepsilon) \leq \frac{\mathrm{Var}[\hat{\theta}]}{\varepsilon^2}$$

Assuming $\hat{\theta}$ is an unbiased estimator of θ, we can rewrite Chebyshev inequality as:

$$\Pr\left(\theta \in (\hat{\theta} - \varepsilon, \hat{\theta} + \varepsilon)\right) \geq 1 - \frac{\mathrm{Var}[\hat{\theta}]}{\varepsilon^2}$$

Warming up [LWYZ16]

Consider the following SQL query Q over the schema $R[A, B]$:

$$\mathrm{SUM}_D(R[A, B] \bowtie S[B, C] \bowtie T[C, D])$$

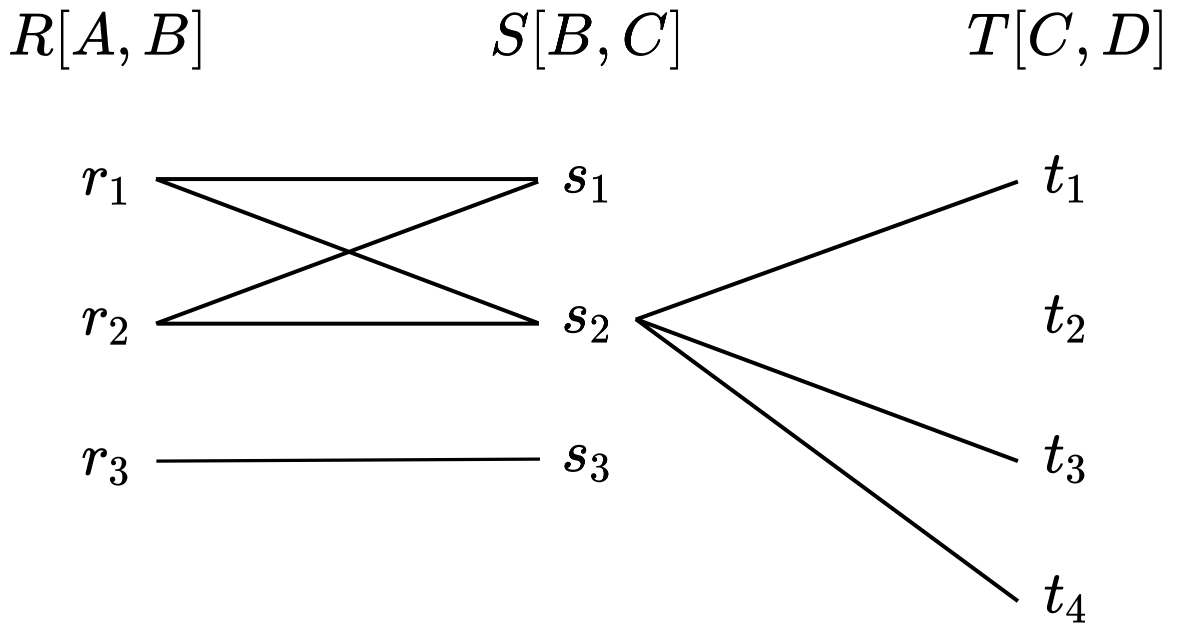
We would like to construct an estimator for the answer to this query

Warming up [LWYZ16]

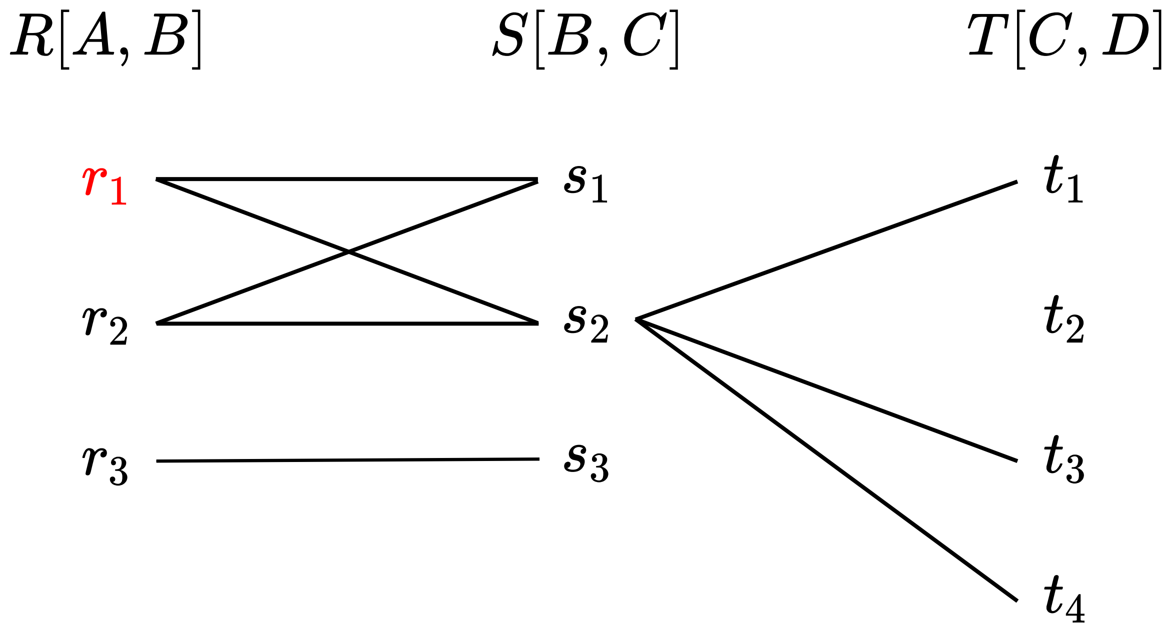
$R[A,B]$	$S[B,C]$	$T[C,D]$
r_1	s_1	t_1
r_2	s_2	t_2
r_3	s_3	t_3
		t_4

$r_1[B] = s_2[B]$

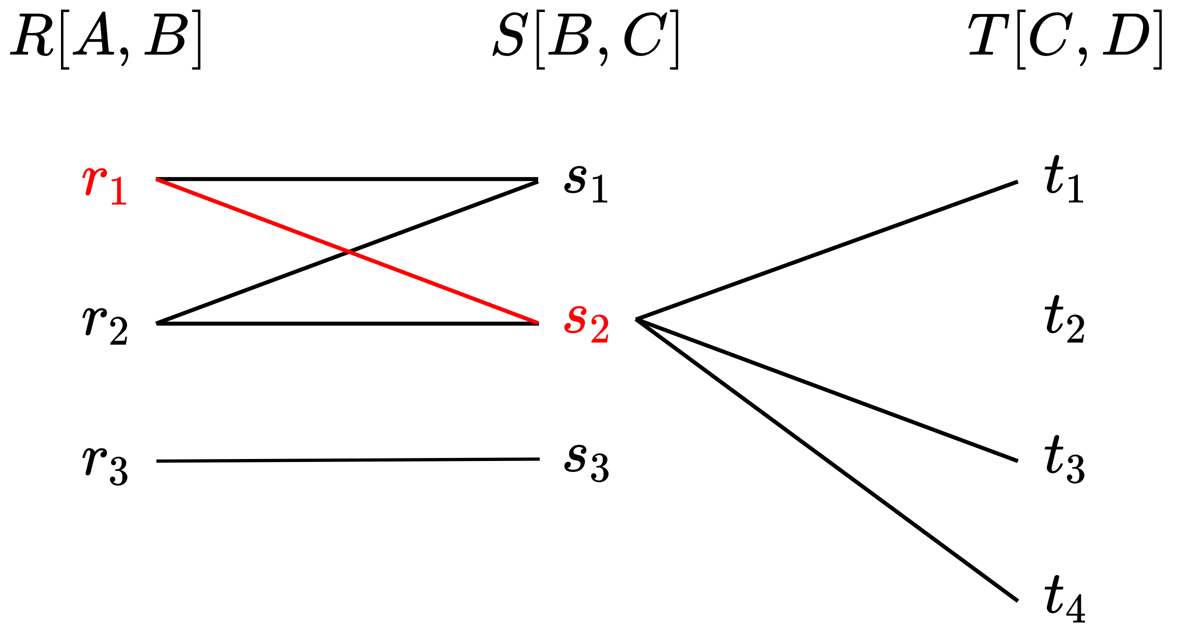
Warming up [LWYZ16]



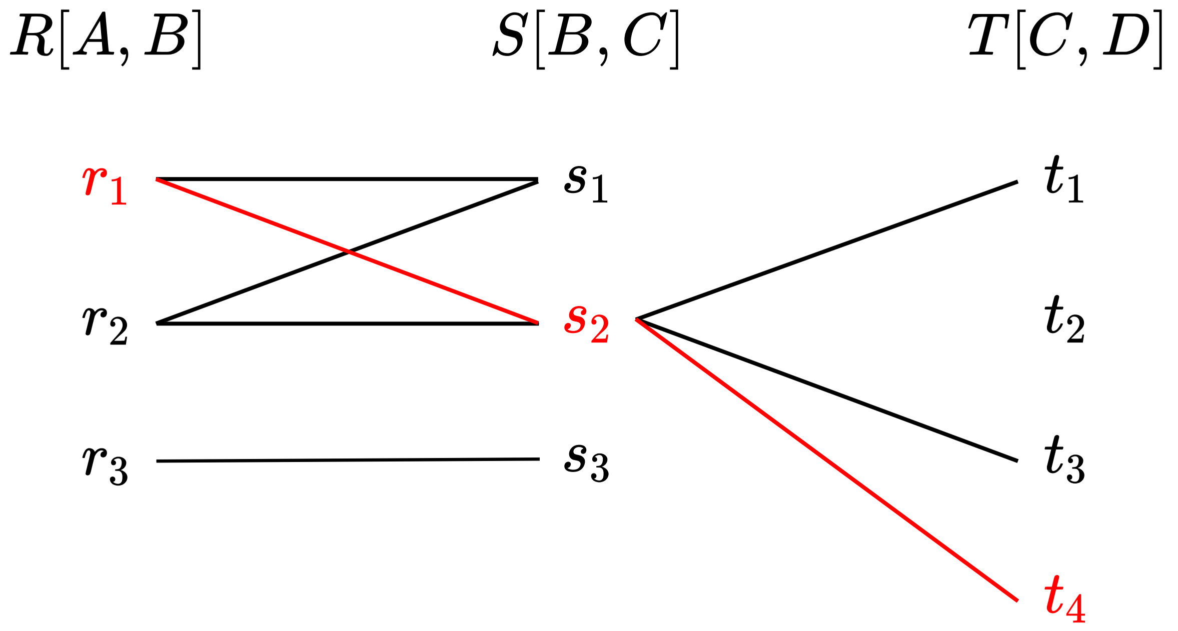
Warming up [LWYZ16]



Warming up [LWYZ16]



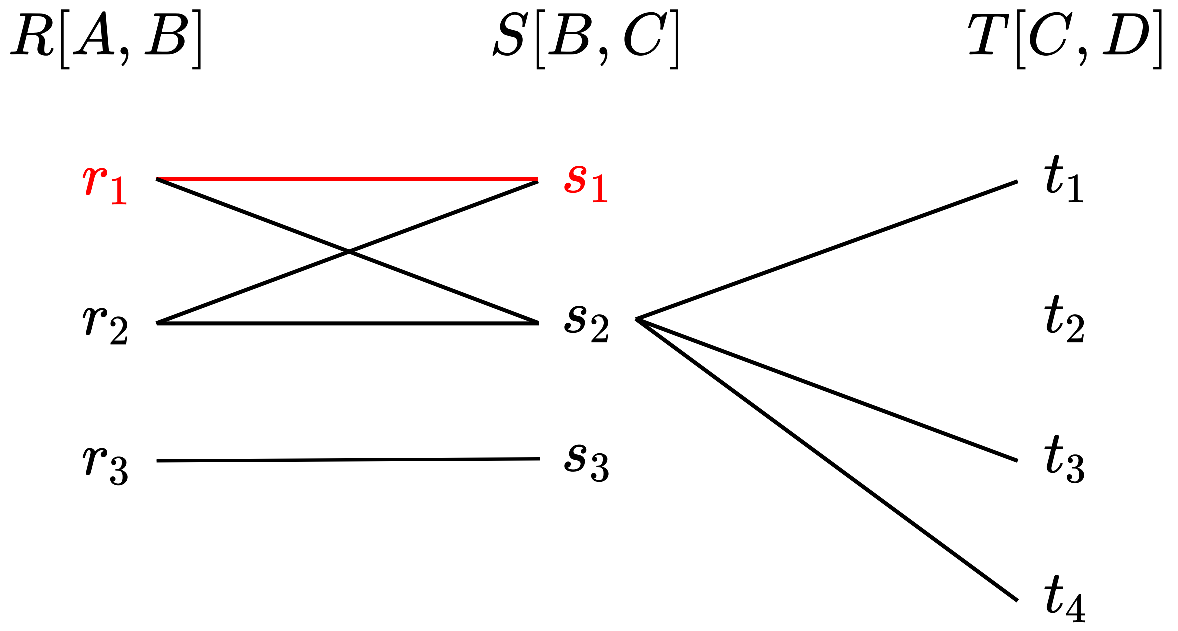
Warming up [LWYZ16]



$$\Pr((r_1, s_2, t_4) \text{ is generated}) = \frac{1}{18}$$

$$v(r_1, s_2, t_4) = t_4[D]$$

Warming up [LWYZ16]



$$\Pr((r_1, s_1) \text{ is generated}) = \frac{1}{6}$$

$$v(r_1, s_1) = 0$$

Warming up [LWYZ16]

How do we estimate $\mathrm{SUM}_D(R[A,B] \bowtie S[B,C] \bowtie T[C,D])$?

