Structures in random graphs: New connections

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The beginning

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There exists graphs on N vertices with no clique or independent set of size $2\log_2 N$.

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- Erdős shows that $G(N, 1/2)$ does not have a clique or independent set of size $n = 2\log_2 N$ by considering the first moment: The expected number of such cliques or independent sets is $\binom{N}{n}2^{-\binom{n}{2}}$ which is small.

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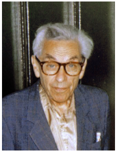
- Much more precise asymptotic understanding of the clique and independence numbers of $G(N, p)$ by Matula and Bollobás and Erdős.

Timeline



Random graphs

Ramsey

First moment prediction

- Suprema of stochastic processes
- Structure of small sets
- Random optimization
- Convex geometry

Sunflowers

Complexity / Random restriction / DNF sparsification

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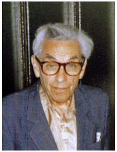

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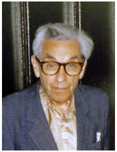

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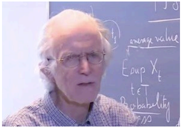

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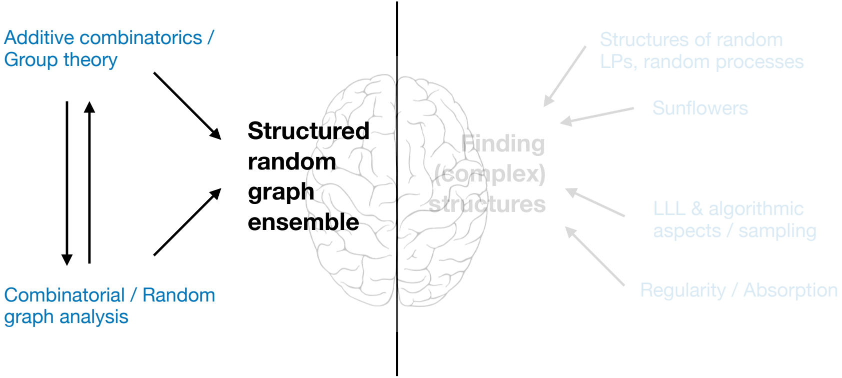

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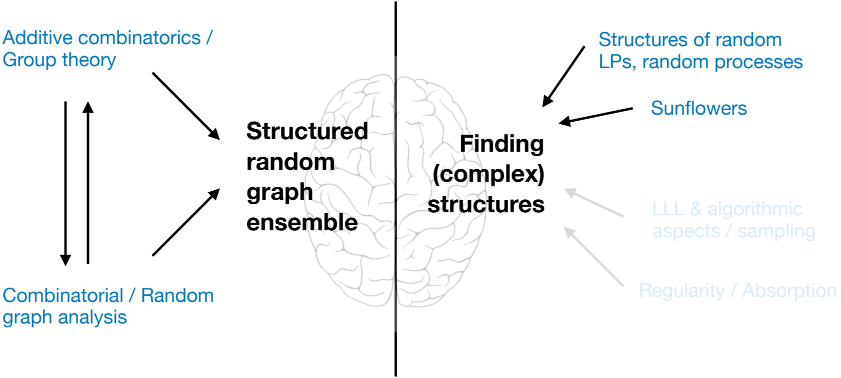

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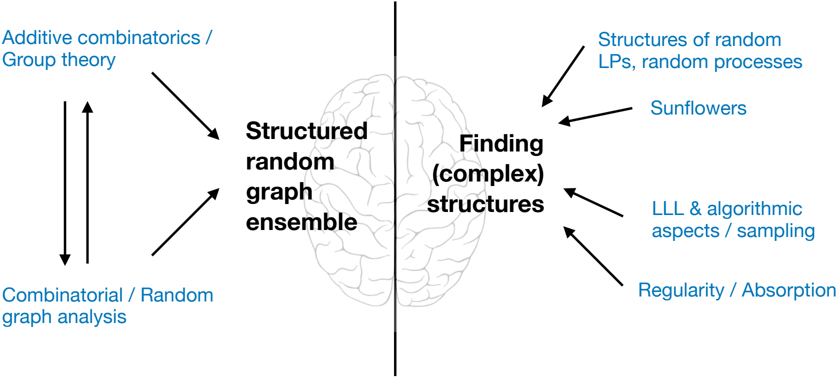

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Problem (Erdős)

Explicitly construct C-Ramsey graphs for some constant C.

