Testing thresholds in random geometric graphs

Siqi Liu
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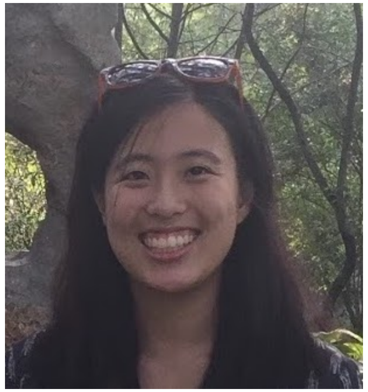
Joint work with



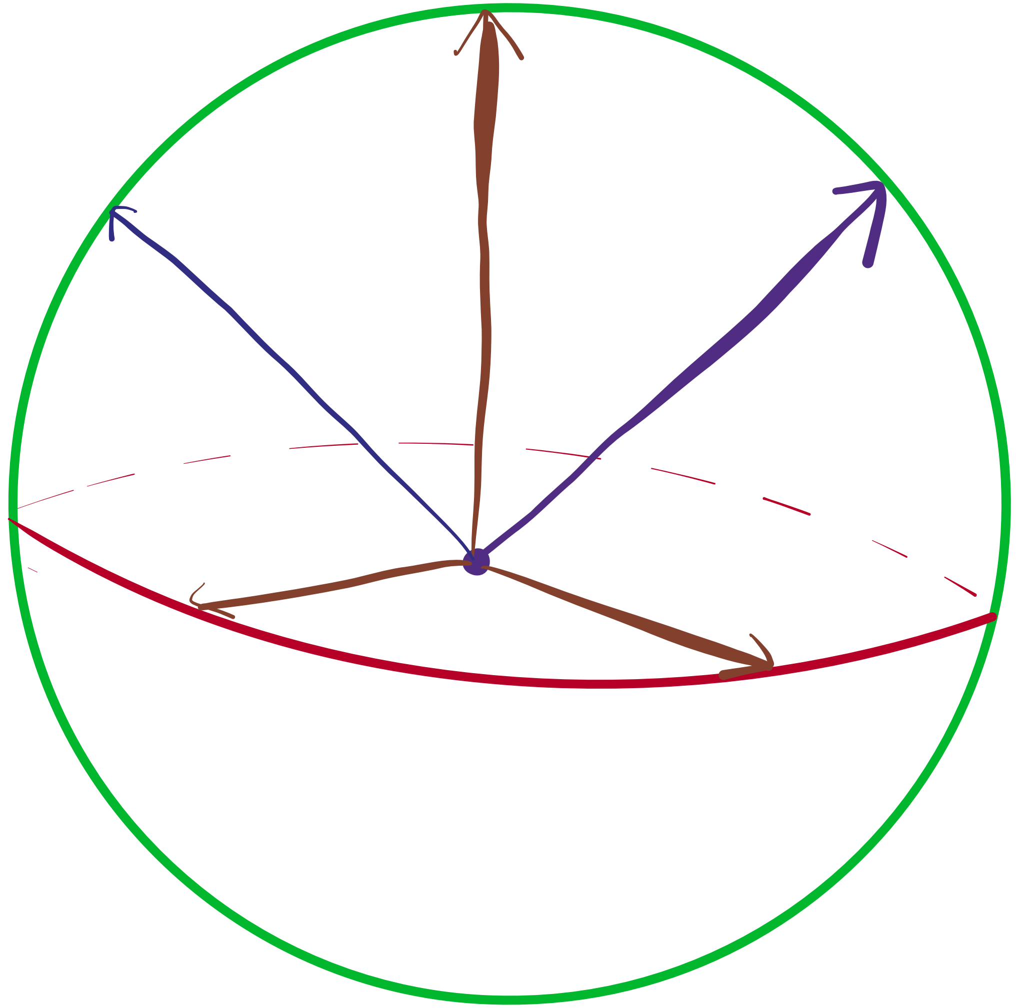
Sidhanth Mohanty
UC Berkeley

Tselil Schramm
Stanford

Elizabeth Yang
UC Berkeley

Random geometric graphs

$v_1, \ldots, v_n \sim \mathbb{S}^{d-1}$

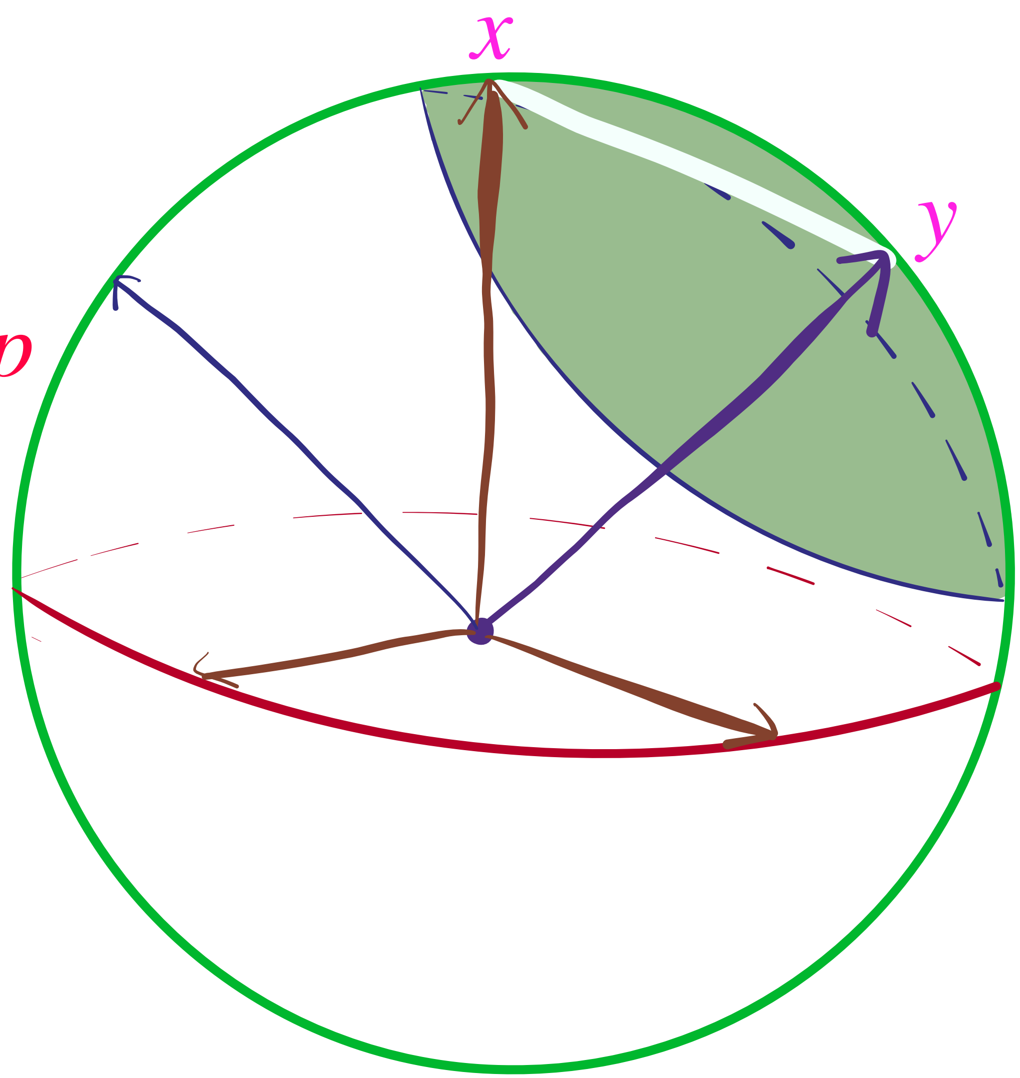


Random geometric graphs

$v_1, \ldots, v_n \sim \mathbb{S}^{d-1}$

xy edge iff $\langle v_x, v_y \rangle \geq \tau(p)$

xy edge with probability p



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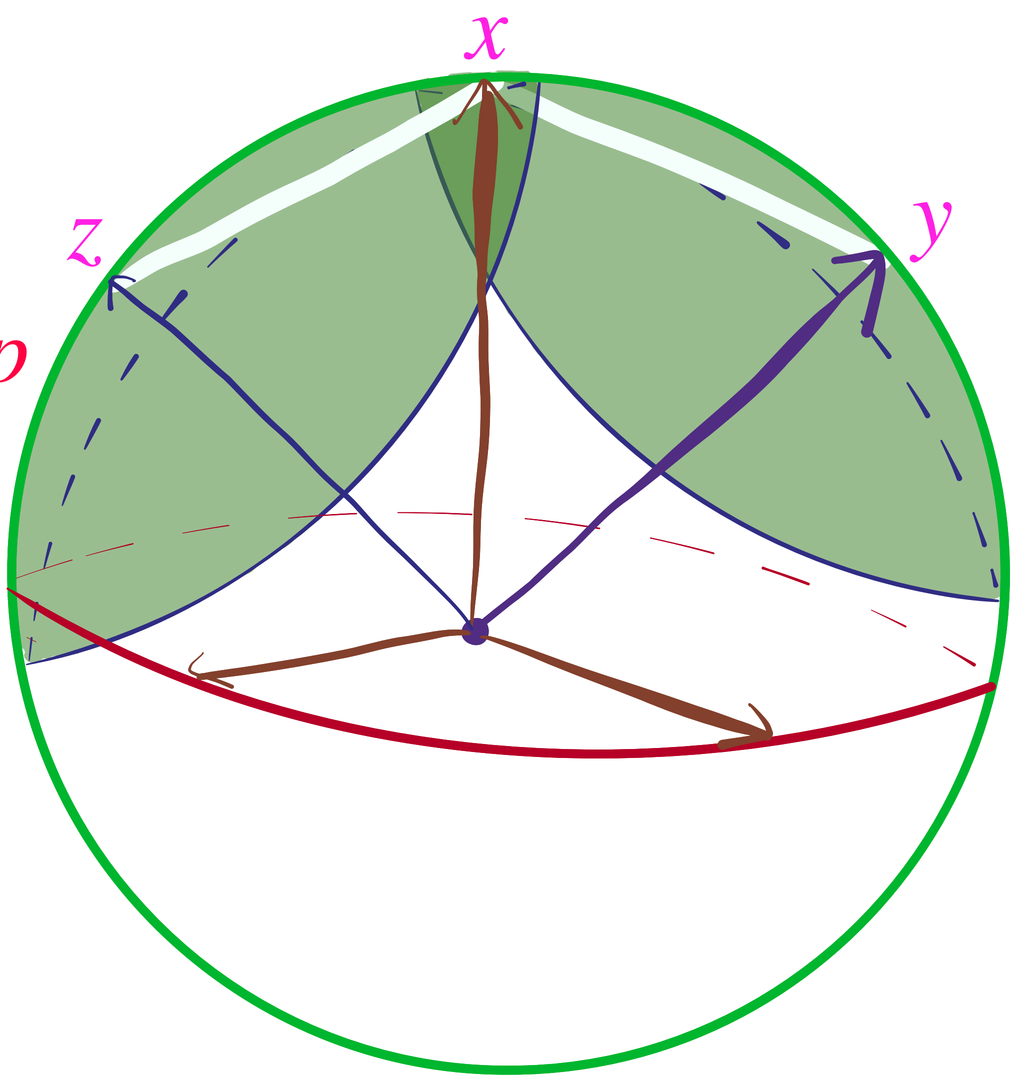