Given a path γ, define $X(\gamma) = v(\gamma)$

We can use X as an estimator

- But this is a biased estimator, as it does not consider that different paths can have different probabilities

How can we transform X into an unbiased estimator?

Warming up [LWYZ16]

Horvitz–Thompson idea:

$$Y(\gamma) = \frac{v(\gamma)}{\Pr(\gamma \text{ is generated})}$$

Warming up [LWYZ16]

Horvitz–Thompson idea:

$$Y(\gamma) = \frac{v(\gamma)}{\Pr(\gamma \text{ is generated})}$$

Y is unbiased:

$$E[Y] = \sum_{\gamma} \Pr(\gamma \text{ is generated}) \cdot Y(\gamma)$$

$$= \sum_{\gamma} \Pr(\gamma \text{ is generated}) \cdot \frac{v(\gamma)}{\Pr(\gamma \text{ is generated})}$$

$$= \sum_{\gamma} v(\gamma)$$

The Horvitz–Thompson estimator [HT52,T12]

Suppose that we have a list of values $(v_1, \ldots, v_N)$, and we need to estimate:

$$\tau = \sum_{i=1}^{N} v_i$$

To do this estimation, we construct a sample of size n of elements from $\{1, \ldots, N\}$

- With or without replacement

The Horvitz–Thompson estimator [HT52,T12]

X_i: number of times element $i \in \{1, \ldots, N\}$ appears in the sample

- If we sample without replacement, then X_i can be 0 or 1

Let $\pi_i = E[X_i]$

The Horvitz–Thompson estimator [HT52,T12]

The Horvitz–Thompson (HT) estimator of τ:

$$Y = \sum_{i=1}^{N} \frac{X_i v_i}{\underbrace{\pi_i}_{\text{inverse weighting}}} = \sum_{i \in \text{sample}} \frac{X_i v_i}{\pi_i}$$

The Horvitz–Thompson estimator [HT52,T12]

The Horvitz–Thompson (HT) estimator of τ:

$$Y = \sum_{i=1}^{N} \frac{X_i v_i}{\pi_i} = \sum_{i \in \text{sample}} \frac{X_i v_i}{\pi_i}$$

HT is unbiased:

$$E[Y] = E\left[\sum_{i=1}^{N} \frac{X_i v_i}{\pi_i}\right] = \sum_{i=1}^{N} \frac{E[X_i] v_i}{\pi_i} = \sum_{i=1}^{N} \frac{\pi_i v_i}{\pi_i} = \tau$$

An example of HT

We sample uniformly with replacemenet: $p = \frac{1}{N}$

We can think of X_i as

$$X_i = \sum_{k=1}^{n} Z_{i,k},$$

where $Z_{i,k}$ is 1 if i is the k-th element sampled, and 0 otherwise

$X_i \sim \mathrm{Binomial}(n, p)$ since each $Z_{i,k} \sim \mathrm{Bernoulli}(p)$ and these random variables are mutually independent

An example of HT

$$\pi_i = E[X_i] = np$$

An example of HT

$$\pi_i = E[X_i] = np$$

HT estimator in this case:

$$Y = \sum_{i=1}^{N} \frac{X_i v_i}{\pi_i} = \sum_{i=1}^{N} \frac{X_i v_i}{np} = \frac{N}{n} \sum_{i \in \text{sample}} X_i v_i$$

What is the variance of HT?

Let $\pi_{i,j} = E[X_i X_j]$

$E[X_i X_j]$ is not necessarily equal to $E[X_i]E[X_j]$

- X_i and X_j are not independent random variables since $X_1 + \cdots + X_N = n$

What is the variance of HT?

$$\sigma^2(Y) = E[Y^2] - E[Y]^2 = E\left[\left(\sum_{i=1}^{N} \frac{X_i v_i}{\pi_i}\right)^2\right] - \tau^2$$

$$= E\left[\sum_{i=1}^{N}\sum_{j=1}^{N} \frac{X_i X_j}{\pi_i \pi_j} v_i v_j\right] - \left(\sum_{i=1}^{N} v_i\right)^2$$

$$= \sum_{i=1}^{N}\sum_{j=1}^{N} \frac{E[X_i X_j]}{\pi_i \pi_j} v_i v_j - \sum_{i=1}^{N}\sum_{j=1}^{N} v_i v_j$$

$$= \sum_{i=1}^{N}\sum_{j=1}^{N} \left(\frac{\pi_{i,j}}{\pi_i \pi_j} - 1\right) v_i v_j$$

But an estimation of $\sigma^2(Y)$ is usually needed in practice

How do we estimate $\sigma^2(Y)$? **We use HT again!**

Define $X_{i,j} = X_i X_j$ and

$$v_{i,j} = \left(\frac{\pi_{i,j}}{\pi_i \pi_j} - 1\right) v_i v_j$$

We have that

$$\sigma^2(Y) = \sum_{(i,j) \in \{1,...,N\} \times \{1,...,N\}} v_{i,j}$$

But an estimation of $\sigma^2(Y)$ is usually needed in practice

The HT estimator of $\sigma^2(Y)$ is

$$\hat{\sigma}^2(Y) = \sum_{(i,j)\in\{1,...,N\}\times\{1,...,N\}} \frac{X_{i,j}v_{i,j}}{\pi_{i,j}},$$

given that $E[X_{i,j}] = E[X_iX_j] = \pi_{i,j}$

But an estimation of $\sigma^2(Y)$ is usually needed in practice

Replacing the values of $v_{i,j}$, we obtain:

$$\hat{\sigma}^2(Y) = \sum_{i=1}^{N}\sum_{j=1}^{N}\frac{X_i X_j}{\pi_{i,j}}\left(\frac{\pi_{i,j}}{\pi_i \pi_j} - 1\right)v_i v_j = \sum_{i,j \in \text{ sample}} \frac{X_i X_j}{\pi_{i,j}}\left(\frac{\pi_{i,j}}{\pi_i \pi_j} - 1\right)v_i v_j$$

Horvitz–Thompson estimators

The idea behind the HT estimator can be used to define unbiased estimators in many different escenarios

In this sense, we can talk about a family of HT estimators

Estimation in databases

Let's put what we learned into practice [CGHJ12]

Consider the following SQL query Q over the schema $R[A, B]$:

$$\mathrm{SUM}_B(R[A, B])$$

The result $Q(R)$ of this query is $\sum_{r \in R} r[B]$, so we need an estimator for this amount

Simple random sampling with replacement (SRSWR)

To produce the sample repeat n times the following steps:

1. Generate uniformly at random $r \in R$
2. Add r to the sample

Simple random sampling with replacement (SRSWR)

X_r: number of times tuple r appears in the sample

- $\pi_r = E[X_r] = \frac{n}{|R|}$

The HT estimator of $Q(R)$:

$$Y = \sum_{r \in R} \frac{X_r \cdot r[B]}{\pi_r} = \frac{|R|}{n} \sum_{r \in \text{sample}} X_r \cdot r[B]$$

The variance for SRSWR

For $i \in \{1, \ldots, n\}$, let W_i be a random variable such that for each possible value v of attribute B:

$$\Pr(W_i = v) = \frac{|\{r \in R \mid r[B] = v\}|}{|R|}$$

We have that:

$$Y = \frac{|R|}{n} \sum_{r \in \text{sample}} X_r \cdot r[B] = \frac{|R|}{n} \sum_{i=1}^{n} W_i$$

The variance for SRSWR

$$E[W_i] = \sum_v v \cdot \Pr(W_i = v) = \frac{1}{|R|} \sum_v v \cdot |\{r \in R \mid r[B] = v\}| \; = \frac{Q(R)}{|R|}$$

The variance for SRSWR

$$E[W_i] = \sum_v v \cdot \Pr(W_i = v) = \frac{1}{|R|} \sum_v v \cdot |\{r \in R \mid r[B] = v\}| \; = \frac{Q(R)}{|R|}$$

Random variables W_i are mutually independent:

$$\sigma^2(Y) = \sigma^2\left(\frac{|R|}{n} \sum_{i=1}^{N} W_i\right) = \frac{|R|^2}{n^2} \sum_{i=1}^{N} \sigma^2(W_i)$$

The variance for SRSWR

$$E[W_i] = \sum_v v \cdot \Pr(W_i = v) = \frac{1}{|R|} \sum_v v \cdot |\{r \in R \mid r[B] = v\}| \; = \frac{Q(R)}{|R|}$$

We have that:

$$\sigma^2(W_i) = E[(W_i - E[W_i])^2] = \sum_{r \in R} \frac{1}{|R|} \left(r[B] - \frac{Q(R)}{|R|} \right)^2 = \sigma^2(R)$$

We conclude that:

$$\sigma^2(Y) = \frac{|R|^2}{n^2} \sum_{i=1}^{n} \sigma^2(W_i) = \frac{|R|^2}{n^2} \sum_{i=1}^{n} \sigma^2(R) = \frac{|R|^2 \sigma^2(R)}{n}$$

Simple random sampling without replacement (SRSWoR)

To produce the sample repeat n times the following steps:

1. Generate uniformly at random $r \in R$
2. Add r to the sample and remove it from R

Simple random sampling without replacement (SRSWoR)

X_r: number of times tuple r appears in the sample, which can be 0 or 1

$X_r \sim \mathrm{Bernoulli}(p)$, where p is the following probability

Assume that s_k is the k-th element sampled, so that:

$$p = \Pr(X_r = 1) = \Pr\left(\bigvee_{i=1}^{n} s_i = r\right)$$

Simple random sampling without replacement (SRSWoR)

$$\Pr\left(\bigvee_{i=1}^{n} s_i = r\right) = \Pr\left(\bigvee_{i=1}^{n}\left[s_i = r \wedge \bigwedge_{j=1}^{i-1} s_j \neq r\right]\right)$$

$$= \sum_{i=1}^{n} \Pr\left(s_i = r \wedge \bigwedge_{j=1}^{i-1} s_j \neq r\right)$$

$$= \sum_{i=1}^{n} \frac{\binom{|R|-1}{i-1}}{\binom{|R|}{i-1}} \cdot \frac{1}{|R| - (i-1)}$$

$$= \sum_{i=1}^{n} \frac{|R| - (i-1)}{|R|} \cdot \frac{1}{|R| - (i-1)} \qquad {\color{red} = \frac{n}{|R|}}$$

Simple random sampling without replacement (SRSWoR)

$\pi_r = E[X_r] = \frac{n}{|R|}$

The HT estimator of $Q(R)$:

$$Y = \sum_{r \in R} \frac{X_r \cdot r[B]}{\pi_r} = \frac{|R|}{n} \sum_{r \in \text{sample}} X_r \cdot r[B] = \frac{|R|}{n} \sum_{r \in \text{sample}} r[B]$$

This is a similar estimator to the one for the case with replacement. But what is the variance of Y?

The variance for SRSWoR

The variance is lower than for the case of SRSWR:

$$\sigma^2(Y) = \frac{|R|(|R| - n)\sigma^2(R)}{n}$$

A second group of estimators [VMZC15,HYPM19]

Now consider the following SQL query Q over the schema $R[A,B], S[B,C]$:

$$\mathrm{SUM}_C(R[A,B] \bowtie S[B,C])$$

Bernoulli sampling: first alternative

To produce the sample do the following for each $(r, s) \in R \times S$:

1. Generate uniformly at random $x \in [0, 1]$
2. If $x \leq p$, then add (r, s) to the sample

Bernoulli sampling: first alternative

$X_{r,s}$: number of times $(r, s) \in R \times S$ appears in the sample

- $X_{r,s} \sim \mathrm{Bernoulli}(p)$, so that $\pi_{r,s} = E[X_{r,s}] = p$

HT estimator of $Q(R, S)$:

$$Y = \sum_{(r,s) \in R \times S} \frac{X_{r,s} \cdot v_{r,s}}{\pi_{r,s}} = \frac{1}{p} \sum_{r \in \mathrm{sample}} v_{r,s}$$

But how is $v_{r,s}$ defined? It cannot always be $s[C]$

- $v_{r,s} = s[C]$ if $r \sim s$, and $v_{(r,s)} = 0$ otherwise

Bernoulli sampling: first alternative

The random variables $X_{r,s}$ are mutually independent, so $\sigma^2(Y)$ is easy to compute

But we have a problem: the loop considers all the tuples, so we could just compute the exact answer to the query

How do we solve this problem?

