HIGH DIMENSIONAL EXPANDERS

- A graph $G=(V,E)$ is a **λ-spectral expander** if $\| A_G - \frac{1}{n}J \| \leq \lambda$ 2-sided
  $\forall f: V \to \mathbb{R}$ $\left| \mathbb{E}_{uv\sim E} f(u)f(v) - \mathbb{E}_{u,v} f(u)f(v) \right| \leq \lambda \cdot \|f\|^2$ (or $A_G - J \lesssim \lambda I$) 1-sided

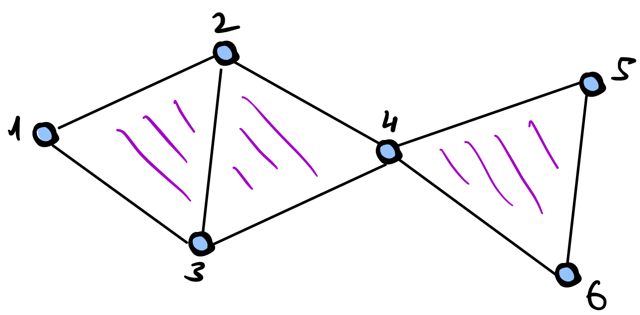
  $\lambda_n \leq \cdots \leq \lambda_2 \leq \lambda_1 = 1$ be the eigenvalues of the RW matrix

  $\lambda_2 \leq \lambda$ (one-sided)

  $-\lambda \leq \lambda_n \leq \cdots \leq \lambda_2 \leq \lambda$ (two-sided)

- A <u>simplicial complex</u> $X = X(0) \cup X(1) \cup \cdots \cup X(d)$

  $X(0)$ - a set of vertices
  $X(1)$ - " " " edges
  $\vdots$
  $X(d)$ - a set of d-faces, each containing $d+1$ vertices.
  if $s \in X$, $t \subset s$ $\longrightarrow$ $t \in X$.



$X(0) = \{1, 2, 3, 4, 5, 6\}$

$X(1) = \{12, 23, 13, 24, 34, 45, 56, 46\}$

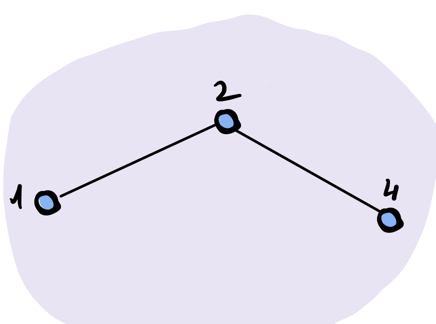
$X(2) = \{123, 234, 456\}$

- A **link** of a face $v \in X$ is a complex whose faces are

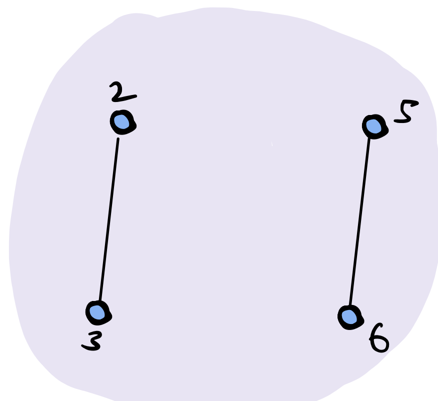
  $X_v = \{ s \setminus v \mid s \in X,\ s \supset v \}$

Examples:

$X_{\{3\}} =$



$X_{\{4\}} =$



Definition of HDX:

- A λ-(one-sided / two-sided) **link expander** is a d-dimensional complex X s.t. for every face $v \in X$, $\dim(v) < d-1$, the graph $G_v = (X_v(0), X_v(1))$ is a λ-(one-sided / two-sided) spectral expander.

  X is sometimes called a λ-HDX.

Some Examples of HDX:

- The complete complex: Let $X(0) = [n]$, $X(i) = \binom{[n]}{i+1}$. $X = \Delta_n^{(d)}$

  e.g. when $d=2$ we have $\binom{n}{2}$ edges & $\binom{n}{3}$ triangles.

  For $v \in X(0)$, $X_v = \Delta_{n-1}^{(d-1)}$. "Johnson scheme $d+1 = pn$"

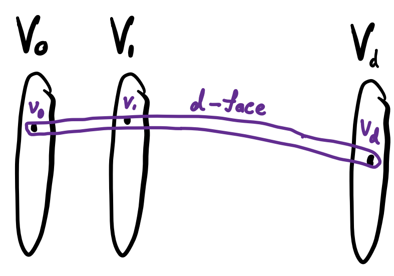
- A random r-regular graph is an expander.

  Is a random model for sparse HDX? NO. Even for $d=2$.

  if we choose edges w probability p small $\longrightarrow$ no triangles

  if we choose Δ's . . . . $\longrightarrow$ disconnected links.