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Conjecture (Alon 1989)

There is a constant C such that every finite group has a Cayley graph which is C-Ramsey.

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Theorem (Green 2005, Mrazović 2017)

The clique number of a uniform random Cayley graph on $\mathbb{F}_2^d$ with $N = 2^d$ is $\Theta(\log N \log \log N)$ whp.

Random graphs meet additive combinatorics

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Theorem (Conlon-Fox-P.-Yepremyan)

In any group G of order N, the number of subsets $A \subset G$ with $|A| = n$ and $|AA^{-1}| \leq Kn$ is at most $N^{C(K+\log n)}(CK)^n$.

- $AA^{-1} := \{ab^{-1} : a, b \in A\}$.
- Note that A is a clique in G_S if and only if $AA^{-1} \setminus \{1_G\} \subset S$.
- Previously analyzed in nice abelian groups via strong structural results/regularity method.

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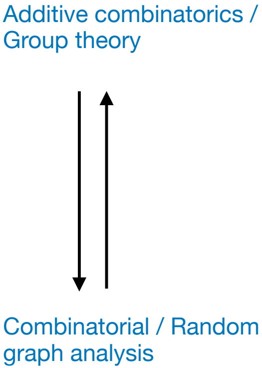
In particular, all N for which the largest factor which is a power of 2 or 3 is at most $(\log N)^{.001}$ has the above property.

Towards additive combinatorics, and back?



Our analysis combines closely purely combinatorial view and additive insights:

- Purely combinatorial view on the role of the group structure, analyzed via exploration reminiscent of classical random graph analysis.
- Relation between solutions to linear equations, expansion (large product sets) and dimension.

A purely combinatorial view on Cayley graphs

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- Constraint: Each color class has bounded degree.

Cliques in entangled graphs & Sets with small product set

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- From Theorem 1, a careful union bound yields Theorem 2.
- Theorem 2 solves a conjecture of Christofides and Markström on the clique number of random Latin square graphs.

Sets with small product set: Greedy exploration

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Greedy process to grow a large component:

- In each step, pick a color that maximally extends the size of the component.
- Guarantee that the component grows roughly by a factor $1/K$ per step. Hence, the entire set is connected in $O(K \log n)$ steps.

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The bound on the number of colors required in a spanning tree is tight.

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- Random exploration: Expose colors randomly and analyze the connected components formed.
- When the set of colors coming out of components is large ($\Omega(Kn)$), large components will merge in $O(\log n)$ steps to a giant component.

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Building large components:

- Pick random colors and grow components using all edges of these colors.
- Keep track of colors going out of each component; as the components grow, the set of colors out of the components also grows.
- Prove that the probability that a component grows increases with the size of the component. Hence, most vertices are in components of size $\Omega(K)$ after $O(K)$ samples of colors.

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There is a self-complementary Ramsey Cayley graph on $\mathbb{F}_5^d$.

Answer a question of Alon and Orlitsky motivated by zero-error capacity and dual-source coding.

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The last condition leads to expansion of any potential clique: $|A + 2 \cdot A| = |A|^2$, so the Plünnecke-Ruzsa inequality implies

$$|A|^2 = |A + 2 \cdot A| \leq |A + A + A| \leq |A - A|^3 |A|^{-2},$$

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yielding $|A - A| \geq |A|^{4/3}$.
Expansion, together with the previous counting result, allows the union bound to work in the large K range without losing the $\log \log N$ factor.