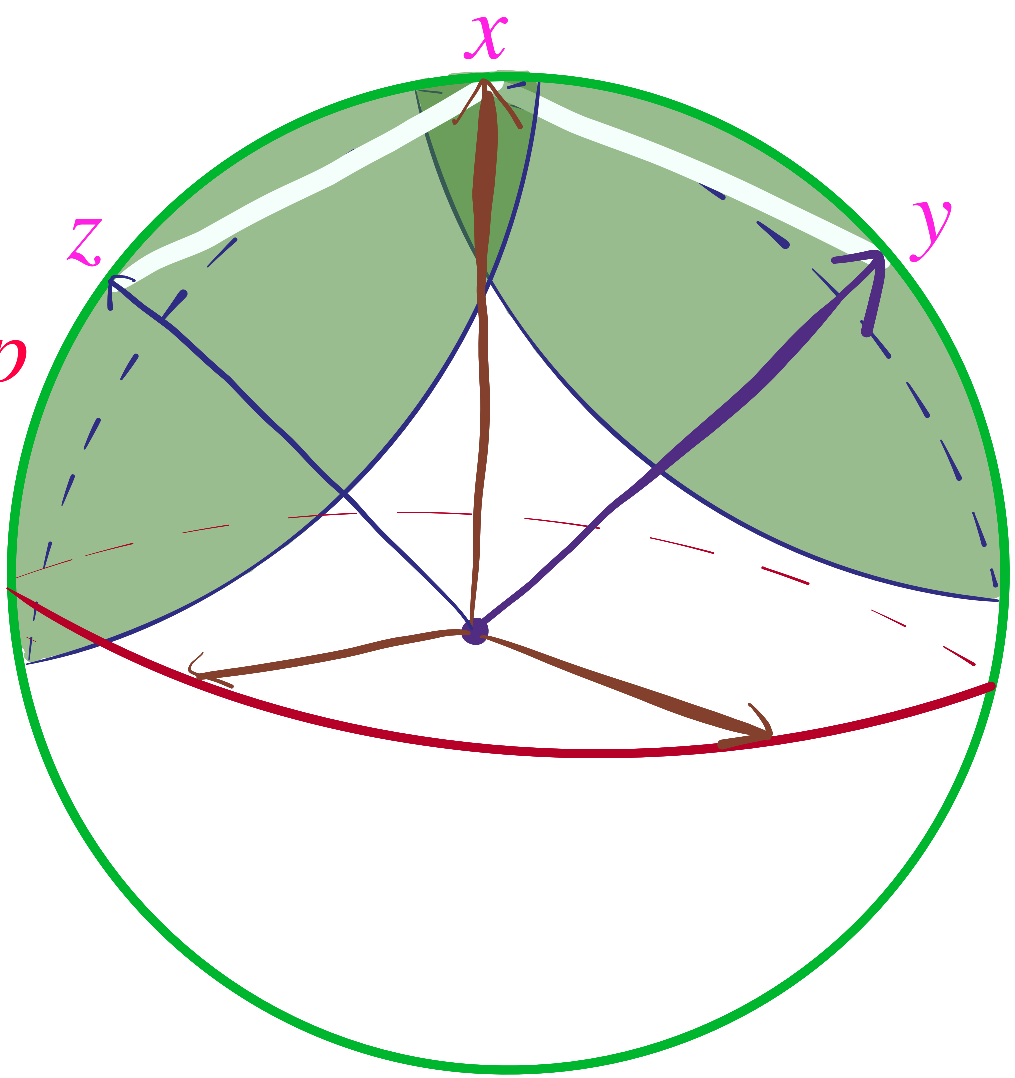
Random geometric graphs

$v_1, \ldots, v_n \sim \mathbb{S}^{d-1}$

xy edge iff $\langle v_x, v_y \rangle \geq \tau(p)$

xy edge with probability p

$$\tau(p) \approx \sqrt{\frac{\log \frac{1}{p}}{d}}$$

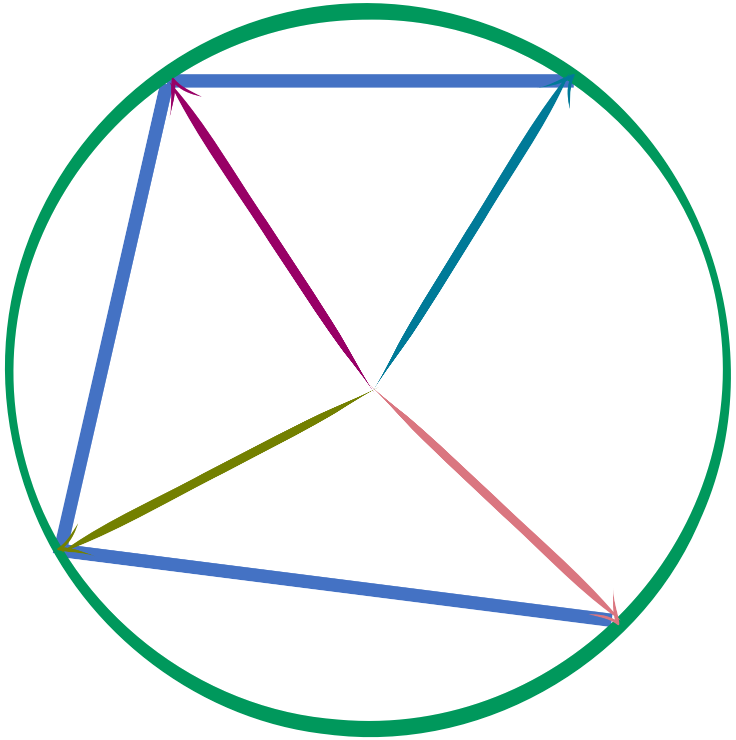


Random geometric graphs

$$v_1, \ldots, v_n \sim \mathbb{S}^{d-1}$$

$$G = \{xy : \langle v_x, v_y \rangle \geq \tau(p)\}$$

Edge xy exists with probability p

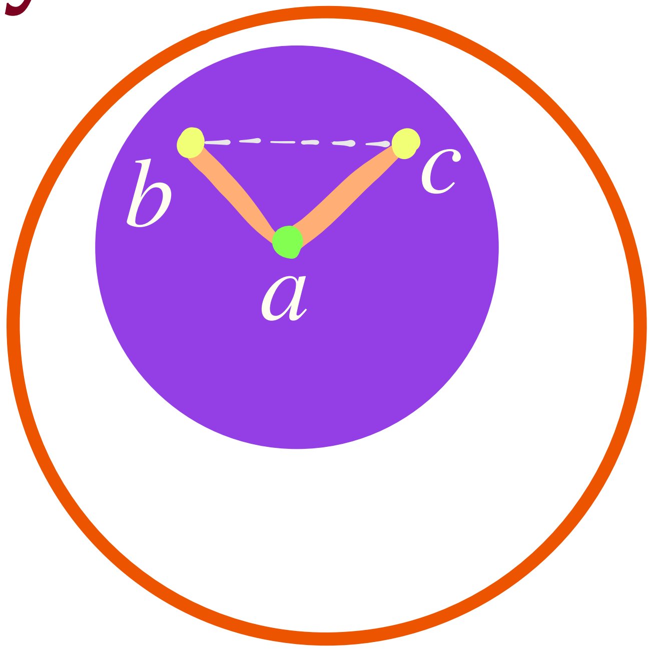


$$G \sim \text{Geo}_d(n, p)$$

Key motivating property

$G \sim \mathsf{Geo}_d(n, p)$

$\Pr[bc \text{ edge} \,|\, ab, ac] > p$



Better approximates real world networks?

More faithful test bed for algorithms on graphs (clustering, partitioning etc.)

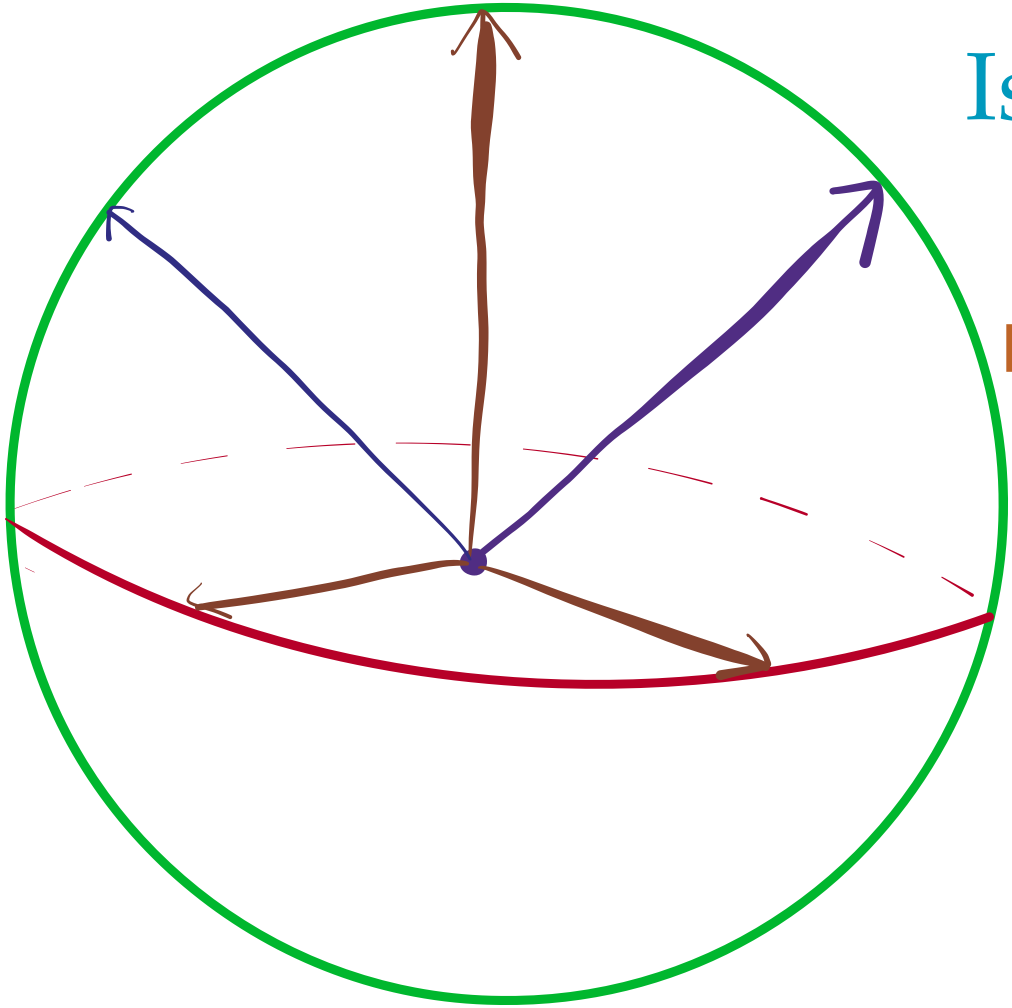
Probabilistic method

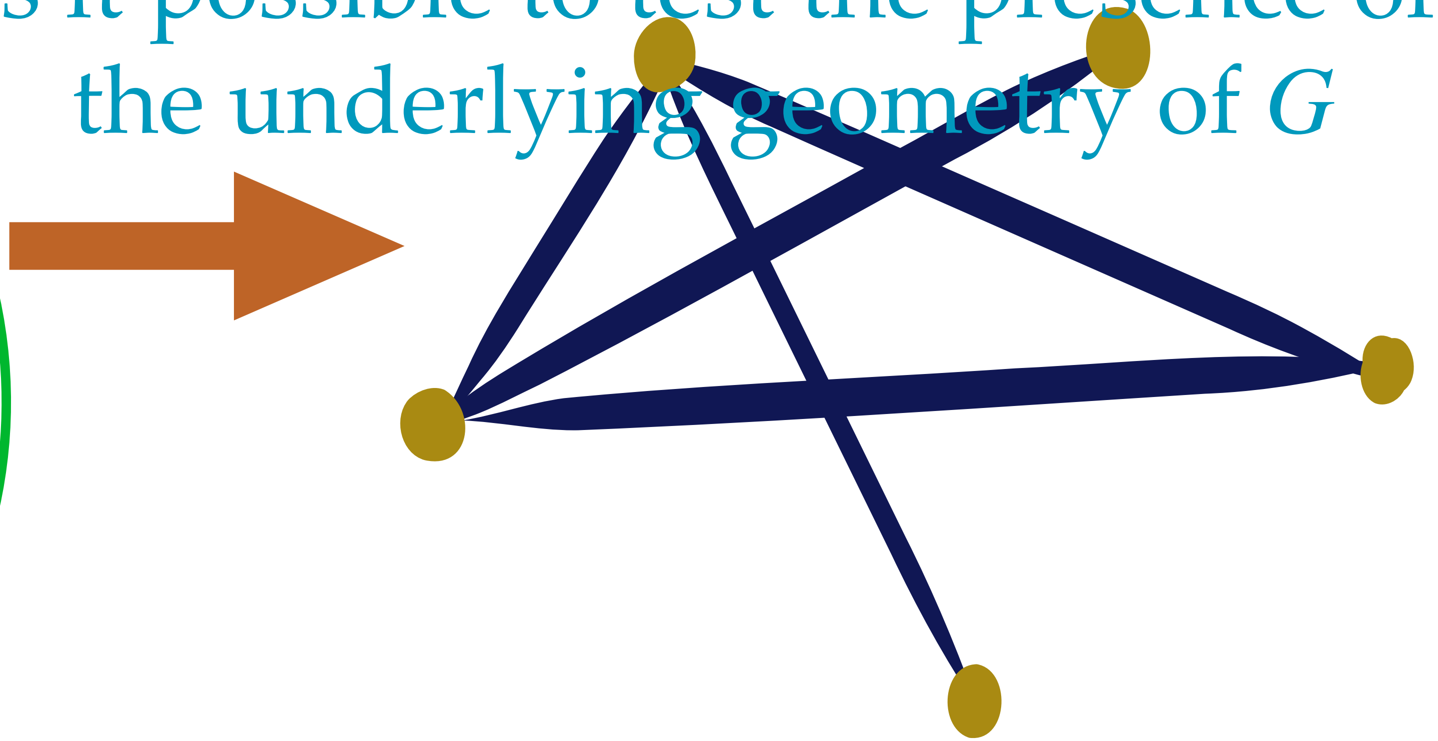
Examples of high-dimensional expanders

(expanders with expanding neighborhoods)