Independent Bernoulli sampling

$p_R \longrightarrow$

R	A	B
	a_1	b_1
	a_2	b_2
	$\vdots$	
	a_N	b_N

sample_R

$p_S \longrightarrow$

S	B	C
	b'_1	c_1
	b'_2	c_2
	$\vdots$	
	b'_M	c_M

sample_S

$$\text{sample} = \text{sample}_R \bowtie \text{sample}_S$$

Independent Bernoulli sampling

To produce the sample do the following:

1. For each $r \in R$, generate uniformly at random $x \in [0, 1]$, and add r to sample_R if $x \leq p_R$
2. For each $s \in S$, generate uniformly at random $x \in [0, 1]$, and add s to sample_S if $x \leq p_S$
3. Let $\mathrm{sample} = \mathrm{sample}_R \bowtie \mathrm{sample}_S$

Independent Bernoulli sampling

$X_{r,s}$ and $v_{r,s}$ are defined as before

- $X_{r,s} \sim \text{Bernoulli}(p_R p_S)$, so that $\pi_{r,s} = E[X_{r,s}] = p_R p_S$

HT estimator of $Q(R, S)$:

$$Y = \sum_{(r,s)\in R\times S} \frac{X_{r,s} \cdot v_{r,s}}{\pi_{r,s}} = \frac{1}{p_R p_S} \sum_{r\in \text{sample}} v_{r,s}$$

The variance of independent Bernoulli sampling

Random variables $X_{r,s}$ are not mutually independent

- If $s \neq s'$, then $\Pr(X_{r,s'} = 1 \mid X_{r,s} = 1) = p_S \neq \Pr(X_{r,s'} = 1)$

The variance of independent Bernoulli sampling

We have that:

$$\mathrm{Var}[Y] = \sum_{(r,s)\in R\times S} \left(\frac{1}{p_R p_S} - 1\right) v_{r,s}^2 +$$

$$\sum_{r\in R} \sum_{s_1,s_2\in S\,:\,s_1\neq s_2} \left(\frac{1}{p_R} - 1\right) v_{r,s_1} v_{r,s_2} +$$

$$\sum_{r_1,r_2\in R\,:\,r_1\neq r_2} \sum_{s\in S} \left(\frac{1}{p_S} - 1\right) v_{r_1,s} v_{r_2,s}$$

The variance of independent Bernoulli sampling

And we also have a simple HT estimator of the variance:

$$
\begin{aligned}
\hat{\mathrm{Var}}[Y] = & \sum_{(r,s)\in R\times S} \frac{X_r X_s}{p_R p_S}\left(\frac{1}{p_R p_S} - 1\right) v_{r,s}^2 + \\
& \sum_{r\in R}\ \sum_{s_1,s_2\in S:\, s_1\neq s_2} \frac{X_r X_s}{p_R p_S}\left(\frac{1}{p_R} - 1\right) v_{r,s_1} v_{r,s_2} + \\
& \sum_{r_1,r_2\in R:\, r_1\neq r_2}\ \sum_{s\in S} \frac{X_r X_s}{p_R p_S}\left(\frac{1}{p_S} - 1\right) v_{r_1,s} v_{r_2,s}
\end{aligned}
$$

The variance of independent Bernoulli sampling

And we also have a simple HT estimator of the variance:

$$\hat{\mathrm{Var}}[Y] = \sum_{r \in \mathrm{sample}_R} \sum_{s \in \mathrm{sample}_S} \frac{X_r X_s}{p_R p_S} \left(\frac{1}{p_R p_S} - 1 \right) v_{r,s}^2 +$$

$$\sum_{r \in \mathrm{sample}_R} \sum_{s_1, s_2 \in r \in \mathrm{sample}_S \,:\, s_1 \neq s_2} \frac{X_r X_s}{p_R p_S} \left(\frac{1}{p_R} - 1 \right) v_{r,s_1} v_{r,s_2} +$$

$$\sum_{r_1, r_2 \in r \in \mathrm{sample}_R \,:\, r_1 \neq r_2} \sum_{s \in r \in \mathrm{sample}_S} \frac{X_r X_s}{p_R p_S} \left(\frac{1}{p_S} - 1 \right) v_{r_1,s} v_{r_2,s}$$

Join size estimation

Consider the schema $R[A, B], S[B, C]$

We can reuse the techniques presented in the previous slides to estimate $|R \bowtie S|$

If we add a column aux to S with value 1 in each tuple, then estimating $|R \bowtie S|$ corresponds to the problem of estimating the answer to the following SQL query:

$$\mathrm{SUM}_{aux}(R[A, B] \bowtie S[B, C, aux])$$

Universe sampling [VMZC15]

$\cancel{p_R} \longrightarrow$

R	A	B
	a_1	b_1
	a_2	b_2
	$\vdots$	
	a_N	b_N

sample_R

$\cancel{p_S} \longrightarrow$

S	B	C
	b'_1	c_1
	b'_2	c_2
	$\vdots$	
	b'_M	c_M

sample_S

$$\text{sample} = \text{sample}_R \bowtie \text{sample}_S$$

Universe sampling [VMZC15]

R	A	B
	a_1	b_1
	a_2	b_2
	$\vdots$	
	a_N	b_N

sample_R

p

S	B	C
	b'_1	c_1
	b'_2	c_2
	$\vdots$	
	b'_M	c_M

sample_S

$$\text{sample} = \text{sample}_R \bowtie \text{sample}_S$$

Universe sampling [VMZC15]

Assume given a (perfect) hash function $h : \mathrm{dom}(B) \to [0, 1]$

To produce the sample do the following:

1. For each $r \in R$, if $h(r[B]) \leq p$, then add r to sample_R
2. For each $s \in S$, if $h(s[B]) \leq p$, then add s to sample_S
3. Let $\mathrm{sample} = \mathrm{sample}_R \bowtie \mathrm{sample}_S$

Universe sampling [VMZC15]

$X_{r,s}$: number of times (r, s) appears in the sample

- $X_{r,s} \sim \mathrm{Bernoulli}(p)$, so that $\pi_{r,s} = E[X_{r,s}] = p$

HT estimator of $Q(R, S)$:

$$Y = \sum_{r \in R} \sum_{s \in S} \frac{X_{r,s} \cdot v_{r,s}}{\pi_{r,s}} = \frac{1}{p} \sum_{r \in \mathrm{sample}_R} \sum_{s \in \mathrm{sample}_S} v_{r,s}$$

where $v_{r,s} = 1$ if $r \sim s$, and $v_{r,s} = 0$ otherwise

The variance of universe sampling

Random variables $X_{r,s}$ are not mutually independent

- If $s \neq s'$ and $s[B] = s'[B]$, then $\Pr(X_{r,s'} = 1 \mid X_{r,s} = 1) = 1$

The variance of universe sampling

But the variance of Y can be computed considered the following representation of this random variable

For $v \in \mathrm{dom}(B)$, let

$$N_R(v) = |\{r \in R \mid r[B] = v\}|$$
$$N_S(v) = |\{s \in S \mid s[B] = v\}|$$

The variance of universe sampling

X_v: random variable such that $X_v = 1$ if v is included as the value of attribute B for some tuple in the sample, and 0 otherwise

- $X_v \sim \text{Bernoulli}(p)$

Then we can represent Y as the following HT estimator:

$$Y = \sum_{v \in \text{dom}(B)} \frac{X_v N_R(v) N_S(v)}{E[X_v]} = \frac{1}{p} \sum_{v \in \text{dom}(B)} X_v N_R(v) N_S(v)$$

The variance of universe sampling

Random variables X_v are mutually independent:

$$\begin{aligned}
\mathrm{Var}[Y] &= \mathrm{Var}\left[\frac{1}{p}\sum_{v\in\mathrm{dom}(B)} X_v N_R(v) N_S(v)\right] \\
&= \frac{1}{p^2}\sum_{v\in\mathrm{dom}(B)} \mathrm{Var}[X_v] N_R^2(v) N_S^2(v) \\
&= \frac{1}{p^2}\sum_{v\in\mathrm{dom}(B)} p(1-p) N_R^2(v) N_S^2(v) \\
&= \left(\frac{1}{p}-1\right)\sum_{v\in\mathrm{dom}(B)} N_R^2(v) N_S^2(v)
\end{aligned}$$

What about other operators?

The previous techniques can be easily extended to consider the selection operator

- We leave this as an exercise for the reader

But the inclusion of projection is more challenging

Part II: Adding projection

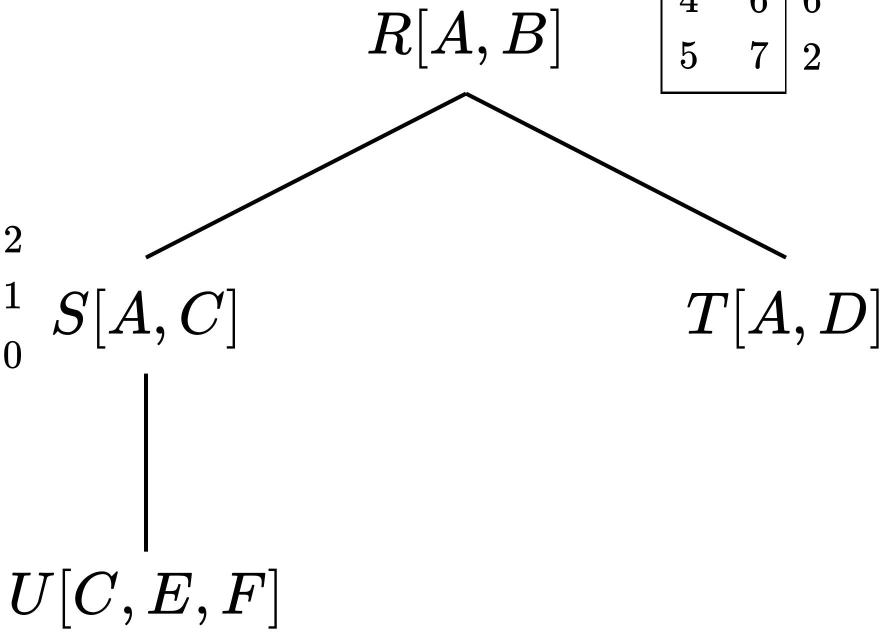
What is left?

We now consider the operators join, selection and projection

- We consider conjunctive queries

Our goal is to show how to do *efficient* cardinality estimation for acyclic conjunctive queries

$$R[A,B] \bowtie S[A,C] \bowtie T[A,D] \bowtie U[C,E,F]$$



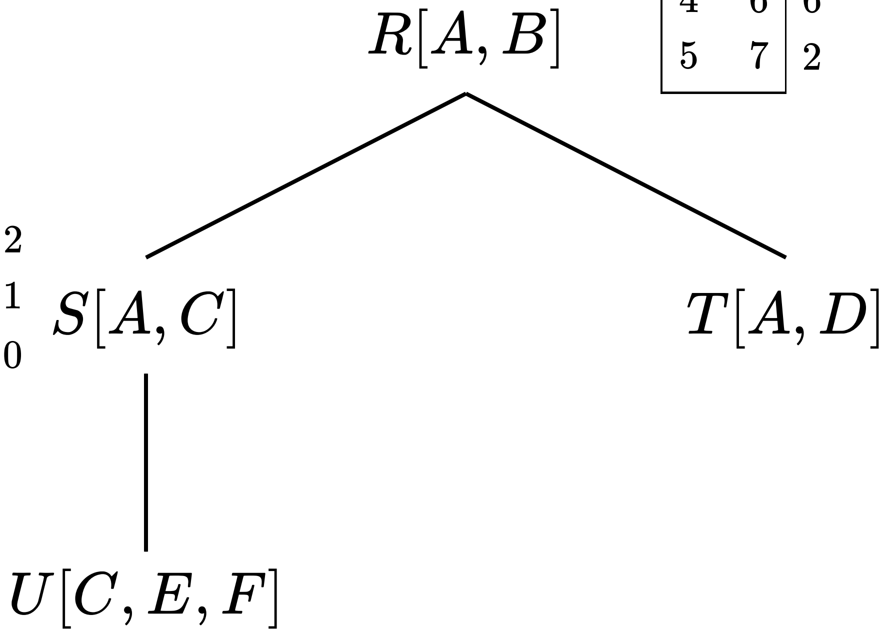
R	A	B	
	4	6	6
	5	7	2

S	A	C	
	4	1	2
	5	2	1
	4	3	0

T	A	D	
	4	1	1
	4	2	1
	4	3	1
	5	4	1
	5	5	1

U	C	E	F	
	1	3	6	1
	1	4	7	1
	2	5	8	1

$$R(x, y) \wedge S(x, z) \wedge T(x, u) \wedge U(z, v, w)$$



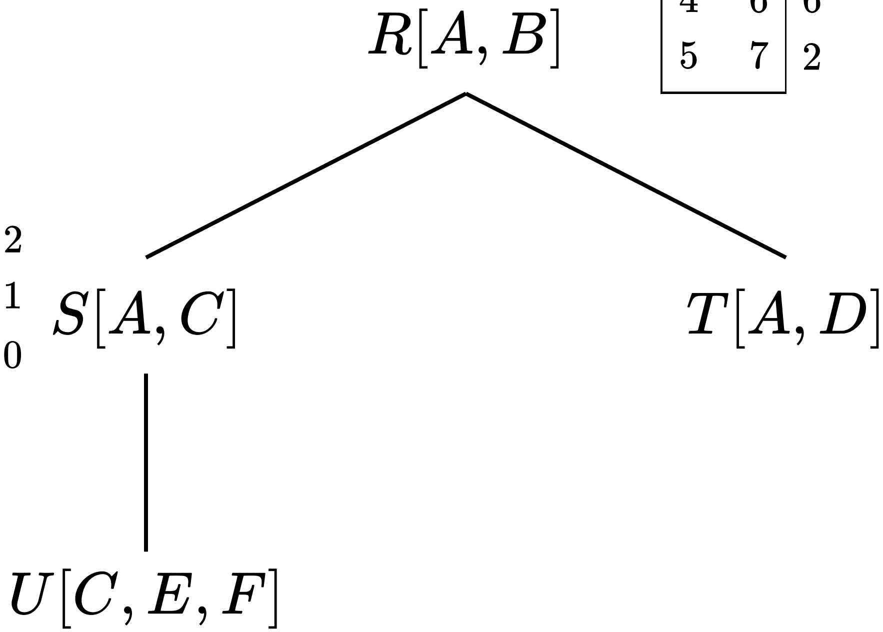
R	A	B	
	4	6	6
	5	7	2

S	A	C	
	4	1	2
	5	2	1
	4	3	0

T	A	D	
	4	1	1
	4	2	1
	4	3	1
	5	4	1
	5	5	1

U	C	E	F	
	1	3	6	1
	1	4	7	1
	2	5	8	1

$$Q(x, y, z, u, v, w) = R(x, y) \wedge S(x, z) \wedge T(x, u) \wedge U(z, v, w)$$