From random graphs to additive combinatorics and back

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- Over vector spaces, sharp dependence on K can be determined through sharp bound on the dimension of A.
- This can be obtained from results on Freiman's conjecture over $\mathbb{F}_p^n$ by Chaim Even-Zohar and Lovett: If $|A - A| \le K|A|$ and $|A| = p^{o(K)}$, then the dimension of A is at most $(1 + o(1))K$.

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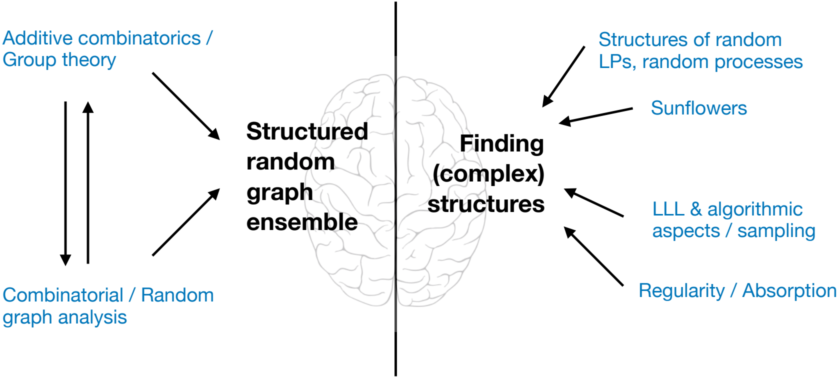
Even sharper asymptotics requires more precise understanding of the additive structure and correspondence with the combinatorial analysis: Make full leverage of the expansion condition and combinatorial insights.

Roadmap

Careful understanding of the role of structure on the first moment is crucial.

Mutual connection between additive combinatorics and random exploration/random graph view.



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- The *threshold* $p_c(\mathcal{H})$ is the value p at which the probability that a structure in $\mathcal{H}$ appears in X_p is $1/2$.

Theorem (the Kahn-Kalai conjecture '06, resolved by Park-P. '22+)

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The expectation threshold is defined as the largest p for which there is $\mathcal{H}'$ with

- $\mathcal{H}'$ covers $\mathcal{H}$: All $H \in \mathcal{H}$ contains some $H' \in \mathcal{H}'$.
- $\mathcal{H}'$ has a small cost: $\sum_{H' \in \mathcal{H}'} p^{|H'|} \leq 1/2$ (naive union bound/first moment).

We say $\mathcal{H}$ is p-small if there exists $\mathcal{H}'$ satisfying the above properties.

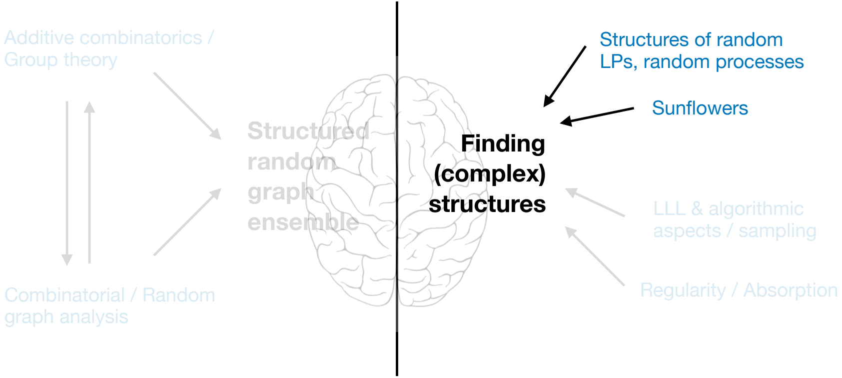
Thresholds and the Kahn-Kalai conjecture

Theorem (the Kahn-Kalai conjecture '06, resolved by Park-P. '22+)

If $\mathcal{H}$ is not p-small, then $X_{Lp \log |X|}$ contains a set from $\mathcal{H}$ with probability at least $1/2$.