Question 0 about random geometric graphs

$G \sim \mathsf{Geo}_d(n, p)$





Is it possible to test the presence of the underlying geometry of G

$G \sim \mathsf{G}(n, p)$

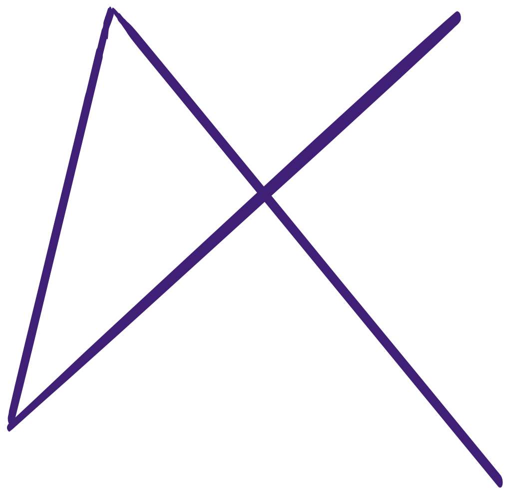
$$\forall xy \in \binom{[n]}{2} : xy \in G \text{ w.p. } p$$

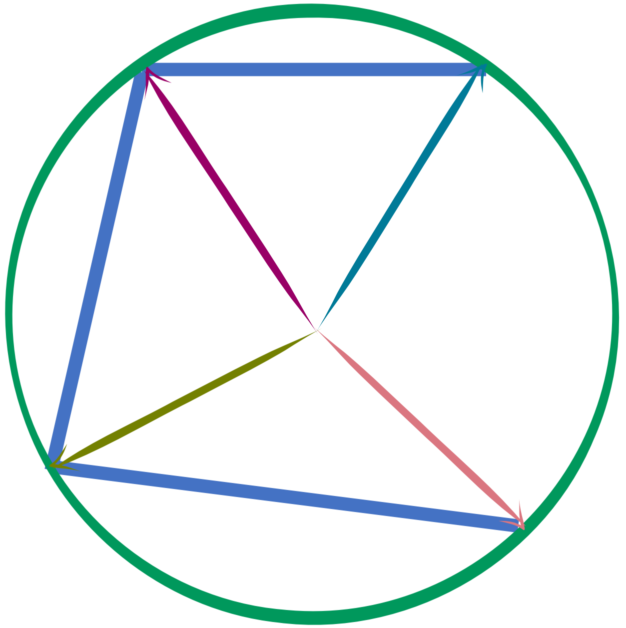
$G \sim \mathsf{Geo}_d(n, p)$

$$v_1, \ldots, v_n \sim \mathbb{S}^{d-1}$$

$$G = \{xy : \langle v_x, v_y \rangle \geq \tau(p)\}$$

vs.



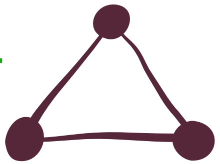
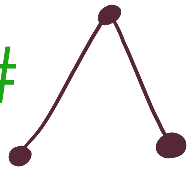


Given G distinguish between models

Known

Distinguish $\mathsf{G}(n,p)$ vs. $\mathsf{Geo}_d(n,p)$

Bubeck-Ding-Eldan-Rácz'16 when $d \ll H(p)^3 n^3$

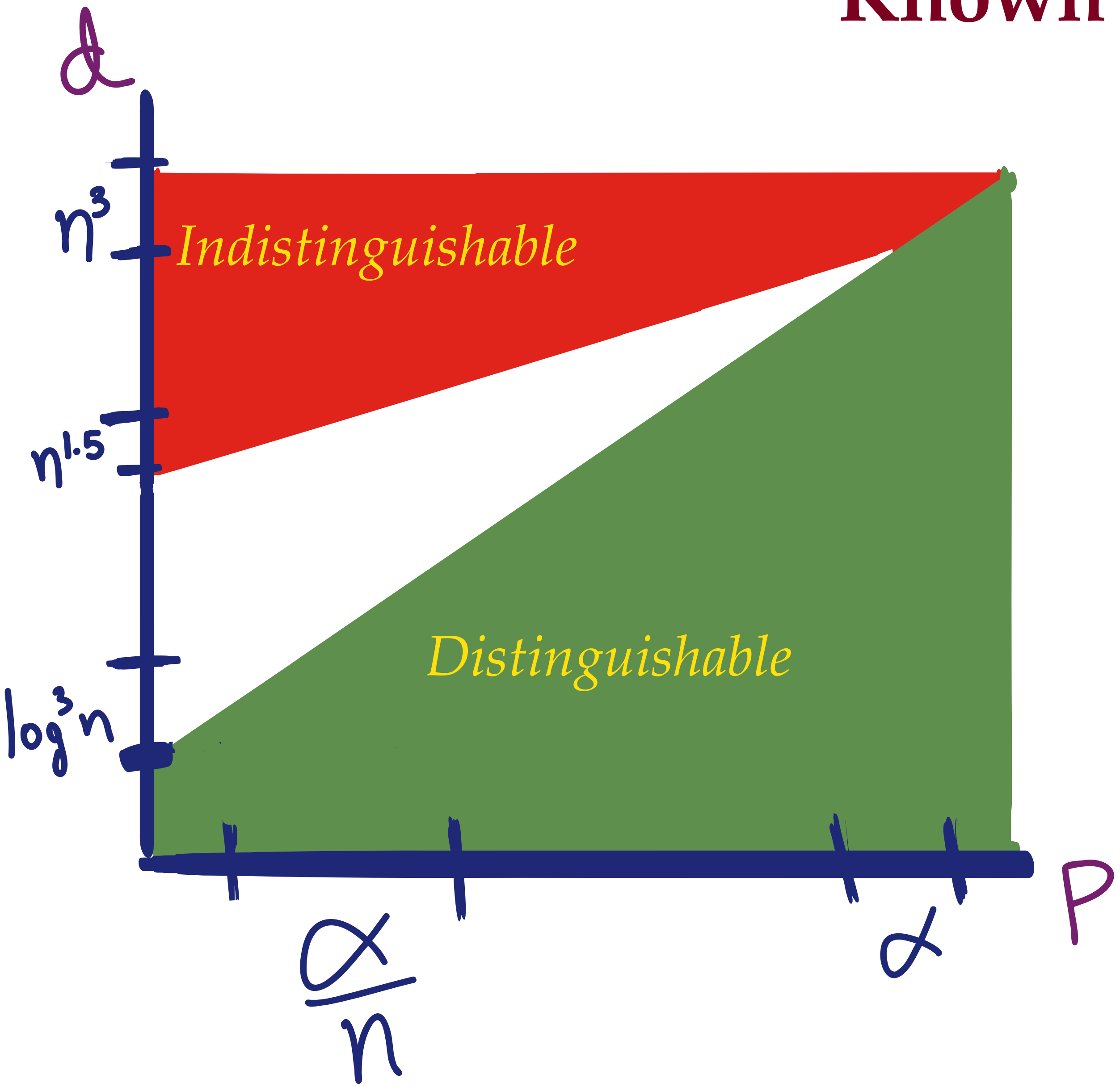
distinguishable via $c_1 \cdot \#$  $+c_2 \cdot \#$  $+c_3 \cdot \#$ 

Brennan-Bresler-Nagaraj'20

indistinguishable when $d \gg n^3$

Indistinguishable when $d \gg \min\{H(p)^3 n^3, H(p)^3 n^{7/2}\}$

Known



Distinguishable when

$$d \ll H(p)^3 n^3$$

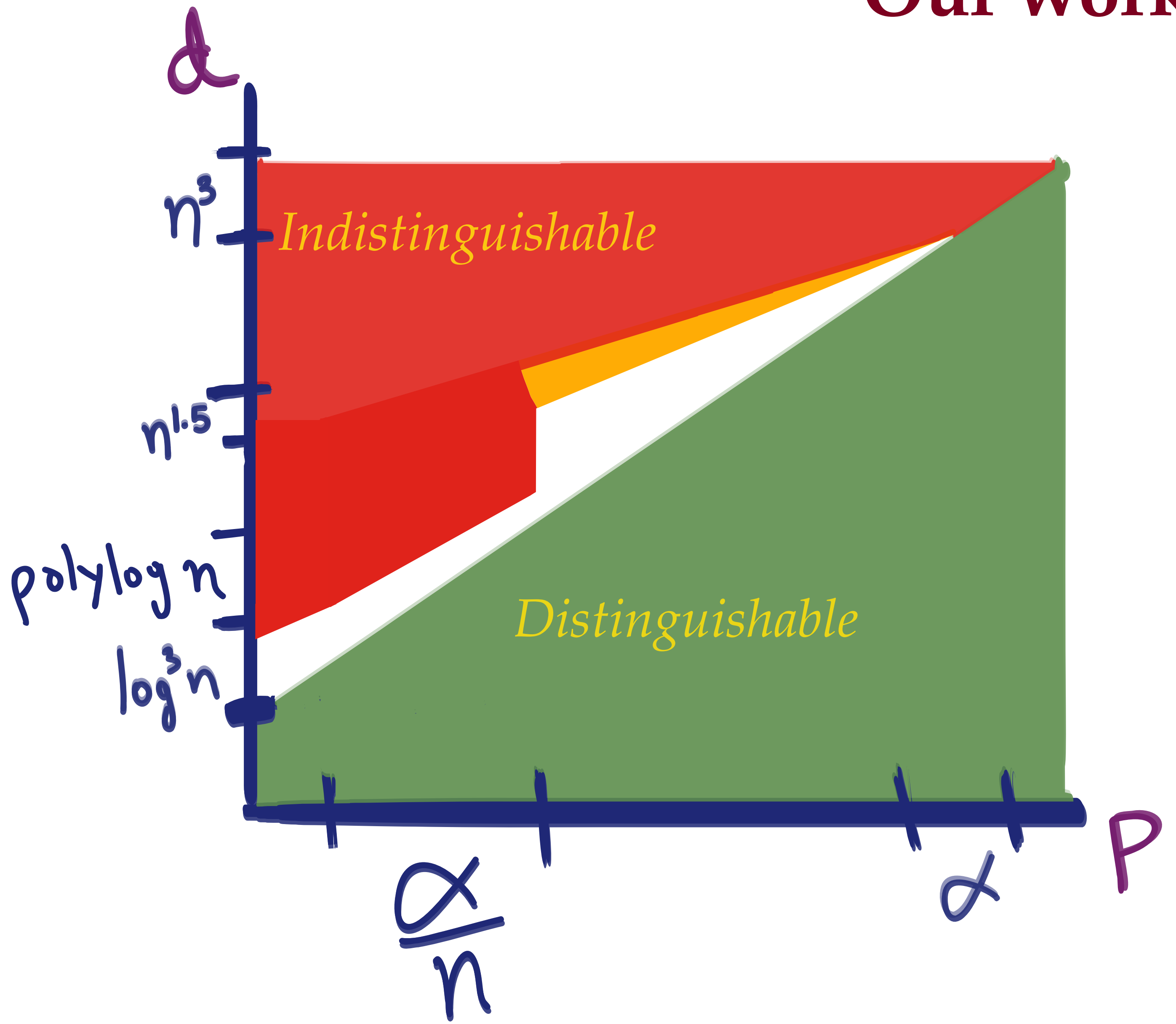
when $p = \dfrac{\alpha}{n}, d \ll \log^3 n$