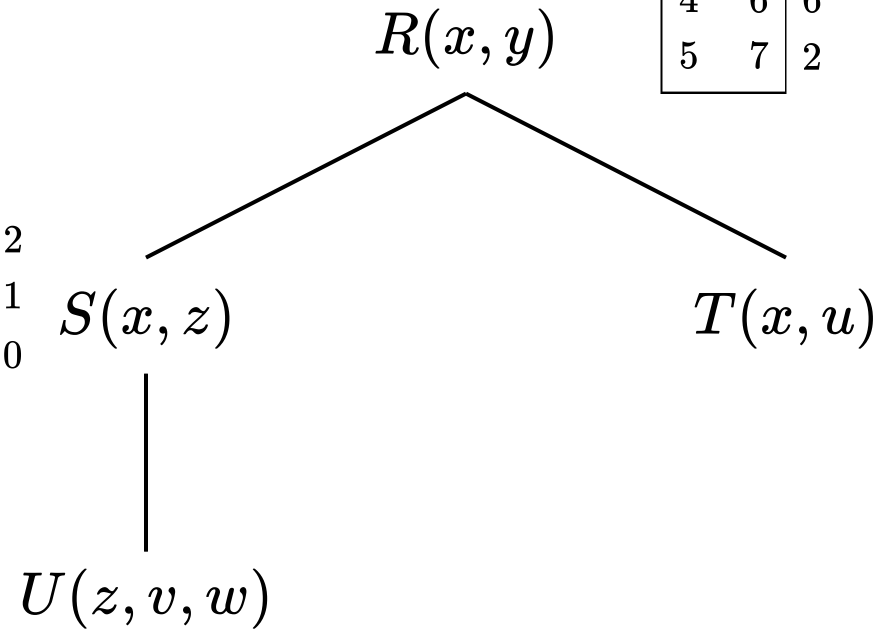
R	A	B	
	4	6	6
	5	7	2

S	A	C	
	4	1	2
	5	2	1
	4	3	0

T	A	D	
	4	1	1
	4	2	1
	4	3	1
	5	4	1
	5	5	1

U	C	E	F	
	1	3	6	1
	1	4	7	1
	2	5	8	1

$$Q(x,y,z,u,v,w) = R(x,y) \wedge S(x,z) \wedge T(x,u) \wedge U(z,v,w)$$



R	x	y	
	4	6	6
	5	7	2

S	x	z	
	4	1	2
	5	2	1
	4	3	0

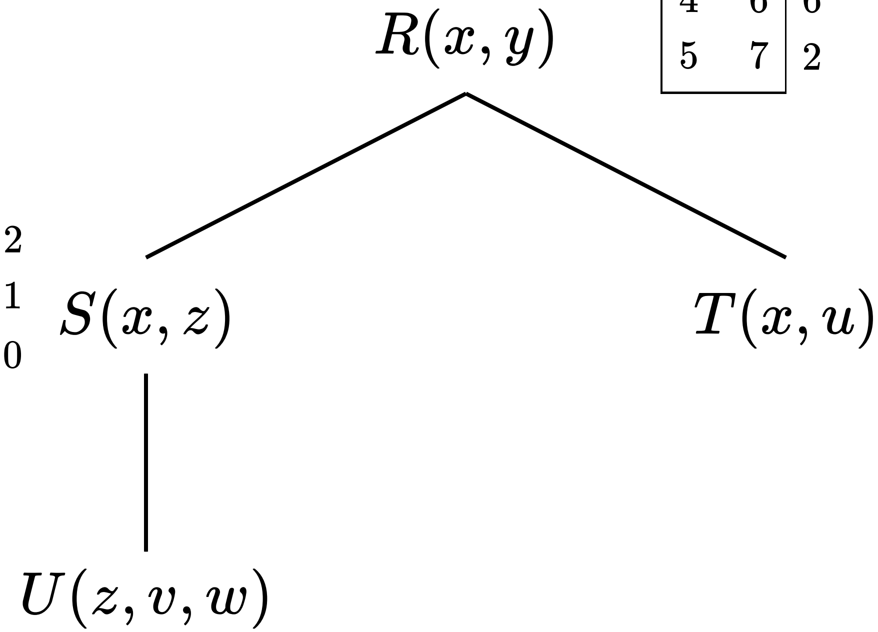
T	x	u	
	4	1	1
	4	2	1
	4	3	1
	5	4	1
	5	5	1

U	z	v	w	
	1	3	6	1
	1	4	7	1
	2	5	8	1

$$Q'(x, w) = \exists y \exists z \exists u \exists v \, [R(x, y) \wedge S(x, z) \wedge T(x, u) \wedge U(z, v, w)]$$

Q'	x	w
	4	6
	4	7
	5	8

R	x	y	
	4	6	6
	5	7	2



S	x	z	
	4	1	2
	5	2	1
	4	3	0

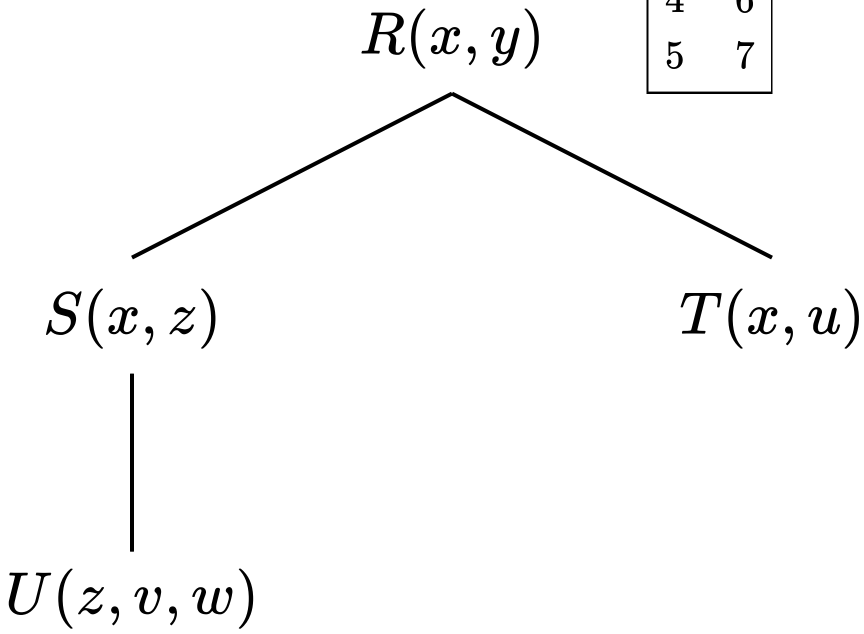
T	x	u	
	4	1	1
	4	2	1
	4	3	1
	5	4	1
	5	5	1

U	z	v	w	
	1	3	6	1
	1	4	7	1
	2	5	8	1

$$Q'(x, w) = \exists y \exists z \exists u \exists v \, [R(x, y) \wedge S(x, z) \wedge T(x, u) \wedge U(z, v, w)]$$

Q'	x	w
	4	6
	4	7
	5	8

R	x	y
	4	6
	5	7



S	x	z
	4	1
	5	2
	4	3

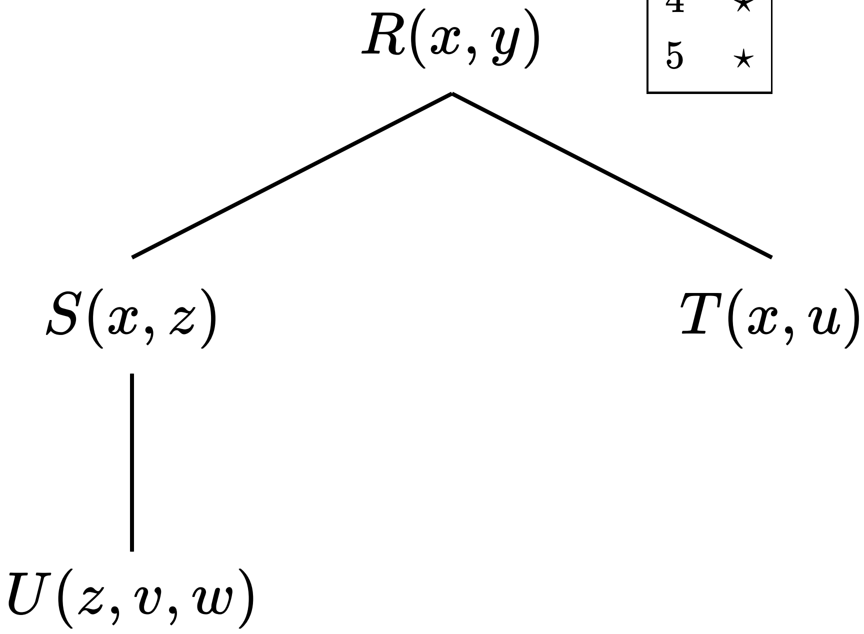
T	x	u
	4	1
	4	2
	4	3
	5	4
	5	5

U	z	v	w
	1	3	6
	1	4	7
	2	5	8

$$Q'(x, w) = \exists y \exists z \exists u \exists v \, [R(x, y) \wedge S(x, z) \wedge T(x, u) \wedge U(z, v, w)]$$

Q'	x	w
	4	6
	4	7
	5	8

R	x	y
	4	$\star$
	5	$\star$



S	x	z
	4	$\star$
	5	$\star$
	4	$\star$

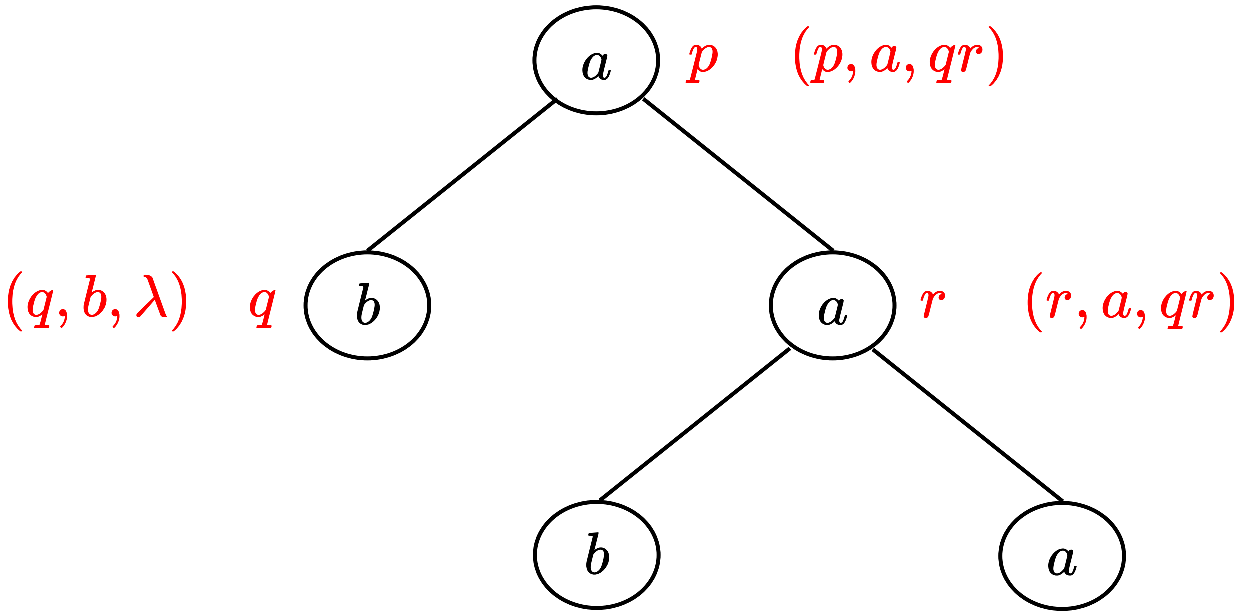
T	x	u
	4	$\star$
	4	$\star$
	4	$\star$
	5	$\star$
	5	$\star$

U	z	v	w
	$\star$	$\star$	6
	$\star$	$\star$	7
	$\star$	$\star$	8

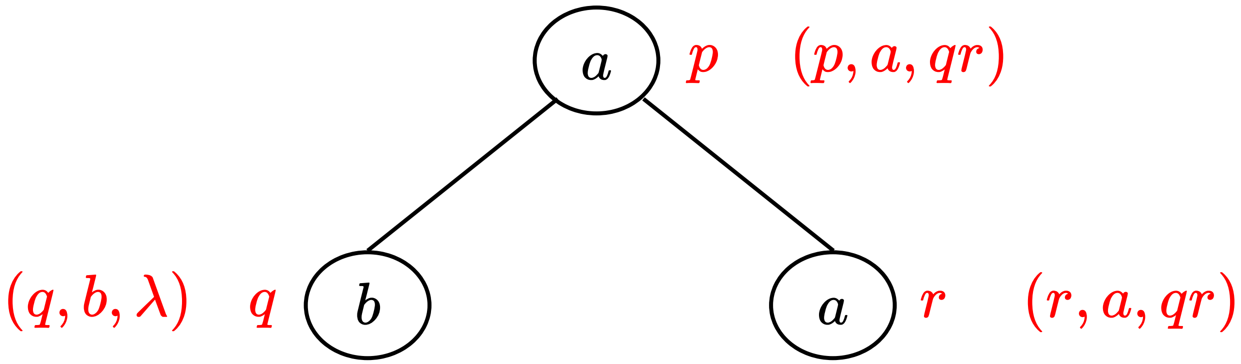
The main ingredient in the solution: Tree automata