- The $(d+1)$-partite complete complex:



  $X(0) = V_0 \dot\cup V_1 \dot\cup \cdots \dot\cup V_d$

  $|V_i| = n \quad \forall i = 0 \dots d$

  $X(d) = \{(v_0, \dots, v_d) : v_i \in V_i\}$ (corresponds to tuples $(v_0, \dots, v_d) \in [n]^{d+1}$

  and $X(i)$ $i<d$ is defined by downward closure.

  $\forall v \in V_j$, X_v is a d-partite complex...

- **The Spherical Building**: Fix $\mathbb{F} = \mathbb{F}_q$ a finite field

$X(0)$ = all linear subspaces of $\mathbb{F}^{d+2}$

$X(0) = V_1 \cup V_2 \cup \cdots \cup V_{d+1}$ where V_i = lin. spaces dim $= i$.
= $Gr(d+2, i)$.

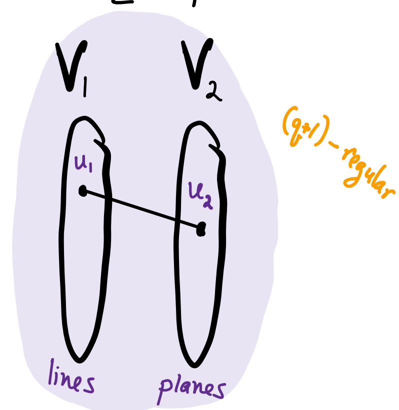
$$X(d) = \{ \{u_1, u_2, \cdots, u_{d+1}\} \mid u_i \in V_i \text{ and } u_1 \subset u_2 \subset u_3 \subset \cdots \subset u_{d+1} \}$$

— For example, $d = 1$: $X(0) = V_1 \cup V_2$ $\quad V_1 = \underbrace{\text{lines}}^{q^2+q+1}$ $\quad V_2 = \underbrace{\text{planes}}^{q^2+q+1}$

$$X(1) = \{ \{u_1, u_2\} \mid u_1 \subset u_2 \}$$

"lines vs. planes incidence graph"

"points vs lines in projective plane"

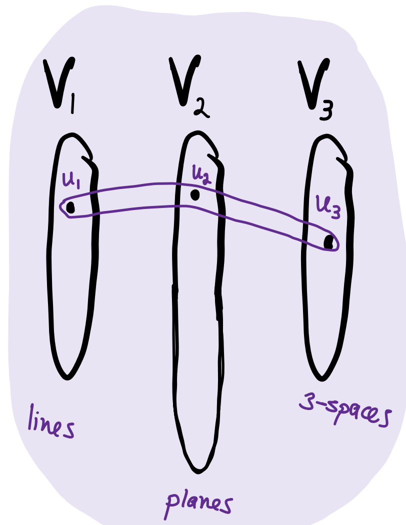


— For example, $d = 2$:

V_1 = lines $\quad V_2$ = planes $\quad V_3$ = 3-spaces

Fix $u_2 \in V_2$. X_{u_2} = complete bipartite graph

Fix $u_1 \in V_1$. X_{u_1} = lines vs. planes ingidence graph



It is a $\frac{1}{\sqrt{q}}$ - one-sided link expander.

- The Lubotzky - Samuels - Vishne **(LSV) Ramanujan Complexes**

For every $d \geq 2$ and prime power q,
there is a family $X^1, X^2, \cdots, X^n, \cdots$ of d-dim complexes on an increasing # of vertices s.t. the link of each vertex is the $(d-1)$-dimensional spherical building over $\mathbb{F}_q$.

Trickling Down theorem [Oppenheim]:

"Local to Global"

Theorem 3.1 (Trickling-Down Theorem, two-dimensional)**.** *Let X be a 2-dimensional simplicial complex such that the graph $(X(0), X(1))$ is connected and $\forall v \in X(0)$ X_v is a one-sided λ-expander. Then $(X(0), X(1))$ is a μ-expander where $\mu = \frac{\lambda}{1-\lambda}$.*

Note that the theorem is useless for $\lambda \geqslant \frac{1}{2}$. By applying the theorem iteratively, we get the following useful corollary:

Corollary 3.2 (Trickling-Down Theorem, d-dimensional)**.** *Let X be a d-dimensional simplicial complex such that the 1-skeleton of every link (including the entire simplicial complex) is connected and $\forall v \in X(d-2)$ X_v is a one-sided λ-expander. Then X is a μ-expander where $\mu = \frac{\lambda}{1-(d-1)\lambda}$.*

Proof of Theorem 3.1. Let A be the adjacency operator associated with the 1-skeleton $(X(0), X(1))$.