Indistinguishable when

$$d \gg \min\{H(p)n^3, H(p)^2 n^{7/2}\}$$

when $p = \dfrac{\alpha}{n}, d \gg n^{1.5}$

Our work



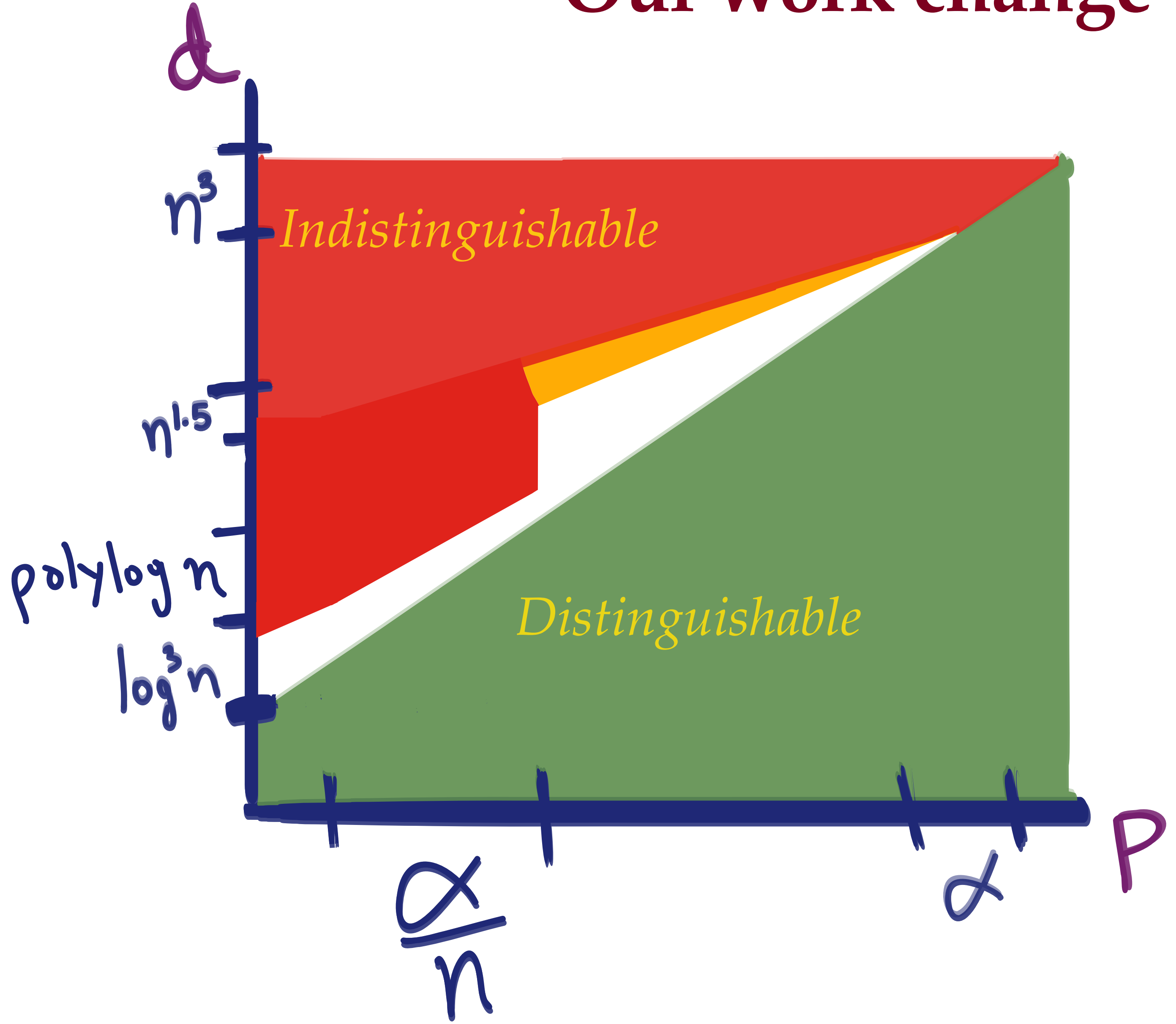
L-Mohanty-Schramm-Yang'22:

Indistinguishable when

$$d \gg H(p)^2 n^3 \text{ for all } p$$

$$\text{when } p = \frac{\alpha}{n}, d \gg \log^{36} n$$

Our work change the figure



L-Mohanty-Schramm-Yang'22:

Indistinguishable when

$d \gg H(p)^2 n^3$ for all p

Loose by $\frac{1}{H(p)}$ factor

when $p = \frac{\alpha}{n}, d \gg \log^{36} n$

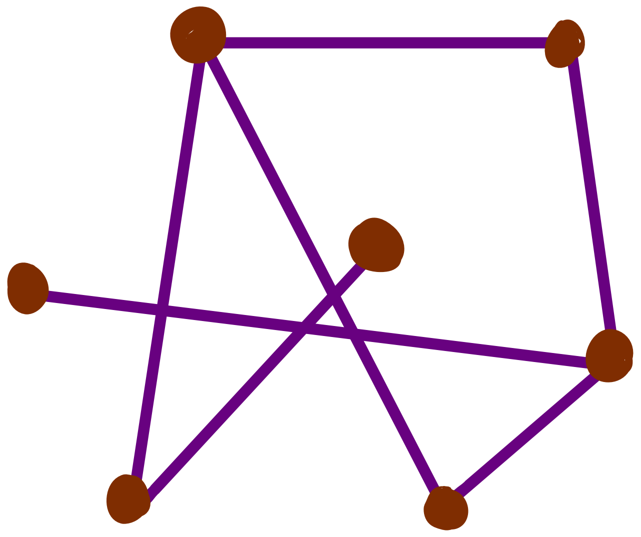
Tight up to polylog(n) factors

Theorem [LMSY'22]: Let $p = \dfrac{\alpha}{n}$, $\alpha = \Theta(1)$, $d \gg \log^{36} n$

$$d_{\mathrm{TV}}(\mathsf{G}(n,p), \mathsf{Geo}_d(n,p)) \leq o_n(1)$$

To prove: $d_{\mathrm{TV}}(\mathsf{G}(n,p), \mathsf{Geo}_d(n,p)) \leq o(1)$

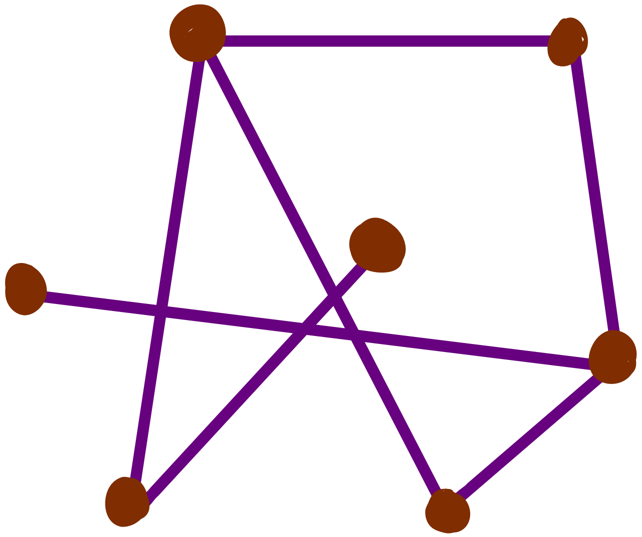
Views of sampling graphs



$G_{\leq t}$

To prove: $d_{\mathrm{TV}}(\mathsf{G}(n,p), \mathsf{Geo}_d(n,p)) \leq o(1)$

Views of sampling graphs



$G_{\leq t}$

$t+1$

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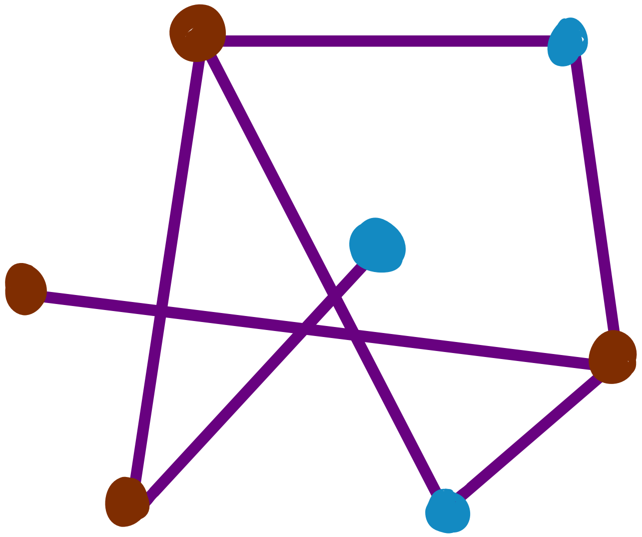


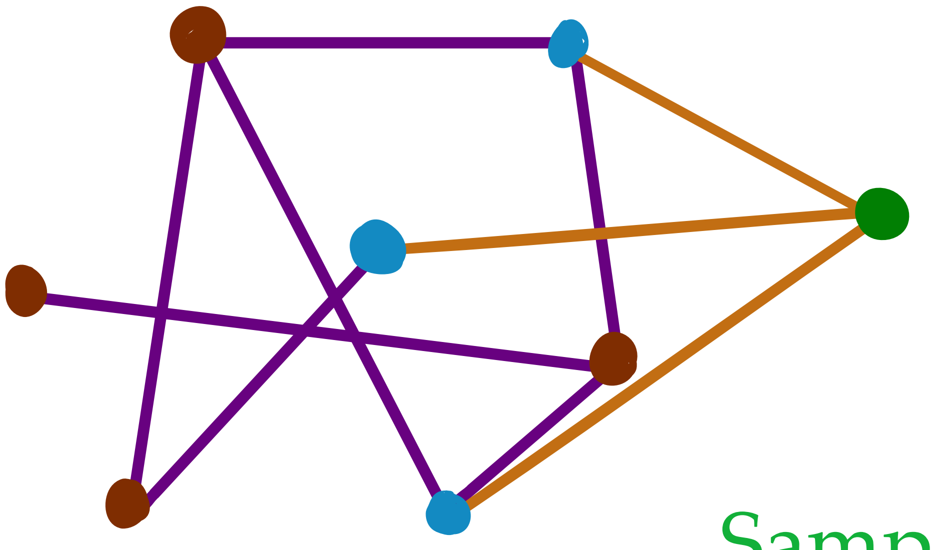
$G_{\le t}$

$\bullet\ t+1$

Sample $S \subseteq [t] \sim \begin{cases} \mathrm{Nbr}(\mathsf{G}(n,p)) \\ \mathrm{Nbr}(\mathsf{Geo}_d(n,p)) \mid G_{\le t} \end{cases}$

To prove: $d_{\mathrm{TV}}(\mathsf{G}(n,p), \mathsf{Geo}_d(n,p)) \leq o(1)$

Views of sampling graphs

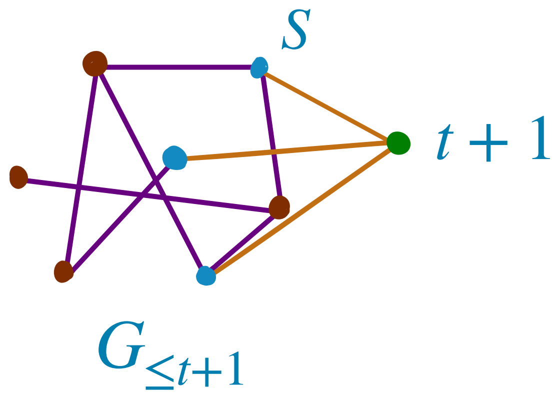


Sample $S \subseteq [t] \sim \begin{cases} \mathrm{Nbr}(\mathsf{G}(n,p)) \\ \mathrm{Nbr}(\mathsf{Geo}_d(n,p)) \mid G_{\leq t} \end{cases}$