This is the right representation for the problem of counting the number of answers to an acyclic conjunctive query

Tree automata



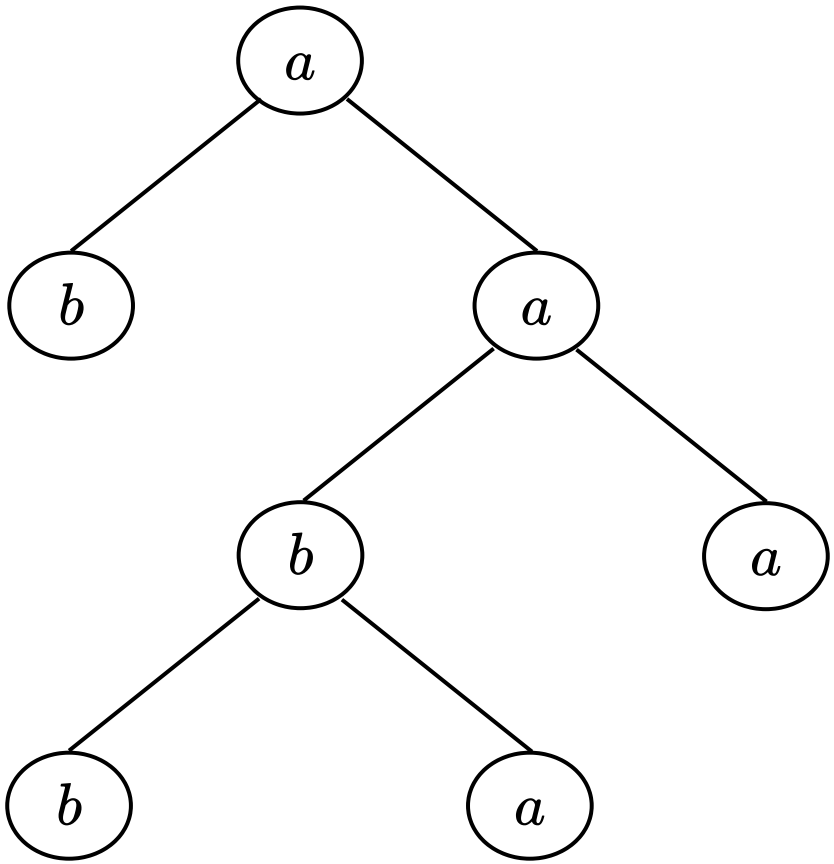
Tree automata



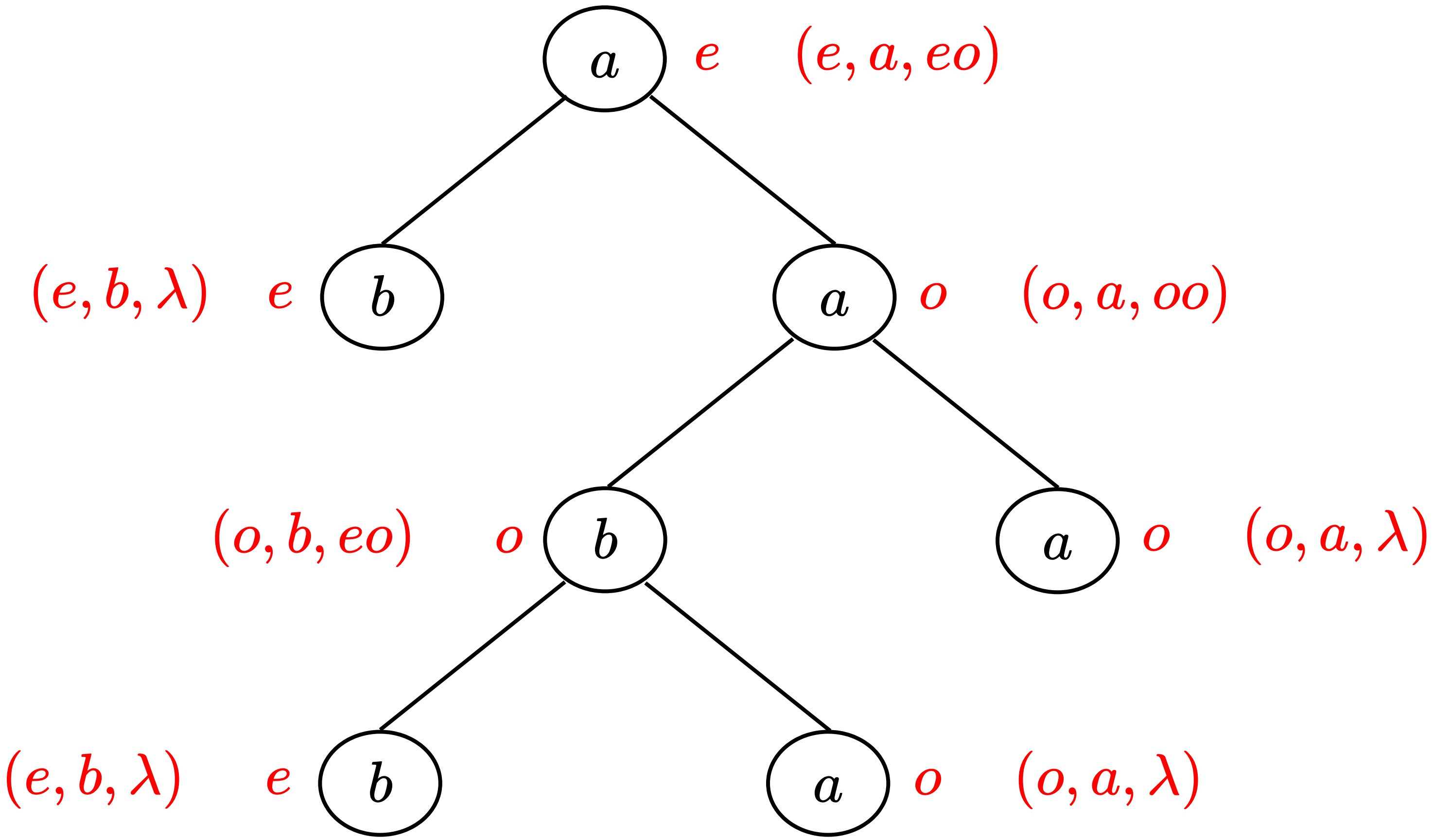
Tree automata: (Q, Σ, Δ, I)

- $Q = \{p, q, r\}$ is the set of states
- $\Sigma = \{a, b\}$ is the alphabet
- $I = \{p\}$ is the set of initial states
- $\Delta = \{(p, a, qr), (q, b, \lambda), (r, a, qr)\}$ is the transition relation

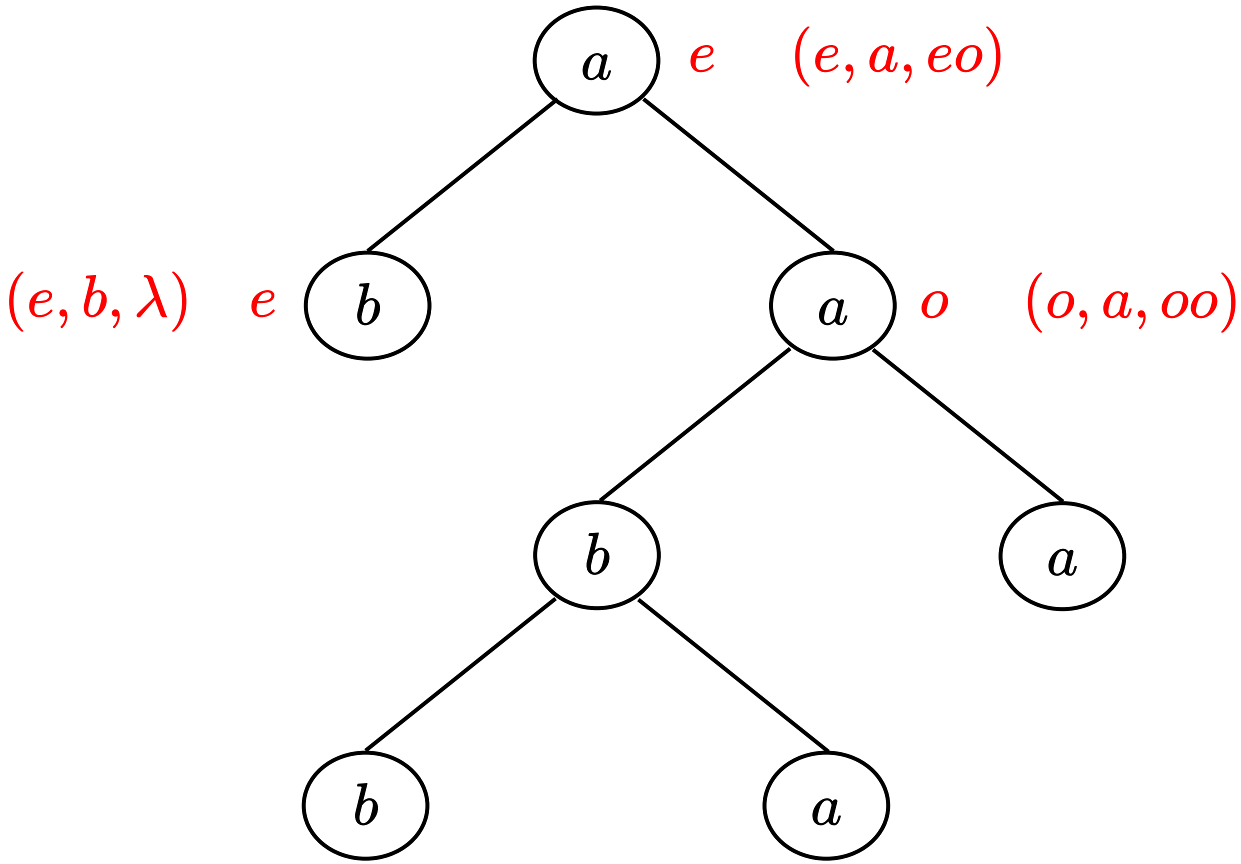
Tree automata



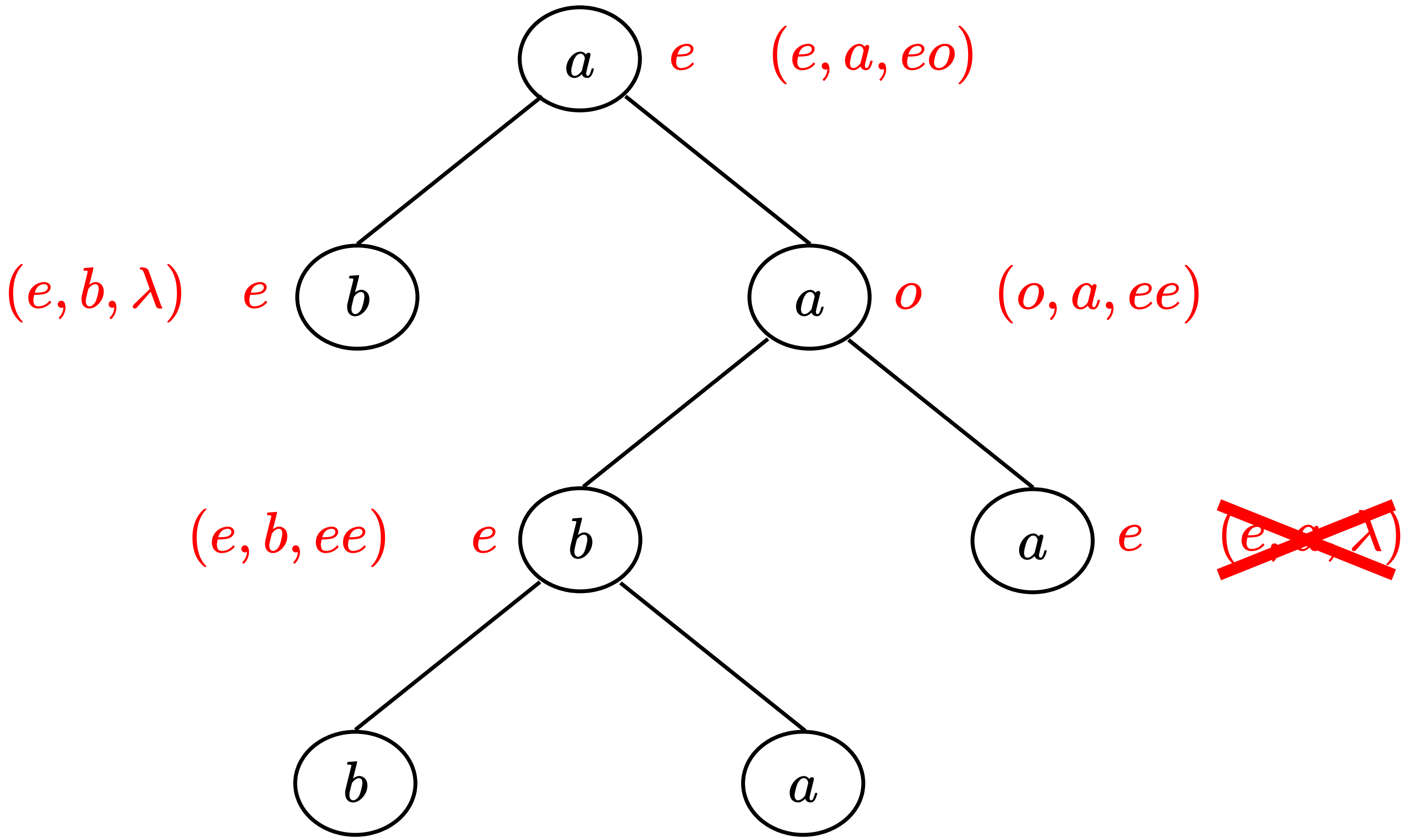
Tree automata



Tree automata



Tree automata



$$Q'(x,w) = \exists y \exists z \exists u \exists v \, [R(x,y) \wedge S(x,z) \wedge T(x,u) \wedge U(z,v,w)]$$