Suppose $f : X(0) \to \mathbb{R}$ is an eigenfunction with eigenvalue γ, and assume $f \perp \mathbf{1}$. Also assume $\|f\| = 1$, namely $\mathbb{E}[f^2] = 1$. We have:

$$\gamma = \langle f, Af \rangle = \mathop{\mathbb{E}}_{\{u,w\} \in X(1)}[f(u)f(w)] = \mathop{\mathbb{E}}_{v \in X(0)} \mathop{\mathbb{E}}_{\{u,w\} \in X_v(1)}[f(u)f(w)] \tag{3.1}$$

choose a vertex v, then an edge $uw \in X_v(1)$

= choose a uniform edge in $X(1)$.

$$\gamma = \mathop{\mathbb{E}}_{v \in X(0)} \underbrace{\mathop{\mathbb{E}}_{\{u,w\} \in X_v(1)}[f(u)f(w)]}_{\text{SS}}$$

$f^v(u) := f(u)$

A^v - adj operator of X_v

(λ-expander by assumption)

$$\langle f^v, A_v f^v \rangle \leq \underbrace{(\mathbb{E}f^v)^2}_{= \gamma^2 f(v)^2} + \lambda \| \overbrace{f^v - \mathbb{E}f^v}^{g^v} \|^2 \quad (= \lambda \cdot \|g^v\|^2)$$

1-sided graph expansion

$$f^v = g^v + \mathop{\mathbb{E}}_u f^v(u) = g^v + \gamma f(v)$$

$$\mathop{\mathbb{E}}_v \|g^v\|^2 = \mathop{\mathbb{E}}_v \|f^v\|^2 - \gamma^2 f(v)^2 = 1 - \gamma^2$$

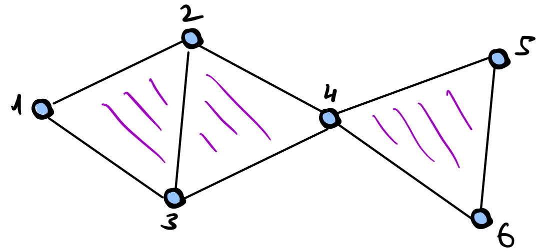
$$\gamma \leq \gamma^2 + \lambda(1-\gamma^2)$$

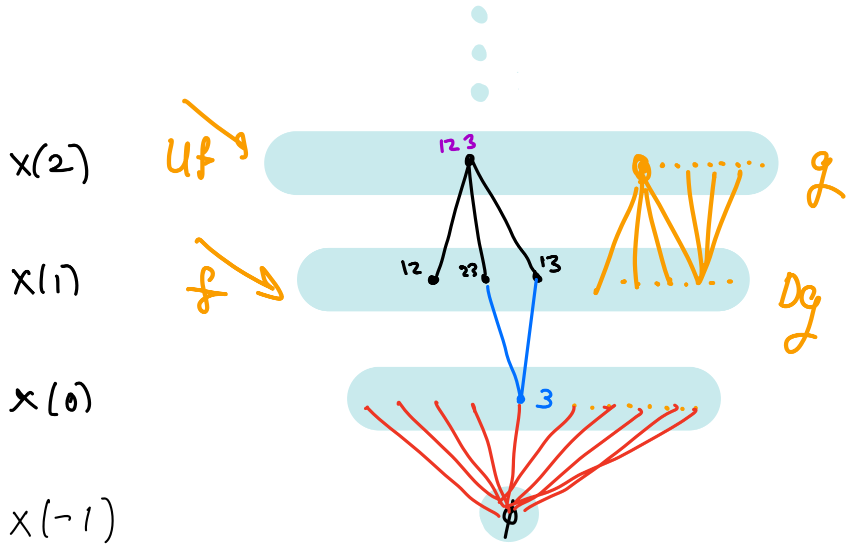
$$\gamma \leq \lambda(1+\gamma)$$ divide by $1-\gamma$ assuming $\gamma < 1$...

$$\gamma \leq \frac{\lambda}{1-\lambda} \qquad \blacksquare$$

&

Up and Down Operators





$C^i(X,\mathbb{R}) = \{ f : X(i) \to \mathbb{R} \}$

$\langle f, f' \rangle_i = \mathbb{E}_{s \sim X(i)} [f(s) f'(s)]$

Def: $U : C^i \to C^{i+1}$ $\quad Uf(s) := \mathbb{E}_{t \lessdot s} f(t)$ where $t \lessdot s$ means $t \subset s$ & $\dim t = \dim s - 1$

$D : C^i \to C^{i-1}$ $\quad Df(s) := \mathbb{E}_{t \gtrdot s} f(t)$

Claim: $f \in C^i$ $\quad g \in C^{i+1}$ $\quad \langle f, Dg \rangle_i = \langle Uf, g \rangle_{i+1}$

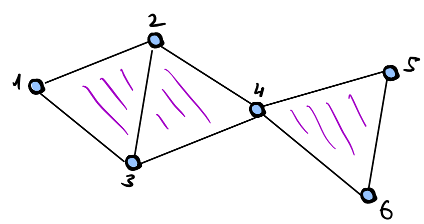
proof: both equal $\mathbb{E}_{s \lessdot t} f(s) g(t)$

Want: basis for C^i that "works well" with U & D (Max...)