Connect $t+1$ to S

To prove: $d_{\mathrm{TV}}(\mathsf{G}(n,p), \mathsf{Geo}_d(n,p)) \leq o(1)$

Views of sampling graphs



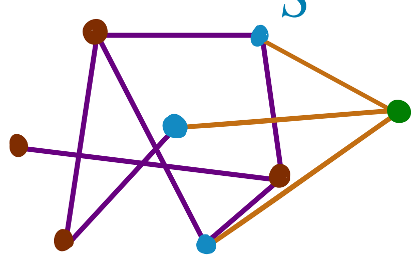
$\mathrm{Nbr}(\mathsf{G}(n,p))$

Choose each $i \in [t]$ independently with probability p

S chosen with probability $p^{|S|}(1-p)^{t-|S|}$

To prove: $d_{\mathrm{TV}}(\mathsf{G}(n,p), \mathsf{Geo}_d(n,p)) \leq o(1)$

Views of sampling graphs



$$\mathrm{Nbr}(\mathsf{Geo}_d(n,p)) \mid G_{\leq t}$$

$$v_1, \ldots, v_t \sim \mathbb{S}^{d-1} \mid \mathsf{gg}(v_1, \ldots, v_t) = G_{\leq t}$$

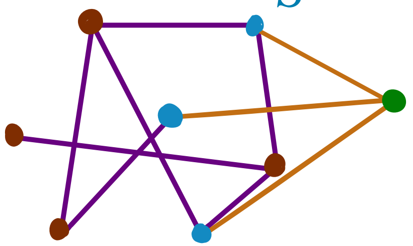
$$v_{t+1} \sim \mathbb{S}^{d-1}$$

$$S := \{i \in [t] : \langle v_i, v_{t+1} \rangle \geq \tau(p)\}$$

To prove: $d_{\mathrm{TV}}(\mathsf{G}(n,p), \mathsf{Geo}_d(n,p)) \leq o(1)$

By standard argument suffices to show

W.h.p. $G_{\leq t} \sim \mathsf{Geo}_d(n,p), S \sim \mathrm{Nbr}(\mathsf{G}(n,p))$



$$\frac{\Pr_{\mathrm{Nbr}(\mathsf{Geo}_d(n,p))|G_{\leq t}}[S]}{\Pr_{\mathrm{Nbr}(\mathsf{G}(n,p))}[S]} = 1 \pm o\left(\frac{1}{\sqrt{n}}\right)$$

Standard argument = Pinsker's inequality + tensorization of relative entropy

What is $\Pr_{\mathsf{Nbr}(\mathsf{Geo}_d(n,p))|G_{\leq t}}[S]$?

To prove: $d_{\mathrm{TV}}(\mathsf{G}(n,p), \mathsf{Geo}_d(n,p)) \leq o(1)$

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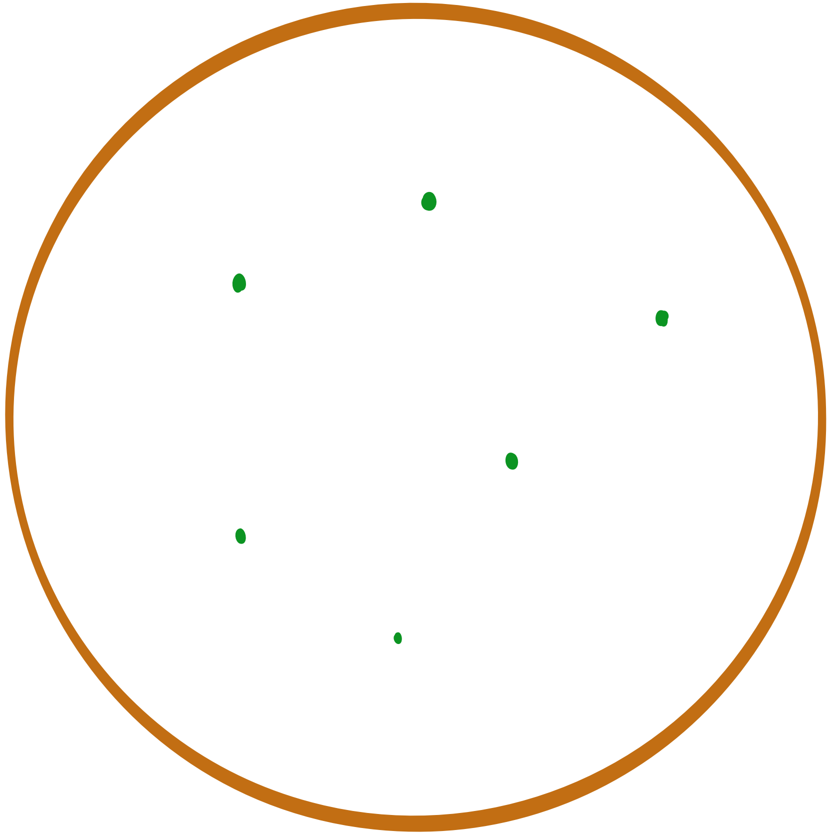

$$v_1, \ldots, v_t \sim \mathbb{S}^{d-1} \,|\, \mathsf{gg}(v_1, \ldots, v_t) = G_{\leq t}$$

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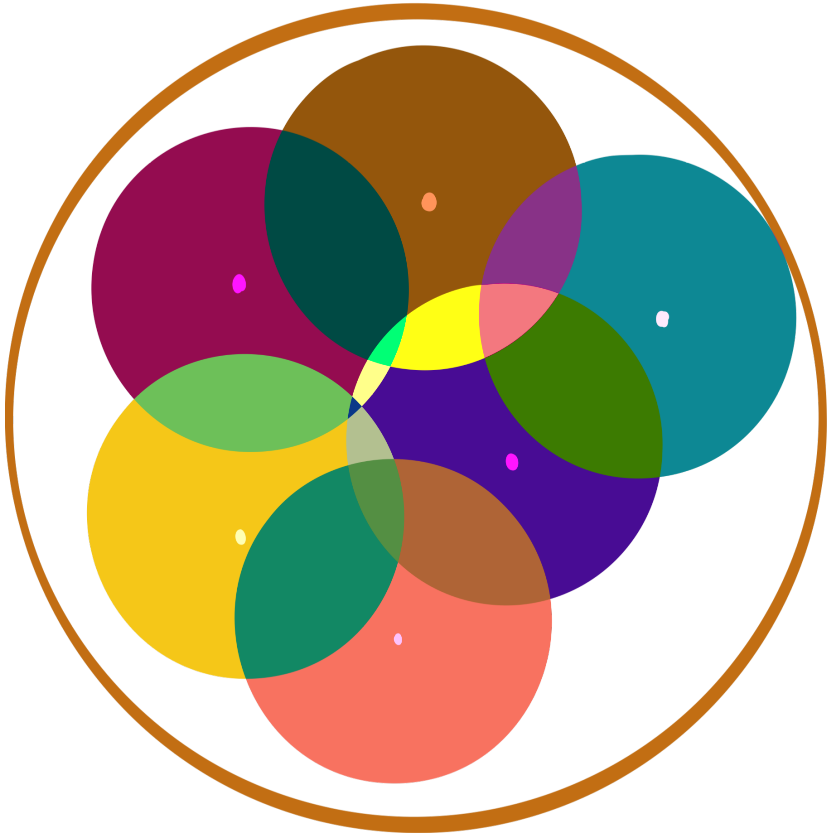

$$v_1, \ldots, v_t \sim \mathbb{S}^{d-1} \,|\, \mathsf{gg}(v_1, \ldots, v_t) = G_{\le t}$$

Choose S with probability Area $\big(\mathrm{Splinter}(S)\big)$

$$\mathrm{Splinter(S)} := \bigcap_{i \in S} \mathrm{cap}(v_i) \cap \bigcap_{i \in \overline{S}} \overline{\mathrm{cap}}(v_i)$$

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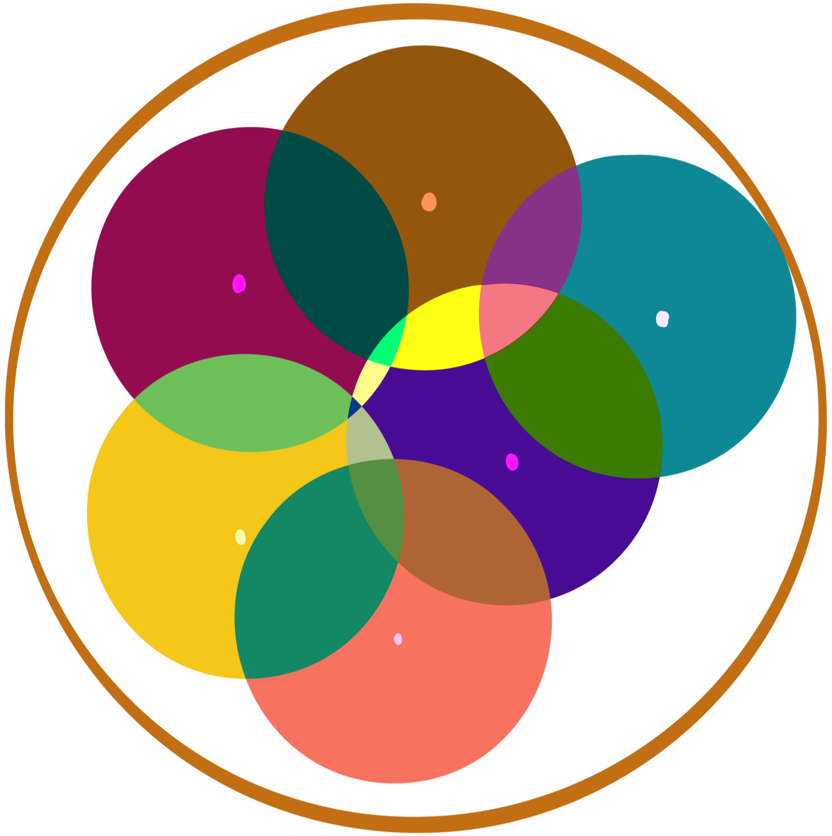

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$$\Pr_{\mathrm{Nbr}(\mathrm{Geo}_d(n,p))|G_{\le t}}[S] = \mathrm{E}_{v_1,\ldots,v_t \sim \mathbb{S}^{d-1}|\mathrm{gg}(v_1,\ldots,v_t)=G_{\le t}}\left[\mathrm{Area}\left(\mathrm{Splinter}(S)\right)\right]$$

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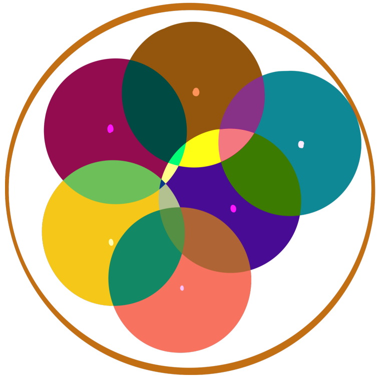

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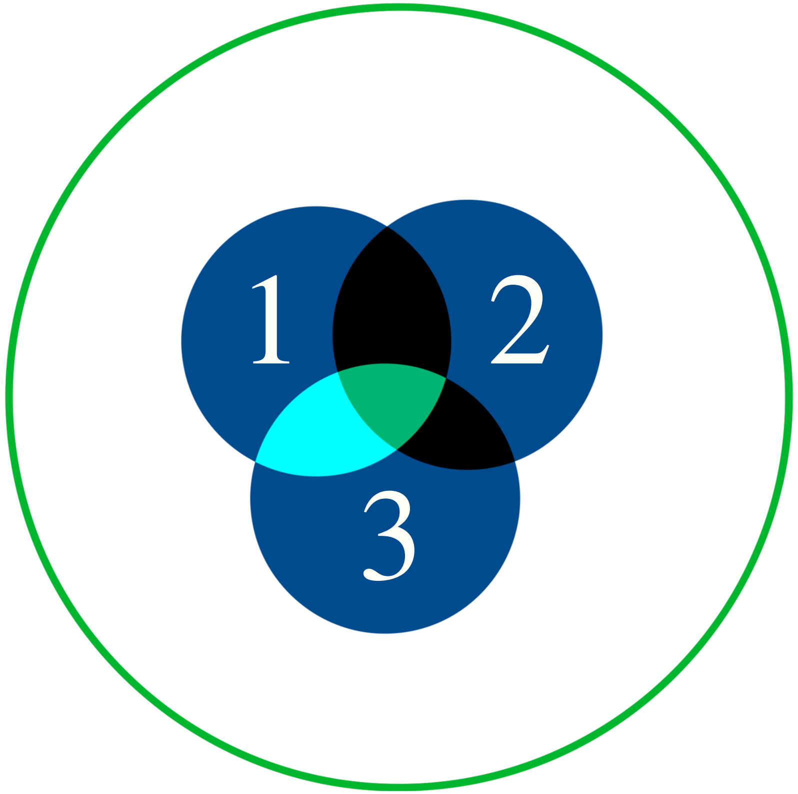
$$\Pr_{\mathrm{Nbr}(\mathrm{G}(n,p))}[S] = \mathbb{E}_{v_1,\ldots,v_t \sim \mathbb{S}^{d-1}}[\mathrm{Area}\left(\mathrm{Splinter}(S)\right)]$$

First attempt:

Try to show $\mathrm{Area}\left(\mathrm{Splinter}(S)\right)$ concentrates under the randomness of $v_1, \ldots, v_t$

What is $\Pr_{\mathsf{Nbr}(\mathsf{Geo}_d(n,p))|G_{\le t}}[S]$?