Q'	x	w
	4	6
	4	7
	5	8

R	x	y
	4	6
	5	7

S	x	z
	4	1
	5	2
	4	3

T	x	u
	4	1
	4	2
	4	3
	5	4
	5	5

U	z	v	w
	1	3	6
	1	4	7
	2	5	8

Alphabet:

$R(4,\star)$	$S(4,\star)$	$T(4,\star)$	$U(\star,\star,6)$
$R(5,\star)$	$S(5,\star)$	$T(5,\star)$	$U(\star,\star,7)$
			$U(\star,\star,8)$

States:

$R(4,6)$	$S(4,1)$		$U(1,3,6)$
$R(5,7)$	$S(5,2)$	$\cdots$	$U(1,4,7)$
	$S(4,3)$		$U(2,5,8)$

$$Q'(x, w) = \exists y \exists z \exists u \exists v \, [R(x, y) \wedge S(x, z) \wedge T(x, u) \wedge U(z, v, w)]$$

Q'	x	w
	4	6
	4	7
	5	8

R	x	y
	4	6
	5	7

S	x	z
	4	1
	5	2
	4	3

T	x	u
	4	1
	4	2
	4	3
	5	4
	5	5

U	z	v	w
	1	3	6
	1	4	7
	2	5	8

$$Q'(x,w) = \exists y \exists z \exists u \exists v \, [R(x,y) \wedge S(x,z) \wedge T(x,u) \wedge U(z,v,w)]$$

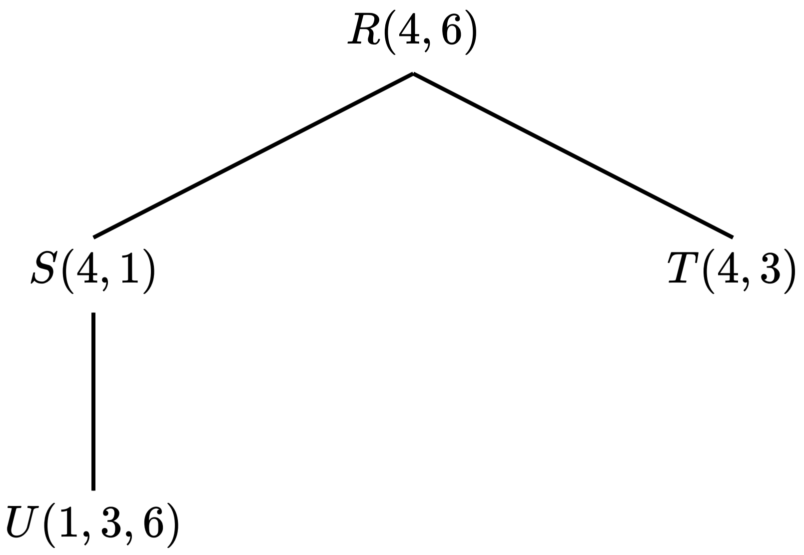
Q'	x	w
	4	6
	4	7
	5	8

R	x	y
	4	6
	5	7

S	x	z
	4	1
	5	2
	4	3

T	x	u
	4	1
	4	2
	4	3
	5	4
	5	5

U	z	v	w
	1	3	6
	1	4	7
	2	5	8



$$Q'(x, w) = \exists y \exists z \exists u \exists v \, [R(x, y) \wedge S(x, z) \wedge T(x, u) \wedge U(z, v, w)]$$

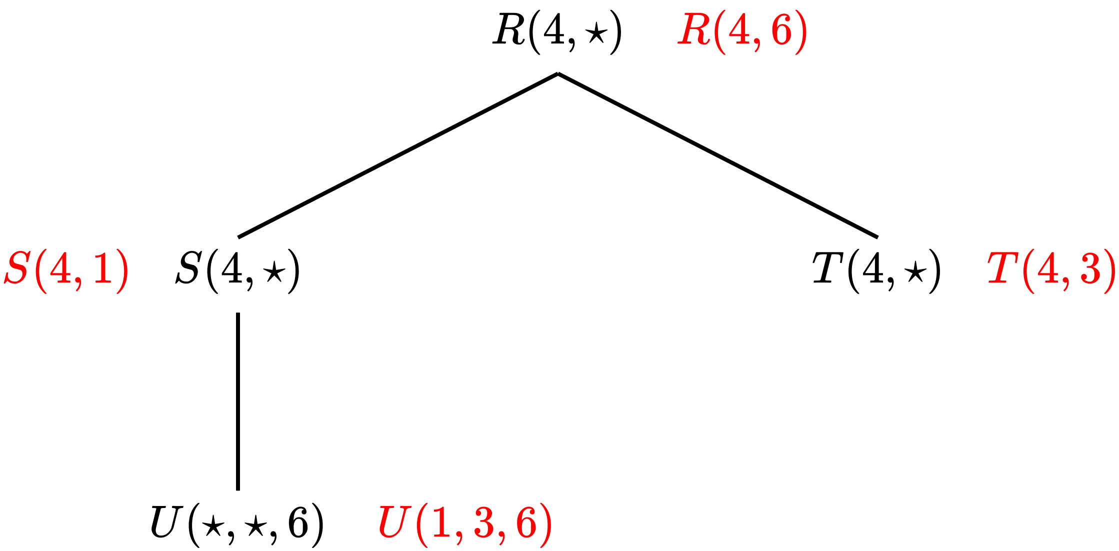
Q'	x	w
	4	6
	4	7
	5	8

R	x	y
	4	6
	5	7

S	x	z
	4	1
	5	2
	4	3

T	x	u
	4	1
	4	2
	4	3
	5	4
	5	5

U	z	v	w
	1	3	6
	1	4	7
	2	5	8



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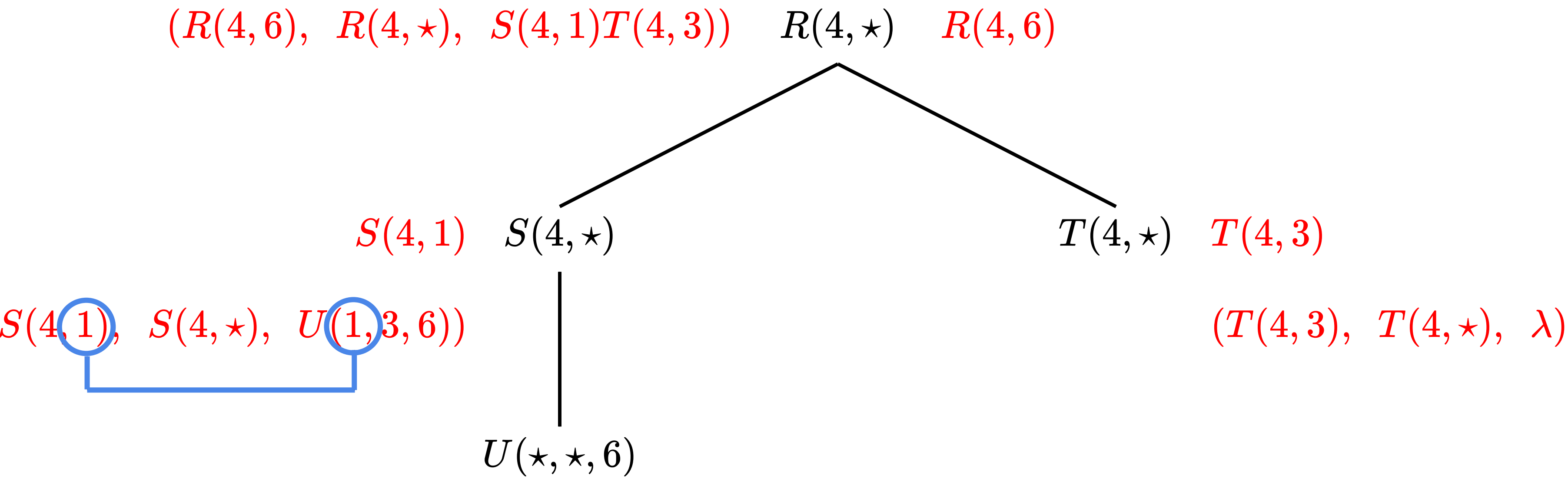
Q'	x	w
	4	6
	4	7
	5	8

R	x	y
	4	6
	5	7

S	x	z
	4	1
	5	2
	4	3

T	x	u
	4	1
	4	2
	4	3
	5	4
	5	5

U	z	v	w
	1	3	6
	1	4	7
	2	5	8



$$Q'(x,w) = \exists y \exists z \exists u \exists v \, [R(x,y) \wedge S(x,z) \wedge T(x,u) \wedge U(z,v,w)]$$

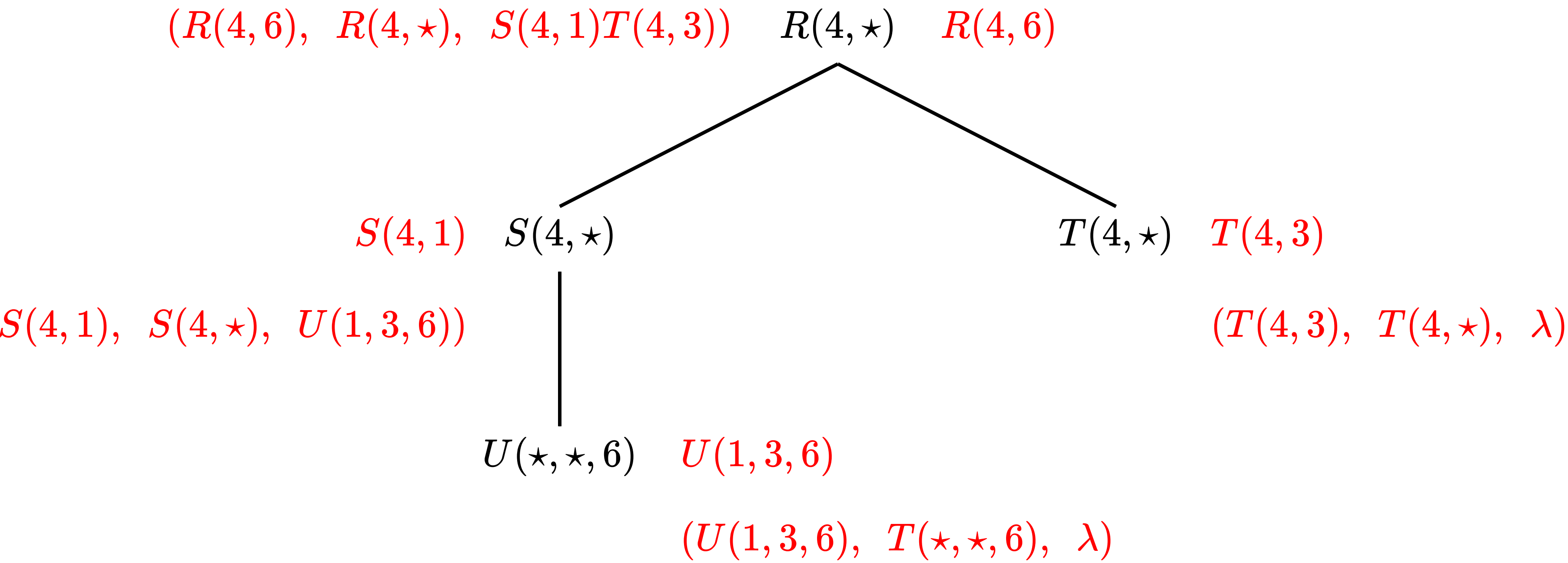
Q'	x	w
	4	6
	4	7
	5	8

R	x	y
	4	6
	5	7

S	x	z
	4	1
	5	2
	4	3

T	x	u
	4	1
	4	2
	4	3
	5	4
	5	5

U	z	v	w
	1	3	6
	1	4	7
	2	5	8



$$Q'(x,w) = \exists y \exists z \exists u \exists v\, [R(x,y) \wedge S(x,z) \wedge T(x,u) \wedge U(z,v,w)]$$

Q'	x	w
	4	6
	4	7
	5	8

R	x	y
	4	6
	5	7

S	x	z
	4	1
	5	2
	4	3

T	x	u
	4	1
	4	2
	4	3
	5	4
	5	5

U	z	v	w
	1	3	6
	1	4	7
	2	5	8

The problem to solve: **count the number of trees with 4 nodes accepted by the tree automaton**