Inexistence of first moment (union bound) obstruction is sufficient to guarantee emergence of structure!

Roadmap



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Thresholds and random LPs

Interesting connections to the structure of random processes and high-dimensional convex geometry (Talagrand '94, '06, '10).

Theorem (Talagrand's selector process conjecture, resolved by Park-P. '22)

Given linear functions $f_i(S) = v_i \cdot S$ on $S \subseteq X$, for which $v_i \geq 0$ and $f_i(X) \geq 1$. If the support of v_i's is not p-small, then

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- $\sup_i f_i(X_p)$ - Fractionally subadditive/XOS functions under random domain subsampling.
- Main idea in a simpler setup gives the proof of the Kahn-Kalai conjecture. Kahn-Kalai conjecture as "structure" of containment.

Estimating expectation threshold

- Bounding the expectation threshold by the dual certificate (Talagrand): $p_E(\mathcal{H}) \le p_f(\mathcal{H})$, the largest p for which there exists a probability measure λ supported on $\mathcal{H}$ which is *p-spread*:

$$\lambda(\{H \in \mathcal{H} : H \supseteq S\}) \le 2p^{|S|} \text{ for all } S.$$

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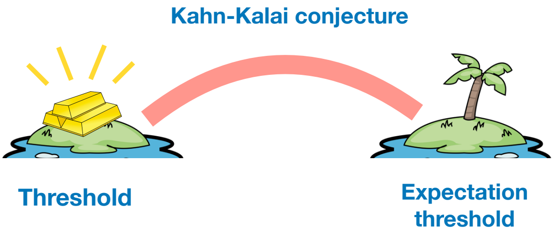
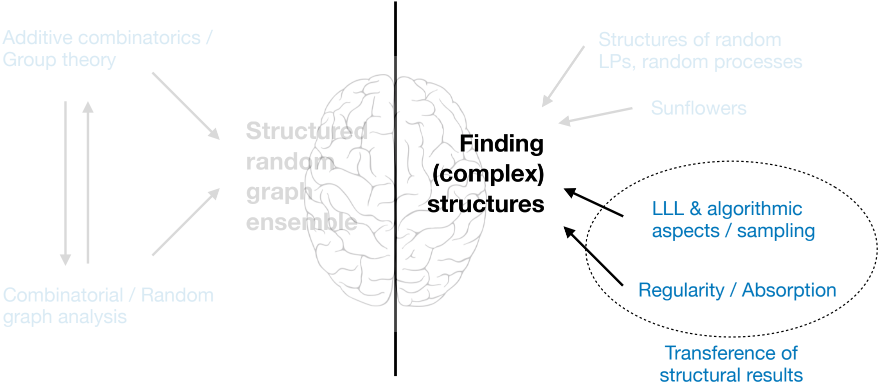
- If one is interested in a specific family of structures $\mathcal{H}$, an imminent question is how to estimate (fractional) expectation thresholds/construct dual certificates.
- Previously restricted to very simple structures in highly symmetric setting, amenable to trivial enumerations.

Where are we?


Kahn-Kalai conjecture
Threshold
Expectation threshold


YOU ARE HERE

Roadmap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- An $n \times n$ Latin square is a matrix $(x_{i,j})_{i,j \in [n]}$ with entries in $[n]$ where each entry appears exactly once in each row and column.

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- For each coordinate (i, j), consider a random subset $X_{i,j}$ of $[n]$ where each element is sampled independently with probability p.
- What is the probability that there exists a Latin square with $x_{i,j} \in X_{i,j}$?

Estimating expectation threshold: Latin squares

Conjecture (Johansson '06, Keevash '14, Luria-Simkin '17)

For $p \geq C(\log n)/n$, with high probability, there exists a Latin square with $x_{i,j} \in X_{i,j}$.