Example: $S = \{1,2\}$ $\quad \overline{S} = \{3\}$



$v_1, \ldots, v_t \sim \mathbb{S}^{d-1} \,|\, \mathsf{gg}(G_{\le t}) = G$

Choose S with probability $\mathrm{Area}\left(\mathrm{Splinter}(S)\right)$

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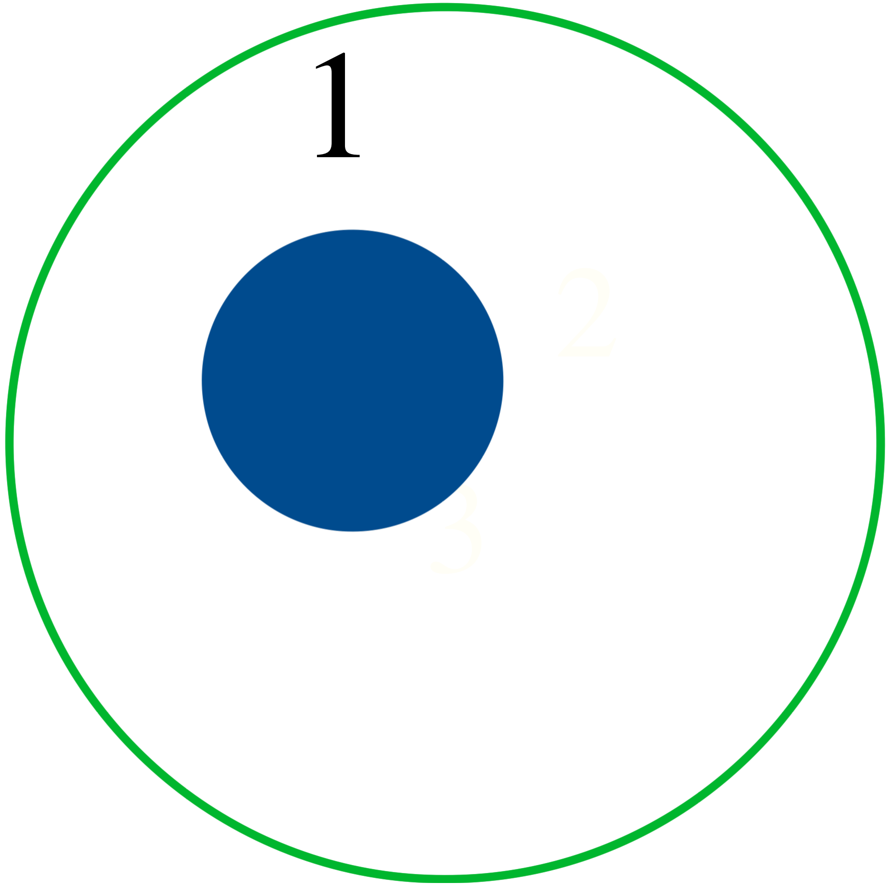

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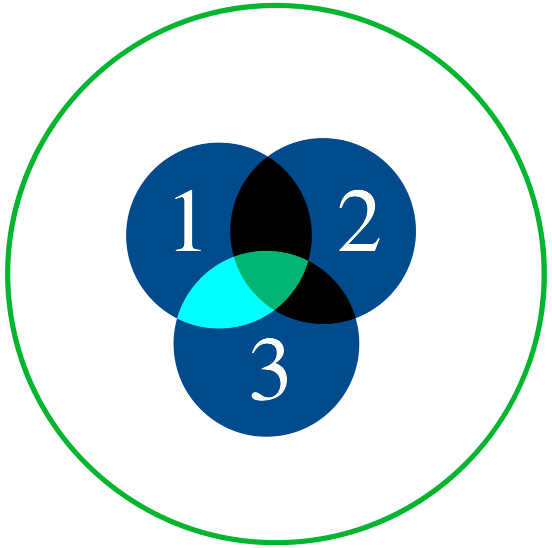

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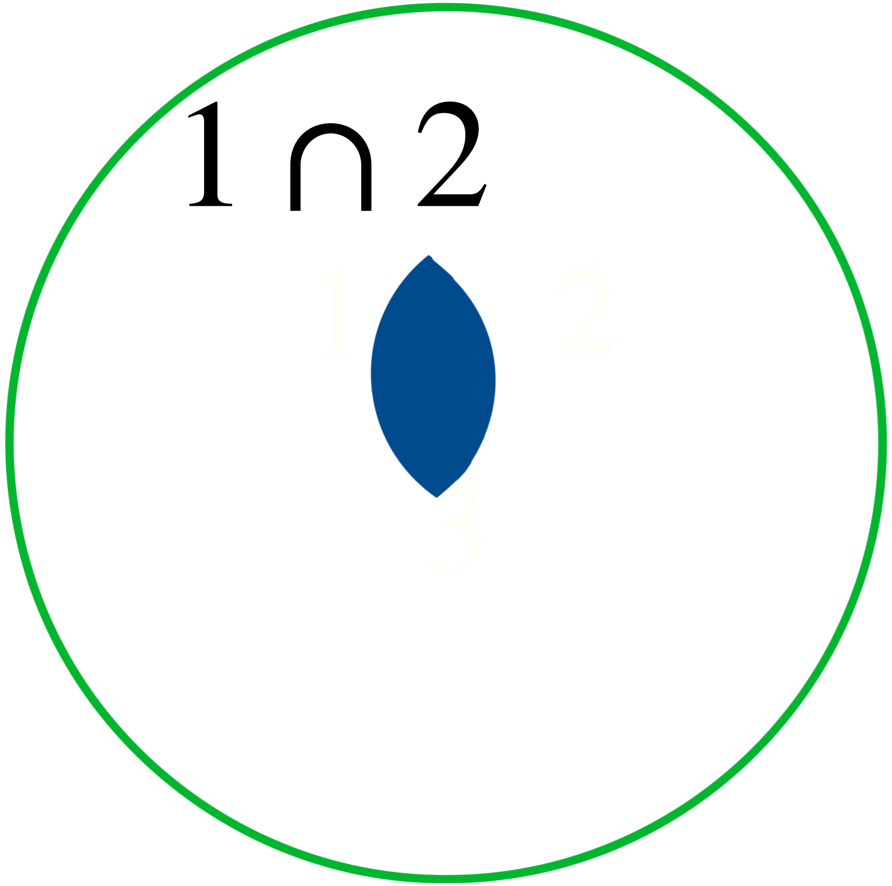

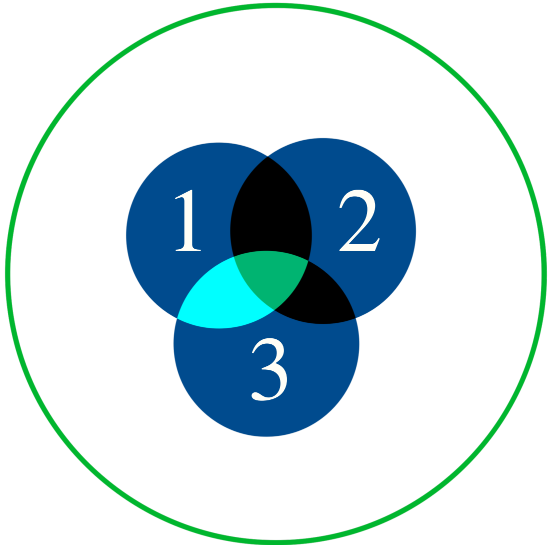

What is $\Pr_{\mathrm{Nbr}(\mathrm{Geo}_d(n,p)) \mid G_{\leq t}}[S]$?

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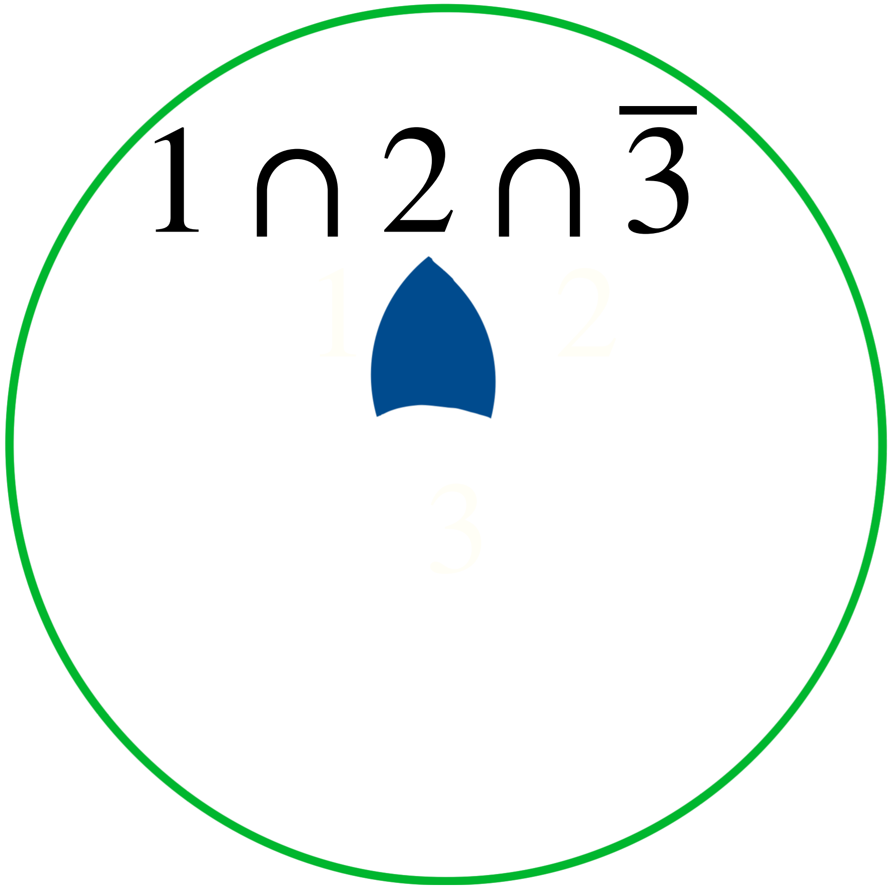
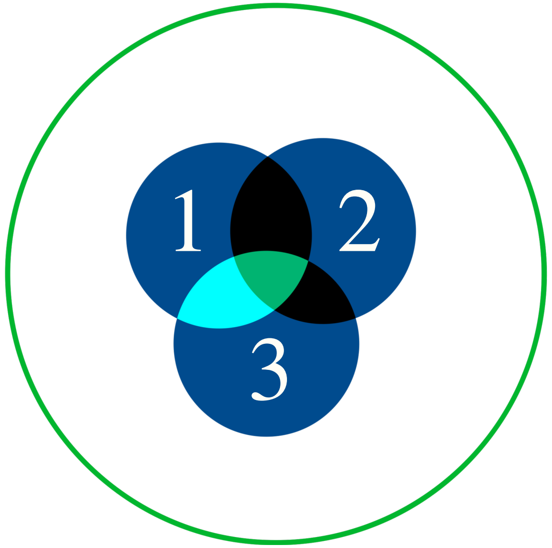
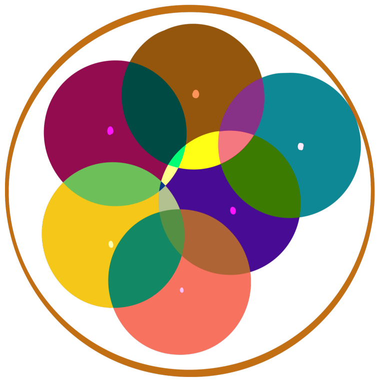