The problem #TA

Input: A tree automaton (TA) T over the alphabet $\{0, 1\}$ and a number n (given in unary)

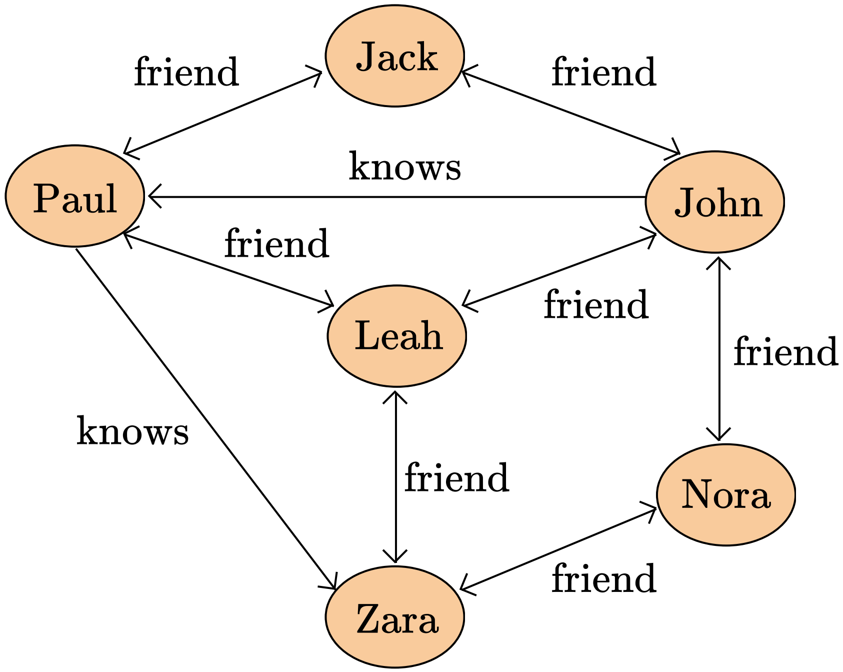
Output: Number of trees t such that $t \in L(T)$ and the number of nodes of t is n

What is the complexity of this problem?

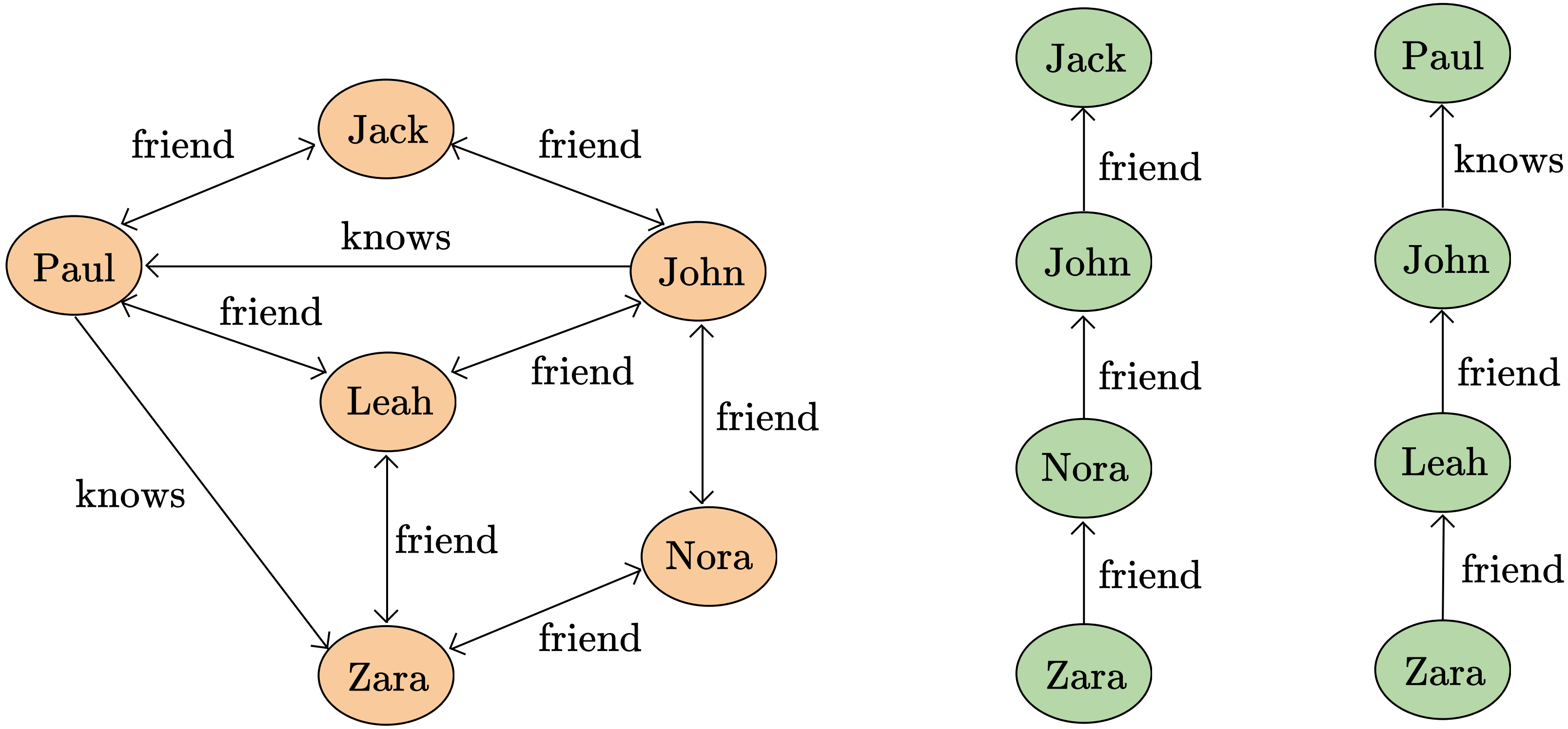
A detour: graph databases

Graph databases

G:



A query: $(\text{friend} + \text{knows})^*$



Two fundamental problems

- COUNT: count the number of paths p in G such that p conforms to regular expression r and the length of p is n
- GEN: generate uniformly at random a path p in G such that p conforms to r and the length of p is n

COUNT is a difficult problem

COUNT is #P-complete

The decision version of the problem can be solved in polynomial time, so this problem *could* admit an FPRAS

The connection with #TA

The problem #NFA:

Input:	A non-deterministic finite automaton (NFA) A over the alphabet $\{0, 1\}$ and a number n (given in unary)
Output:	Number of words w such that $w \in L(A)$ and the length of w is n

The connection with #TA

COUNT and #NFA are polynomially equivalent under parsimonious reductions

- This implies that if an FPRAS exists for one of them, then it exists for the other

#TA is #P-complete

- The construction of an FPRAS for #NFA seems to be a natural step to construct an FPRAS for #TA

Existence of an FPRAS for #NFA

How do we obtain such an approximation algorithm?

- We use the techniques learned in the previous part of the tutorial!

An FPRAS for #NFA

Input: An NFA A over the alphabet $\{0, 1\}$ and a number n (given in unary)

Output: Number of words w such that $w \in L(A)$ and the length of w is n

Assume that $L_n(A) = \{w \in L(A) \mid |w| = n\}$, so that the output of #NFA is $|L_n(A)|$

An FPRAS for #NFA

The input of the approximation algorithm: A, n and $\varepsilon \in (0, 1)$

The task is to compute a number N that is a $(1 \pm \varepsilon)$-approximation of $|L_n(A)|$:

$$\mathbf{Pr}\left((1-\varepsilon)|L_n(A)| \leq N \leq (1+\varepsilon)|L_n(A)|\right) \geq \frac{3}{4}$$

Moreover, number N has to be computed in time $\mathrm{poly}(m, n, \frac{1}{\varepsilon})$, where m is the number of states of A

An FPRAS for #NFA

If we think of the approximation algorithm as an estimator $\hat{N}$ for $|L_n(A)|$, then we need to construct the following confidence interval:

$$\Pr\left(|L_n(A)| \in \left[\frac{\hat{N}}{1+\varepsilon}, \frac{\hat{N}}{1-\varepsilon}\right]\right) \geq \frac{3}{4}$$

Constructing an FPRAS for #NFA [ACJR21a]

Assume that $A = (Q, \{0,1\}, \Delta, I, F)$

- Q is a finite set of states
- $\Delta \subseteq Q \times \{0,1\} \times Q$ is the transition relation
- $I \subseteq Q$ is the set of initial states
- $F \subseteq Q$ is the set of final states

First component: unroll automaton A

Construct A_{unroll} from A:

- for each state $q \in Q$, include copies $q_0, q_1, \ldots, q_n$ in A_{unroll}
- for each transition $(p, a, q) \in \Delta$ and $i \in \{0, 1, ..., n-1\}$, include transition (p_i, a, q_{i+1}) in A_{unroll}

Besides, eliminate from A_{unroll} unnecessary states: each state q_i is reachable from an initial state p_0 ($p \in I$)

Second component: a sketch to be used in the estimation

Define $L(q_i)$ as the set of strings w such that there is a path from an initial state p_0 to q_i labeled with w

- Notice that $|w| = i$

Besides, define for every $X \subseteq Q$:

$$L(X^i) = \bigcup_{q \in X} L(q^i)$$

Then the task is to compute an estimation of $|L(F^n)|$

Second component: a sketch to be used in the estimation

From now assume that $m = |Q|$, and let

$$\kappa = \left\lceil \frac{nm}{\varepsilon} \right\rceil$$

We maintain for each state q_i :

- $N(q^i)$: a $(1 \pm \kappa^{-2})^i$-approximation of $|L(q^i)|$
- $S(q^i)$: a multiset of uniform samples from $L(q^i)$ of size $2\kappa^7$

Second component: a sketch to be used in the estimation

Data structure to be inductively computed:

$$\mathrm{Sketch}[i] = \{N(q^j), S(q^j) \mid 0 \leq j \leq i \text{ and } q \in Q\}$$

The algorithm template

1. Construct A_{unroll} from A
2. For each state $q \in I$, set $N(q^0) = |L(q^0)| = 1$ and $S(q^0) = L(q^0) = \{\lambda\}$
3. For each $i \in \{0, \ldots, n-1\}$ and state $q \in Q$:

    3.1. Compute $N(q_{i+1})$ given $\mathrm{Sketch}[i]$

    3.2. Sample polynomially many uniform elements from $L(q^{i+1})$ using $N(q^{i+1})$ and $\mathrm{Sketch}[i]$, and let $S(q^{i+1})$ be the multiset of uniform samples obtained

4. Return an estimation of $|L(F^n)|$ given $\mathrm{Sketch}[n]$

Computing an estimation $N(F^n)$ of $|L(F^n)|$

We use notation $N(X^i)$ for an estimation $|L(X^i)|$

Such an estimation is not only needed in the last step of the algorithm, but also in the inductive construction of $\mathrm{Sketch}[i]$:

3. For each $i \in \{0, \ldots, n-1\}$ and state $q \in Q$:
    3.1. Compute $N(q_{i+1})$ given $\mathrm{Sketch}[i]$
    3.2. Sample polynomially many uniform elements from $L(q^{i+1})$ using $N(q^{i+1})$ and $\mathrm{Sketch}[i]$, and let $S(q^{i+1})$ be the multiset of uniform samples obtained

Computing an estimation $N(X^i)$ of $|L(X^i)|$

Recall that

$$L(X^i) = \bigcup_{p \in X} L(p^i)$$

Notice that $L(X^i) = \sum_{p \in X} |L(p^i)|$ is not true in general

But the following holds, given a linear order $<$ on Q:

$$|L(X^i)| = \sum_{p \in X} \left| L(p^i) \setminus \bigcup_{q \in X \,:\, q < p} L(q^i) \right|$$

Computing an estimation $N(X^i)$ of $|L(X^i)|$

We have that:

$$|L(X^i)| = \sum_{p \in X} \Big| L(p^i) \setminus \bigcup_{q \in X \,:\, q < p} L(q^i) \Big|$$

$$= \sum_{p \in X} |L(p^i)| \frac{\big| L(p^i) \setminus \bigcup_{q \in X \,:\, q < p} L(q^i) \big|}{|L(p^i)|}$$

So we will use the following approximation:

$$= \sum_{p \in X} |L(p^i)| \frac{\big| L(p^i) \setminus \bigcup_{q \in X \,:\, q < p} L(q^i) \big|}{|L(p^i)|}$$

Computing an estimation $N(X^i)$ of $|L(X^i)|$

We have that:

$$
\begin{aligned}
|L(X^i)| &= \sum_{p \in X} \Big| L(p^i) \setminus \bigcup_{q \in X \,:\, q < p} L(q^i) \Big| \\
&= \sum_{p \in X} |L(p^i)| \frac{\left| L(p^i) \setminus \bigcup_{q \in X \,:\, q<p} L(q^i) \right|}{|L(p^i)|}
\end{aligned}
$$

So we will use the following approximation:

$$
= \sum_{p \in X} \qquad \frac{\left| \qquad \setminus \bigcup_{q \in X \,:\, q<p} L(q^i) \right|}{| \qquad |}
$$

Computing an estimation $N(X^i)$ of $|L(X^i)|$

We have that:

$$|L(X^i)| = \sum_{p \in X} \Big| L(p^i) \setminus \bigcup_{q \in X\,:\,q<p} L(q^i) \Big|$$

$$= \sum_{p \in X} |L(p^i)| \frac{\left| L(p^i) \setminus \bigcup_{q \in X\,:\,q<p} L(q^i) \right|}{|L(p^i)|}$$

So we will use the following approximation:

$$= \sum_{p \in X} {\color{red} N(p^i)} \frac{\left| \qquad\quad \setminus \bigcup_{q \in X\,:\,q<p} L(q^i) \right|}{\left| \qquad\quad \right|}$$

Computing an estimation $N(X^i)$ of $|L(X^i)|$

We have that:

$$|L(X^i)| = \sum_{p \in X} \Big| L(p^i) \setminus \bigcup_{q \in X \,:\, q < p} L(q^i) \Big|$$

$$= \sum_{p \in X} |L(p^i)| \frac{\left| L(p^i) \setminus \bigcup_{q \in X \,:\, q < p} L(q^i) \right|}{|L(p^i)|}$$

So we will use the following approximation:

$$N(X^i) = \sum_{p \in X} N(p^i) \frac{\left| S(p^i) \setminus \bigcup_{q \in X \,:\, q < p} L(q^i) \right|}{|S(p^i)|}$$

Computing an estimation $N(X^i)$ of $|L(X^i)|$

$N(X^i)$ can be computed in polynomial time in the size of $\mathrm{Sketch}[i]$

- $S(p^i) \setminus \bigcup_{q \in X \,:\, q<p} L(q^i)$ is constructed by checking for each $w \in S(p^i)$ whether w is not in $L(q^i)$ for every $q \in X$ with $q < p$

What guarantees that $N(X^i)$ is a good estimation of $|L(X^i)|$?