- Related conjectures by Simkin, Casselgren-Häggkvist.

Theorem (Jain-P., Keevash '22+)

There exists a C/n-spread probability distribution on Latin squares. As a corollary, for $p \geq C(\log n)/n$, with high probability, there exists a Latin square with $x_{i,j} \in X_{i,j}$.

- Previous partial progress by Sah-Sawhney-Simkin ('22), Kang-Kelly-Kühn-Methuku-Osthus ('22).

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- Summary: Existence of (rigid) object leveraging on LLL. Spread is guaranteed by local uniformity property of the distribution of solutions to the constraint satisfaction problem.
- Local uniformity property: Under LLL setting, given bad events with probability at most p, and assume that the maximum degree of the dependency graph is Δ with $4p\Delta \leq 1$. For any event $\mathcal{F}$ depending on at most N bad events, the probability of $\mathcal{F}$ under a random satisfying solution is at most the probability of $\mathcal{F}$ under the product measure up to an error $\exp(pN)$.
  - Key property in previous works on algorithmic LLL and recent works on sampling algorithms for the distribution of solutions.

Constructing spread measures

- We view Latin squares as edge decompositions of $K_{n,n}$ into perfect matchings, and construct the desired distribution in progressive steps, decomposing $K_{n,n}$ into regular subgraphs of decreasing degrees.
- Each step employs a random partition that naturally has optimal spread. However, this is not compatible with the rigid (regular) nature of the objects.
- We condition the random partition on satisfying a constraint satisfaction problem, which allows to correct the random object to a regular object.
- Show spread by using local uniformity property, and bootstrap on spread to show success of the iterations.

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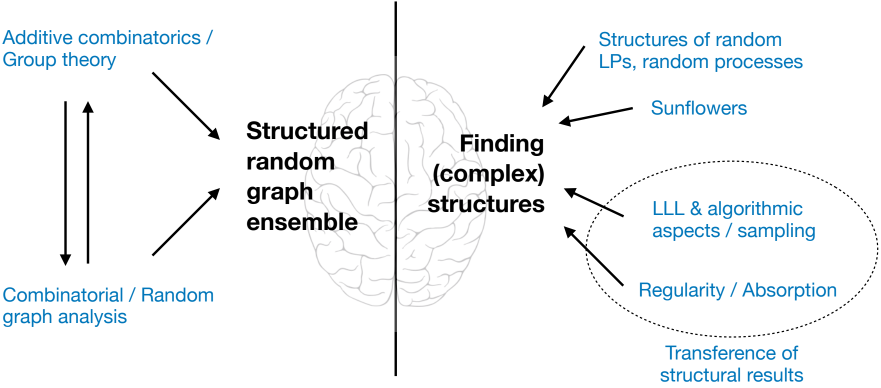
- Interesting future direction: Obtaining robust (threshold) versions of other properties given by constraint satisfaction problems.

Roadmap



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- Max-Cut/Max-Bisection in $G(N, p)$: The max-cut value is $N(pN/4 + (1 + o_{pN}(1))P_* \sqrt{pN}/2)$ (Dembo-Montanari-Sen '17).

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Question: How does p-dependence to p-independence transition happen?

Further connections: An open invitation

Conjecture (Alon 1989)

There is a constant C such that every finite group has a Cayley graph which is C-Ramsey.

An important step in this direction is the following:

Toy Conjecture

There is a two-coloring of $\mathbb{F}_2^d \setminus \{0\}$ such that there is no subspace of size Cd whose nonzero elements are monochromatic.

The trouble in small characteristic indicates interesting relationship with Ramsey theory, additive combinatorics.

Further connections: An open invitation

Conjecture (Alon 1989)

Consider a random Cayley graph with density p. The independence number is almost surely $\tilde{O}(p^{-1})$.

- Random Cayley graphs should have similar behaviors to random regular graphs.
- Relations to spectral graph theory, random matrix theory/suprema of processes.

Thank you!