Concentration of Area $\left(\text{Splinter}(S)\right)$

$$\text{Splinter(S)} := \bigcap_{i \in S} \text{cap}(v_i) \cap \bigcap_{i \in \overline{S}} \overline{\text{cap}}(v_i)$$



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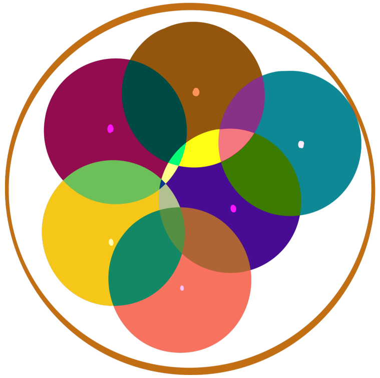

More general question $L \subseteq \mathbb{S}^{d-1}$, $w \sim \mathbb{S}^{d-1}$

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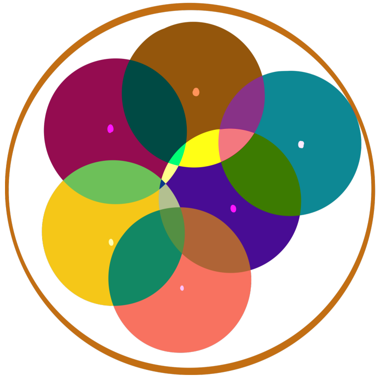
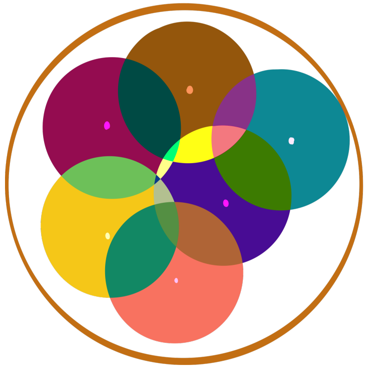


More general question $L \subseteq \mathbb{S}^{d-1}$, $w \sim \mathbb{S}^{d-1}$

Concentration of Area$(L \cap \text{cap}(w))$?

$$\mathrm{E}\left[\text{Area}(L \cap \text{cap}(w))\right] = p \cdot \text{Area}(L)$$

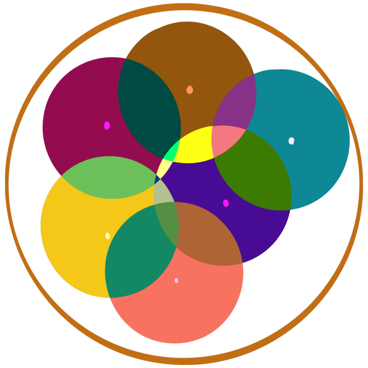
$L \subseteq \mathbb{S}^{d-1}$, $w \sim \mathbb{S}^{d-1}$



$$\text{Theorem: } \text{Area}(L \cap \text{cap}(w)) \in \left(1 \pm \widetilde{O}\left(\sqrt{\frac{\log \frac{1}{\text{Area}(L)}}{d}}\right)\right) \cdot p \cdot \text{Area}(L)$$

$$\text{Area}(L \cap \overline{\text{cap}}(w)) \in \left(1 \pm \widetilde{O}\left(p\sqrt{\frac{\log \frac{1}{\text{Area}(L)}}{d}}\right)\right) \cdot (1-p) \cdot \text{Area}(L)$$

For most S : $\text{Area}\left(\text{Splinter}(S)\right) \in \left(1 \pm \widetilde{O}\left(\frac{1}{\sqrt{d}}\right)\right) \Pr_{\text{Nbr}(\text{G}(n,p))}[S]$



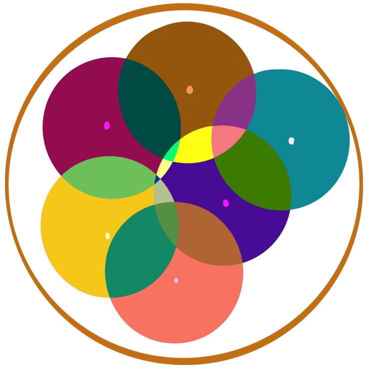
$$\text{Implies } \frac{\Pr_{\text{Nbr}(\text{Geo}_d(n,p))|G_{\leq t}}[S]}{\Pr_{\text{Nbr}(\text{G}(n,p))}[S]} = 1 \pm \widetilde{O}\left(\frac{1}{\sqrt{d}}\right)$$

$$\text{We need } \frac{\Pr_{\text{Nbr}(\text{Geo}_d(n,p))|G_{\leq t}}[S]}{\Pr_{\text{Nbr}(\text{G}(n,p))}[S]} = 1 \pm o\left(\frac{1}{\sqrt{n}}\right)$$

Requirement met when $d \gg n$ polylog n

But we care about $d \geq$ polylog n

$$\text{For most } S: \quad \text{Area}\left(\text{Splinter}(S)\right) \in \left(1 \pm \widetilde{O}\left(\frac{1}{\sqrt{d}}\right)\right) \Pr_{\text{Nbr}(\mathsf{G}(n,p))}[S]$$



$$\text{We need } \frac{\Pr_{\text{Nbr}(\mathsf{Geo}_d(n,p))|G_{\le t}}[S]}{\Pr_{\text{Nbr}(\mathsf{G}(n,p))}[S]} = 1 \pm o\left(\frac{1}{\sqrt{n}}\right)$$

$$\Pr_{\text{Nbr}(\mathsf{Geo}_d(n,p))|G_{\le t}}[S] = \mathrm{E}_{v_1,\ldots,v_t \sim \mathbb{S}^{d-1}|\mathsf{gg}(v_1,\ldots,v_t)=G_{\le t}}\left[\text{Area}\left(\text{Splinter}(S)\right)\right]$$

Studied concentration of Area(Splinter(S)) under randomness of $v_1, \ldots, v_t$

$$\mathrm{E}_{v_1,\ldots,v_t \sim \mathbb{S}^{d-1}|\mathsf{gg}(v_1,\ldots,v_t)=G_{\le t}} \text{ is key}$$

Need to study concentration of $\mathrm{E}_{v_1,\ldots,v_t \sim \mathbb{S}^{d-1}|\mathsf{gg}(v_1,\ldots,v_t)=G_{\le t}}[\text{Area}(\text{Splinter}(S)]$

under randomness of $G_{\le t}$

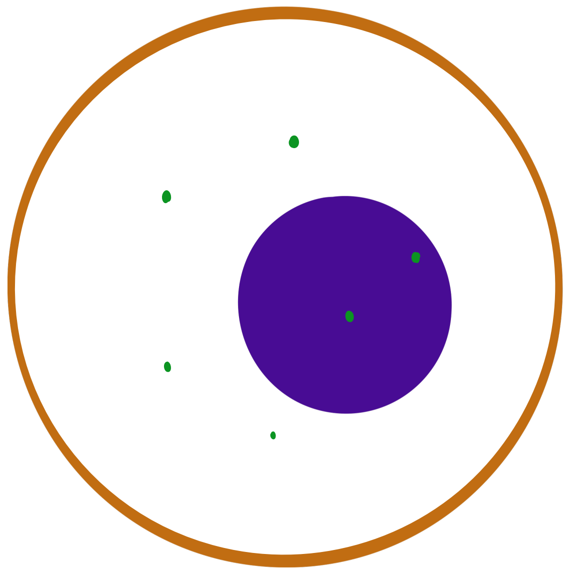
$$\text{New goal: show } \frac{\Pr_{\mathrm{Nbr}(t+1)|G_{\leq t}}[\supseteq S]}{p^{|S|}} = 1 \pm o\left(\frac{1}{\sqrt{n}}\right) \text{ w.h.p. over } G_{\leq t}$$

$$\mathrm{Nbr}(t+1) \mid G_{\leq t}$$

$$v_{t+1} \sim \mathbb{S}^{d-1}$$

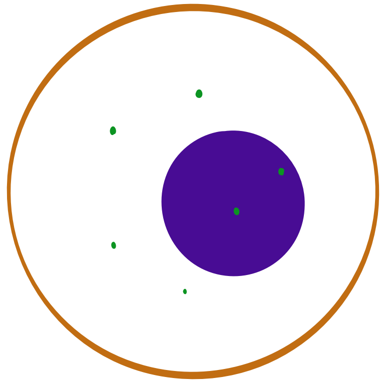
$$v_1, \ldots, v_t \sim \mathbb{S}^{d-1} \mid \mathrm{gg}(v_1, \ldots, v_t) = G_{\leq t}$$

$$\mathrm{Nbr}(t+1) := \{i : v_i \in \mathrm{cap}(v_{t+1})\}$$



New goal: show $\dfrac{\Pr_{\mathrm{Nbr}(t+1)|G_{\leq t}}[\supseteq S]}{p^{|S|}} = 1 \pm o\left(\dfrac{1}{\sqrt{n}}\right)$ w.h.p. over $G_{\leq t}$

$\mathrm{Nbr}(t+1) := \{i : v_i \in \mathrm{cap}(v_{t+1})\}$



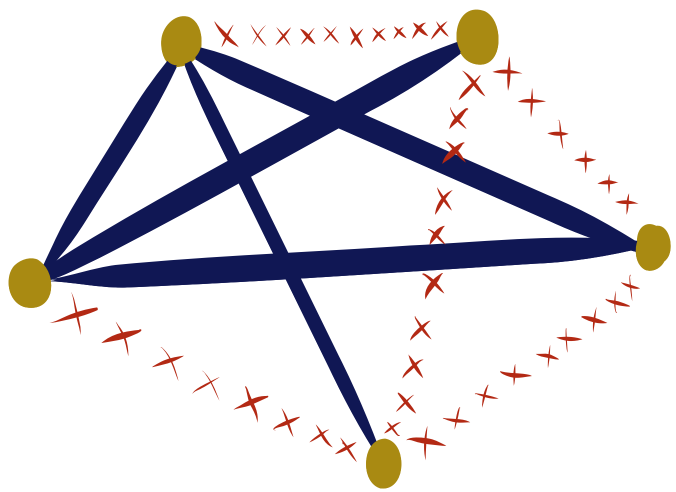
Suppose $(v_i)_{i\in S} \,|\, G_{\leq t}$ i.i.d. & uniform

$$\Pr_{\mathrm{Nbr}(t+1)|G_{\leq t}}[\supseteq S] = p^{|S|}$$

Will show $(v_i)_{i\in S} \,|\, G_{\leq t}$ w.h.p i.i.d. and approximately uniform

Goal: understand marginals $(v_i)_{i\in S} \,|\, G_{\leq t}$

Goal: understand marginals $(v_i)_{i \in S} \mid G_{\leq t}$



$G_{\leq t}$

Constraint Satisfaction Problem on $G_{\leq t}$

$\forall ij$ edge: $\langle v_i, v_j \rangle \geq \tau(p)$

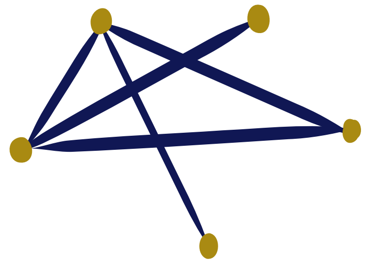
$\forall ij$ non edge: $\langle v_i, v_j \rangle < \tau(p)$

$v_1, \ldots, v_t \sim \mathbb{S}^{d-1} \mid G_{\leq t}$ uniform on solutions to above CSP

Constraint Satisfaction Problem on $G_{\leq t}$

$\forall ij$ edge: $\langle v_i, v_j \rangle \geq \tau(p)$

~~$\forall ij$ non edge: $\langle v_i, v_j \rangle < \tau(p)$~~



$G_{\leq t}$

$v_1, \ldots, v_t \sim \mathbb{S}^{d-1} \mid G_{\leq t}$ uniform on solutions to above CSP