An invariant to be mantained

$\mathcal{E}(i)$ holds if for every $p \in Q$ and $X \subseteq Q$:

$$\left| \frac{\left| L(p^i) \setminus \bigcup_{q \in X} L(q^i) \right|}{\left| L(p^i) \right|} - \frac{\left| S(p^i) \setminus \bigcup_{q \in X} L(q^i) \right|}{\left| S(p^i) \right|} \right| < \frac{1}{\kappa^3}$$

The use of the main property

3. For each $i \in \{0, \ldots, n-1\}$ and state $q \in Q$:
   3.1. Compute $N(q_{i+1})$ given $\mathrm{Sketch}[i]$
   3.2. Sample polynomially many uniform elements from $L(q^{i+1})$ using $N(q^{i+1})$ and $\mathrm{Sketch}[i]$, and let $S(q^{i+1})$ be the multiset of uniform samples obtained

Lemma: If $\mathcal{E}(i)$ holds and $N(p^i)$ is a $(1 \pm \kappa)^i$-approximation of $|L(p^i)|$ for every $p \in Q$, then $N(X^i)$ is a $(1 \pm \kappa^{-2})^{i+1}$-approximation of $|L(X^i)|$ for every $X \subseteq Q$

The use of the main property

$\mathcal{E}(0)$ holds and $N(p^0)$ is a $(1 \pm \kappa^{-2})^0$-approximation of $|L(p^0)|$ for every $p \in Q$

- Recall that $N(p^0) = |L(p^0)|$ and $S(p^0) = L(p^0)$ for every $p \in Q$

Then $N(X^0)$ is a $(1 \pm \kappa^{-2})$-approximation of $|L(X^0)|$ for every $X \subseteq Q$

- We want to use the values $N(X^0)$ to estimate the values $N(p^1)$

The use of the main property

For $p \in Q$, define:

$$\begin{aligned} Y &= \{q^0 \mid (q^0, 0, p^1) \text{ is a transition in } A_{\text{unroll}}\} \\ Z &= \{q^0 \mid (q^0, 1, p^1) \text{ is a transition in } A_{\text{unroll}}\} \end{aligned}$$

Then $L(p^1) = L(Y) \cdot \{0\} \uplus L(Z) \cdot \{1\}$

- So that $|L(p^1)| = |L(Y)| + |L(Z)|$

The use of the main property

For $p \in Q$, define:

$$\begin{aligned} Y &= \{q^0 \mid (q^0, 0, p^1) \text{ is a transition in } A_{\text{unroll}}\} \\ Z &= \{q^0 \mid (q^0, 1, p^1) \text{ is a transition in } A_{\text{unroll}}\} \end{aligned}$$

Then given that $N(Y)$ is a $(1 \pm \kappa^{-2})$-approximation of $|L(Y)|$ and $N(Z)$ is a $(1 \pm \kappa^{-2})$-approximation of $|L(Z)|$:

$N(Y) + N(Z)$ is a $(1 \pm \kappa^{-2})$-approximation of

$$N(p^1) = |L(Y)| + |L(Z)|$$

Main property: a summary

$\mathcal{E}(0)$ holds and $N(p^0)$ is a $(1 \pm \kappa^{-2})^0$-approximation of $|L(p^0)|$ for every $p \in Q$

$$\Downarrow$$

$N(X^0)$ is a $(1 \pm \kappa^{-2})^1$-approximation of $|L(X^0)|$ for every $X \subseteq Q$

$$\Downarrow$$

$N(p^1) = N(R_0(p^1)) + N(R_1(p^1))$ is a $(1 \pm \kappa^{-2})^1$-approximation of $L(p^1)$ for every $p \in Q$

where $R_b(p^1) = \{q^0 \mid (q^0, b, p^1) \text{ is a transition in } A_{\text{unroll}}\}$

Main property: a summary

$N(p^1)$ is a $(1 \pm \kappa^{-2})^1$-approximation of $|L(p1)|$ for every $p \in Q$

Main property: a summary

$\mathcal{E}(1)$ holds and $N(p^1)$ is a $(1 \pm \kappa^{-2})^1$-approximation of $|L(p1)|$ for every $p \in Q$

$$\Downarrow$$

$N(X^1)$ is a $(1 \pm \kappa^{-2})^2$-approximation of $|L(X^1)|$ for every $X \subseteq Q$

$$\Downarrow$$

$N(p^2) = N(R_0(p^2)) + N(R_1(p^2))$ is a $(1 \pm \kappa^{-2})^2$-approximation of $L(p^2)$ for every $p \in Q$

where $R_b(p^2) = \{q^1 \mid (q^1, b, p^2) \text{ is a transition in } A_{\text{unroll}}\}$

The final result

Proposition: If $\mathcal{E}(i)$ holds for every $i \in \{0, 1, ..., n\}$, then $N(F^n)$ is a $(1 \pm \varepsilon)$-approximation of $|L(F^n)|$

How can we maintain property $\mathcal{E}(i)$?

Sampling from a state

We need to construct the multiset $S(q^{i+1})$ of uniform samples

Recall that:

- $S(q^{i+1})$ contains $2\kappa^7$ words from $L(q^{i+1})$
- $S(q^{i+1})$ is computed assuming that $N(q^{i+1})$ and $\text{Sketch}[i] = \{N(q^j), S(q^j) \mid 0 \leq j \leq i\}$ have already been constructed

To recall

1. Construct A_{unroll} from A
2. For each state $q \in I$, set $N(q^0) = |L(q^0)| = 1$ and $S(q^0) = L(q^0) = \{\lambda\}$
3. For each $i \in \{0, \ldots, n-1\}$ and state $q \in Q$:

    3.1. Compute $N(q_{i+1})$ given $\mathrm{Sketch}[i]$

    3.2. Sample polynomially many uniform elements from $L(q^{i+1})$ using $N(q^{i+1})$ and $\mathrm{Sketch}[i]$, and let $S(q^{i+1})$ be the multiset of uniform samples obtained

4. Return an estimation of $|L(F^n)|$ given $\mathrm{Sketch[n]}$

Sampling from q^{i+1}

To generate a sample in $L(q^{i+1})$, we construct a sequence of words $w^{i+1}, w^i, \ldots, w^1, w^0$ such that

- $w^{i+1} = \lambda$
- $w^j = b^j w^{j+1}$ with $b^j \in \{0, 1\}$
- $w^0 \in L(q^{i+1})$

To choose $w^i = bw^{i+1}$, construct for $b = 0, 1$:

$$P_b = \{p^i \mid (p^i, b, q^{i+1}) \text{ is a transition in } A_{\text{unroll}}\}$$

Sampling from q^{i+1}

P_0 and P_1 are sets of states at layer i

Sampling from q^{i+1}

P_0 and P_1 are sets of states at layer i

We compute $N(P_0)$ and $N(P_1)$ as follows:

$$N(X^i) = \sum_{p \in X} N(p^i) \frac{\left| S(p^i) \setminus \bigcup_{q \in X \,:\, q < p} L(q^i) \right|}{|S(p^i)|}$$

We choose $b \in \{0, 1\}$ with probability:

$$\frac{N(P_b)}{N(P_0) + N(P_1)}$$

We could have started from a set of states

Previous procedure works for every set of states P^{i+1}:

$$P_b = \{p^i \mid \exists r^{i+1} \in P^{i+1} : (p^i, b, r^{i+1}) \text{ is a transition in } A_{\text{unroll}}\}$$

In particular, we applied the procedure for $P^{i+1} = \{q^{i+1}\}$

The sampling algorithm

1. $\text{prob} = \varphi_0$
2. $P^{i+1} = \{q^{i+1}\}$
3. for $j = i + 1$ to 1 do
    3.1. $P_{j,0} = \{p^{j-1} \mid \exists r^j \in P^j : (p^{j-1}, 0, p^j) \text{ is a transition in } A_{\text{unroll}}\}$
    3.2. $P_{j,1} = \{p^{j-1} \mid \exists r^j \in P^j : (p^{j-1}, 1, p^j) \text{ is a transition in } A_{\text{unroll}}\}$
    3.3. Generate $b \in R_i \in \{0,1\}$ with probability $\frac{N(P_{j,b})}{N(P_{b,0})+N(P_{b,1})}$
    3.4. $w^{j-1} = bw^j$
    3.5. $P^{j-1} = P_{j,b}$
    3.6. $\text{prob} = \text{prop} \cdot \frac{N(P_{j,0})+N(P_{j,1})}{N(P_{j,b})}$
4. reject with probability $1 - \text{prob}$
5. return w^0

As before ...

Let $x = x_1 \cdots x_{i+1}$ be a word in $L(q^{i+1})$

$$\Pr(\text{the output of the procedure is } x)$$

$$= \Pr(w^0 = x \wedge \text{the procedure does not reject})$$

$$= \Pr(\text{the procedure does not reject} \mid w^0 = x) \Pr(w^0 = x)$$

$$= \left(\prod_{j=1}^{i+1} \frac{N(P_{j,x_j})}{N(P_{j,0})\varphi + N(P_{j,1})} \right)^{-1} \cdot \varphi_0 \cdot \left(\prod_{j=1}^{i+1} \frac{N(P_{j,x_j})}{N(P_{j,0}) + N(P_{j,1})} \right)$$

$$= \varphi_0$$

The value of the initial probability φ_0

Lemma: Assume that $\mathcal{E}(j)$ holds for each $j < i + 1$. If $\varphi_0 = \frac{e^{-5}}{N(q^{i+1})}$, then

- $\text{prob} \leq 1$ in each step in the loop
- $\Pr(\text{procedure rejects}) \leq 1 - e^{-9}$
- $\Pr(w^0 = x) = \frac{e^{-5}}{N(q^{i+1})}$ for every $x \in L(q^{i+1})$

Bounding the probability of breaking the main assumption

Recall that $\mathcal{E}(i)$ holds if for every $q \in Q$ and $P \subseteq Q$:

$$\left| \frac{\left|L(q^i) \setminus \bigcup_{p \in P} L(p^i)\right|}{\left|L(q^i)\right|} - \frac{\left|S(q^i) \setminus \bigcup_{p \in P} L(p^i)\right|}{\left|S(q^i)\right|} \right| < \frac{1}{\kappa^3}$$

Bounding the probability of breaking the main assumption

By using Hoeffding's inequality, it is possible to obtain that:

$$\Pr(\mathcal{E}(0) \wedge \cdots \wedge \mathcal{E}(n)) \leq 1 - e^{-\kappa}$$

The complete algorithm: final comments [ACJR21a]

Putting all together, we obtain that the probability that the algorithm returns a wrong estimate is at most $\frac{1}{4}$

The algorithm runs in time $\mathrm{poly}(m, n, \frac{1}{\varepsilon})$

Back to conjunctive queries

The ideas used for the case of NFA can be extended to the case of TA

Theorem [ACJR21b]: #TA admits an FPRAS

Theorem [ACJR21b]: The problem of counting the number of answers to an acyclic conjunctive query admits an FPRAS

- The same holds for each class of conjunctive queries with bounded hypertree width

Research questions

- Development of a general theory for estimation in query optimization [HYPM19]
  - Which estimator should be used given a budget? What is an appropriate notion of budget? What are optimal estimators?

- Understand for which relational algebra operators and aggregates it is posible to develop sampling techniques with (strong) guarantees
  - Develop (very) efficient algorithms to compute these estimators
  - Understand the complexity of computing such estimators (fine-grained complexity)

- Understand for which relational algebra operators and aggregates it is **not** posible to develop sampling techniques with (strong) guarantees
    - What can of guarantees can be provided in these cases?

- Could sample techniques be used for some fundamental tasks for *K*-relations? For first-order logic with semiring semantics?

- Does #CFG admits an FPRAS?

Thanks!

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