The short-code graph is a ***SSE4GS***

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The short version

- Problem: Satisfy a constraint graph
- PCP theorem: The $(\delta, 1)$-gap case is NP-hard
- Khot 02 conjectures:
  - UGC: $(\delta, 1-\delta)$-gap is NP-hard even for … nstrains
  - 222: $(\delta, 1)$-gap is NP-hard for 2-to-2 constraints.

"Proved 222 conjecture"

- [KMS 16, DKKMS 16, BKS 17, DKKMS 17, KMS 18]:

  $(\delta, 1-\delta)$-gap is NP-hard for 2-to-2 constraints.
- We **Generalize**, *streamline*, **shorten**, *conceptualize* DKKMS 17+KMS 18

A bit more details

- [KMS 16, DKKMS 16]: proved 222 conjecture,
  *if Grassmann test is **sound***
- (requires: Grassmann graph is ***SSE4GS***)

“Proved 222 conjecture”

- [DKKMS 17]: Grassmann graph is weakly ***SSE4GS***
- [BKS 17] : If Grassmann graph is ***SSE4GS***, then it is ***sound***
- [KMS 18]: Grasmann graph is actual

“Proved 222 conjecture”

- We prove: Short-code graph is ***SSE4GS***
- [BKS 17] : If short-code graph is ***SSE4GS***, then
  Grassmann graph is ***SSE4GS***

SSE : small set expander

- G=(V, E): a family of (weighted) graphs
- G is ***SSE***: if for $S \subseteq V$,

$$\frac{|S|}{|V|} \leq \delta \quad \rightarrow \quad \Pr_{v \in S, u \in N(v)}[u \in V \setminus S] \geq 1 - \epsilon(\delta)$$

- Noisy cube example:

V = $\{0,1\}^n$

Edge *distribution*: Pick $x \in \{0,1\}^n$

$$\text{Pick } y_i = \begin{cases} x_i & w.p. \quad \rho \\ random. & w.p. \quad 1-\rho \end{cases}$$

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- Thm: ρ-noisy cube is an ***SSE***.
- Follows from [Bonami 72, Gross 73, Beckner 73]
- Used in [KKL 92] (collective coin flipping), [Friegut 98] (low-influence functions are juntas), [DS 02] (VC hardness), [MOO 05] (majority is stablest), etc. etc.

Not ***SSE***: noisy mesh

$\mathrm{V} = \{0,1, \dots, q(n)\}^n$

Edge *distribution*: Pick $x \in \{0,1, \dots, q(n)\}^n$

$$\text{Pick } y_i = \begin{cases} x_i & w.p. \quad \rho \\ random. & w.p. \quad 1-\rho \end{cases}$$

- ρ-noisy mesh is NOT an ***SSE***:

  Dictatorship: S=$\{x : x_i = 17\}$

  $|\mathrm{S}|/|\mathrm{V}| = 1/q(n)$, but

  Probability of leaving S is $\leq 1-\rho$

- But: Dictatorships (and juntas) are *local*.

Global sets and functions

- Let $f\colon \{0,..,q\}^n \to \mathbb{R}$.   (for a set S, take $f = \mathbf{1}_S$)
- $\|f\|_2^2 = \mathbb{E}_x[f(x)^2]$
- Restrictions: $f_{T\to y}$(x)=f(x,y)
- Global set/function: f is (d,ϵ)-*global* if

$$\left\|f_{T\to y}\right\|_2^2 \le \epsilon \quad \text{for any } |T| \le d.$$

SSE4GS !

- Dictatorship is not even (1,1/2)-global! Juntas also not global.
- [KLLM 19] The ρ-noisy mesh is a ***SSE*** for global-enough sets.
- We reprove this with slightly worse parameters
- We also prove this for the (degree 2) short-code graph

The short-code graph

- Vertices: Linear functions in $\mathcal{L}(\mathbb{F}_2^n, \mathbb{F}_2^m)$
- Alternatively: $Mat_{m\times n}(\mathbb{F}_2)$
- Edges: (A, B) edge if $rank(B - A) = 1$
- Not ***SSE***: S=$\{A : A \cdot v = w\}$

  or T=$\{A : w^t \cdot A = v^t\}$
- If $A \in S$ then $\Pr_{B\in N(A)}[B \cdot v = w] =$

$$\Pr_{\psi,\xi}[(A + \psi \cdot \xi^t) \cdot v = A] = \Pr[\xi^t v = 0] = \frac{1}{2}$$

- We show: Short-code graph is ***SSE4GS***

Can replace 2 by q

What are restrictions?

The short-code graph

- Vertices: Linear functions in $\mathcal{L}(\mathbb{F}_2^n, \mathbb{F}_2^m)$
- Alternatively: $Mat_{m\times n}(\mathbb{F}_2)$
- Edges: (A, B) edge if $rank(B - A) = 1$
- Not ***SSE***: S=$\{A : A \cdot v = w\}$

  or T=$\{A : w^t \cdot A = v^t\}$
- d-restriction: An intersection of at most d sets of type S or T as above.
- (d, ϵ)-global: define as before

What are restrictions?

The short-code graph is a ***SSE4GS***

- We prove:

  A $(Cr, 2^{-C`r^2})$-global set of matrices is $1 - 2^{-r}$ expanding.

- We *actually* prove:

  If f: $\mathcal{L}(\mathbb{F}_2^n, \mathbb{F}_2^m) \to \mathbb{R}$ is (d, δ)-global, then

  $$\left\| f^{\le d} \right\|_4^4 \le 2^{cd^2} \cdot \delta \cdot \left\| f^{\le d} \right\|_2^2$$

- Bonami lemma: Same, for general functions on $\{0,1\}^n$
- Degree d function: A linear combination of d-restrictions
- $f^{\le d}$: The projection of f on degree d functions.

“Proof”

- For any function of degree d, we prove something of the form

$$\|f\|_4^4 \leq 2^{d^2}\|f\|_2^2 + \sum_S d^{|s|}\|L_S[f]\|_4^4$$

- L_S part is “Laplacians”, which we can apply induction to using ”derivatives” which are of smaller degree.
- These terms are easy to define for the noisy cube and mesh
- We also define them for the short-code graph, not so easily…


That's all Folks!