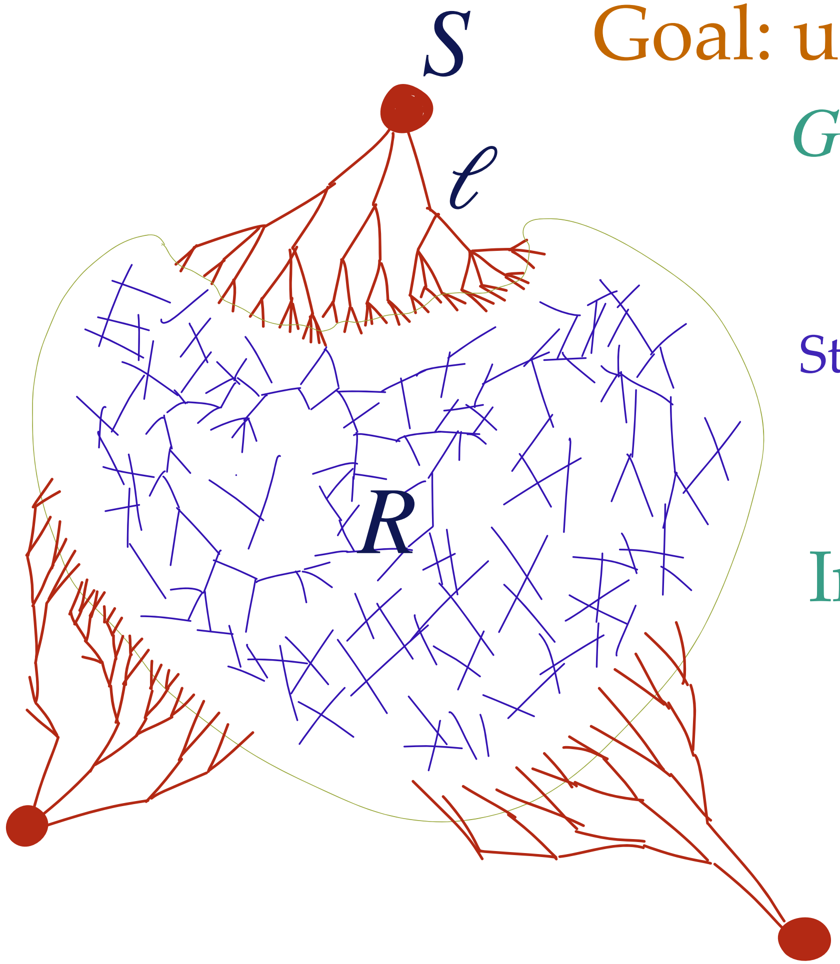
Goal: understand marginals $(v_i)_{i \in S} \mid G_{\leq t}$

Technical simplification for talk: Drop non-edge constraints!

Constraint Satisfaction Problem on $G_{\leq t}$

$\forall ij$ edge: $\langle v_i, v_j \rangle \geq \tau(p)$

$v_1, \ldots, v_t \sim \mathbb{S}^{d-1} \mid G_{\leq t}$ uniform on solutions to above CSP



Goal: understand marginals $(v_i)_{i \in S} \mid G_{\leq t}$

$G_{\leq t}$ sparse, locally-treelike

vertices in S pairwise far

Strategy: show for a typical fixing of vectors in purple part

$(v_i)_{i \in S}$ independent and uniform

Independence for free because pieces are disjoint!

Need to show (approximate) uniformity of v_i

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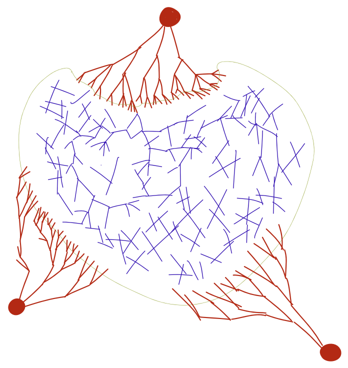
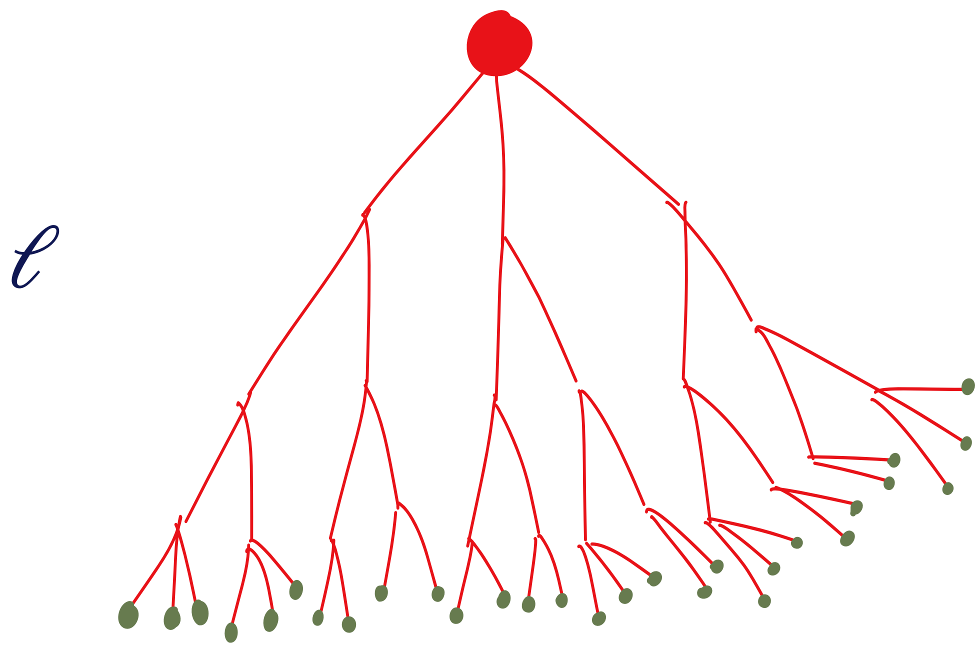


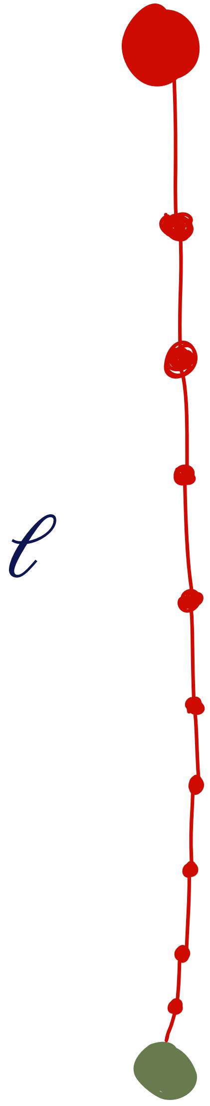


Green leaves receive "typical" vector assignment

Will show: Red root conditioned on leaves is approximately uniform

Strategy: compute distribution of $v_{\text{root}} \mid (v_i)_{i \in \text{Leaves}}$ via belief propagation

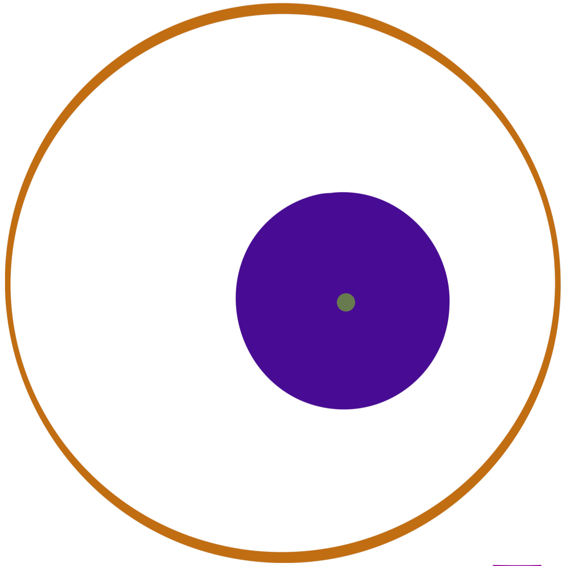
Illustration on special case: tree is length-ℓ path



Green leaf has vector v_{leaf}

Will show: $v_{\text{root}} \mid v_{\text{leaf}}$ is approximately uniform

$$\text{Distribution } f_{\text{Parent}(x)} \mid f_x \begin{cases} \text{Sample } w \sim f_x \\ \text{Walk to random vector in cap}(w) \end{cases}$$



P Markov operator

$f_y :=$ PDF of v_y $\qquad f_{\text{Parent}(x)} = P \cdot f_x$

$$f_{\text{root}} = P^{\ell} \cdot f_{\text{Leaf}}$$

Uniform stationary for P

Exhibit contraction properties of P

Exhibit contraction properties of P

Key lemma: For "smooth" f

"nice" means density $\leq 1/p$

$$d_{\mathrm{TV}}(Pf, \mathrm{Unif}) \leq \widetilde{O}\left(\frac{1}{\sqrt{d}}\right) d_{\mathrm{TV}}(f, \mathrm{Unif})$$

$$d_{\mathrm{TV}}(f_{\mathrm{root}}, \mathrm{Unif}) \leq \widetilde{O}\left(\frac{1}{\sqrt{d}}\right)^{\ell-1}$$

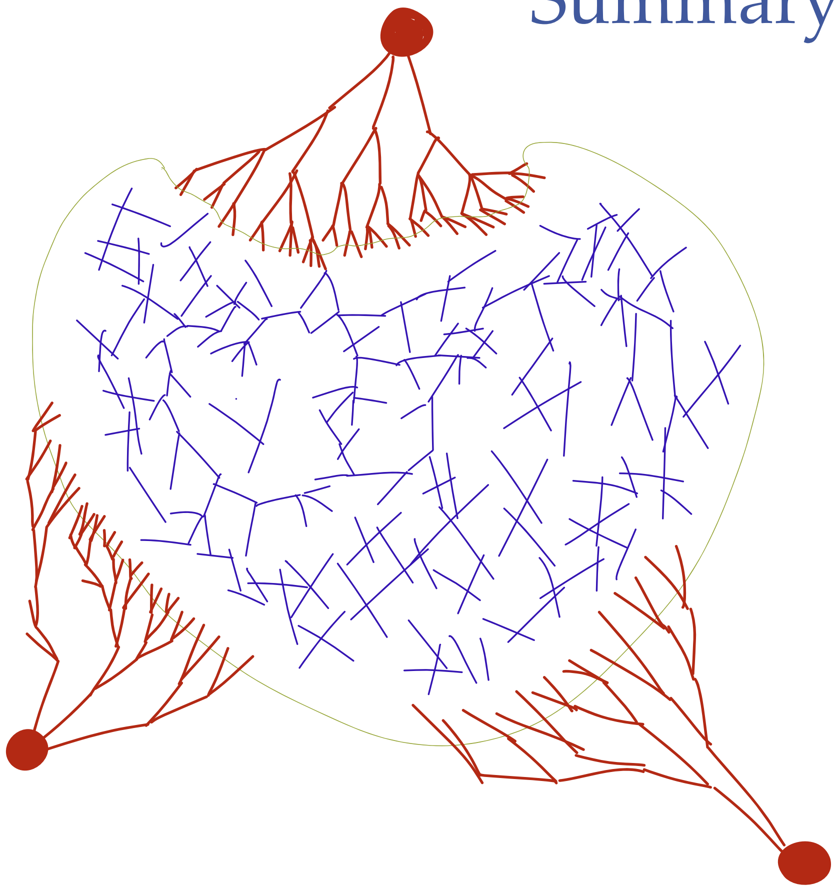
when $d \geq \mathrm{polylog}\, n$, set $\ell = \dfrac{\log n}{\log\log n}$, and then TV distance is $o\left(\dfrac{1}{\sqrt{n}}\right)$

Constraint Satisfaction Problem on $G_{\leq t}$

$\forall ij$ edge: $\langle v_i, v_j \rangle \geq \tau(p)$

$v_1, \ldots, v_t \sim \mathbb{S}^{d-1} \,|\, G_{\leq t}$ uniform on solutions to above CSP

Summary: $(v_i)_{i \in S}$ independent and $o\left(\frac{1}{\sqrt{n}}\right)$-close to uniform



$$\frac{\Pr_{\mathrm{Nbr}(\mathsf{Geo}_d(n,p))|G_{\leq t}}[\supseteq S]}{p^{|S|}} = 1 \pm o\left(\frac{1}{\sqrt{n}}\right)$$

$$d_{\mathrm{TV}}(\mathsf{G}(n,p), \mathsf{Geo}_d(n,p)) \leq o(1)$$

Thank you! Questions?