High order Random Walks

$U_{i-1} D_i : C^i \longrightarrow C^i$ "lower" RW

$D_{i+1} U_i : C^i \longrightarrow C^i$ "upper" RW



case $i = 0$: $\quad UD_0 = \frac{1}{n} J$

$DU_0 = \frac{1}{2} M + \frac{1}{2} I$

Notation : M_i^+ = non-lazy upper walk $:= \frac{i+2}{i+1} DU_i - \frac{1}{i+2} I$

(so $DU_i = \frac{i+1}{i+2} M_i^+ + \frac{1}{i+2} I$)

$M - J$

<u>Def</u>: X is a γ-RW HDX if $\| M_i^+ - UD_i \| \leq \gamma$ (2-sided)

$M_i - UD_i \preceq \gamma \cdot I$ (1-sided)

<u>Thm</u> 1: [DDFH, KO] : X λ-link expander $\Longrightarrow$ X λ-RW-HDX

<u>Thm</u> 2 : [DDFH]: X λ-RW-HDX (2-sided) $\Longrightarrow$ X $3\lambda d$-HDX (2-sided)

<u>Pf 1</u>: Let $f \in C^i$ $i < d$. Show: $\langle M_i^+ f, f \rangle \leq \langle UDf, f \rangle + \lambda \|f\|^2$

$$\langle M_i^+ f, f \rangle = \mathop{\mathbb{E}}_{s \sim X(i)} f(s) \mathop{\mathbb{E}}_{t > s} \mathop{\mathbb{E}}_{\substack{s' < t \\ s' \neq s}} f(s')$$

$t \in X(i+1)$

$$= \mathop{\mathbb{E}}_{t} \mathop{\mathbb{E}}_{\substack{s, s' < t \\ s \neq s'}} f(s) f(s')$$

$s = t \setminus \{x\}$

$s' = t \setminus \{y\}$

$r := t \setminus \{x, y\}$

$$= \mathop{\mathbb{E}}_{r} \mathop{\mathbb{E}}_{\substack{x \sim y \\ xy \sim X_r(1)}} f(r \cup x) f(r \cup y)$$

$f_r(x) := f(r \cup \{x\})$

$$= \mathop{\mathbb{E}}_{r} \langle A_r f_r, f_r \rangle$$

A_r – adj of link

$A_r \preceq UD_0 + \gamma I$

γ-link expansion

$$\leq \mathop{\mathbb{E}}_{r} \langle UD_0 f_r, f_r \rangle + \gamma \|f_r\|^2$$

$$= \mathop{\mathbb{E}}_{r} Df(r)^2 + \gamma \cdot \|f\|^2$$

$$= \langle UDf, f \rangle + \gamma \cdot \|f\|^2$$

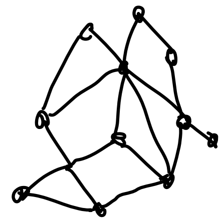
Application: [Anari - Liu - Oveis-Gharan - Vinzant]:

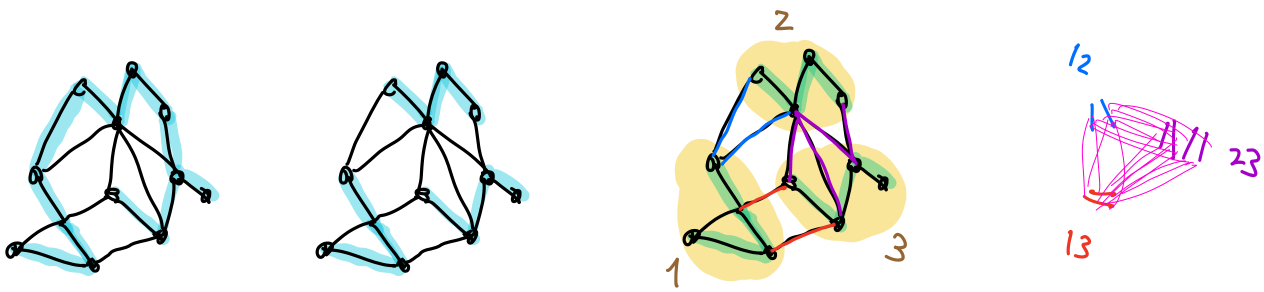
"Matroid basis exchange" RW converges

Given $G = (V, E)$ $|V| = n$, $|E| = m$

$X(0) = E$

$X(n-2) =$ all spanning trees of G





to see convergence, check

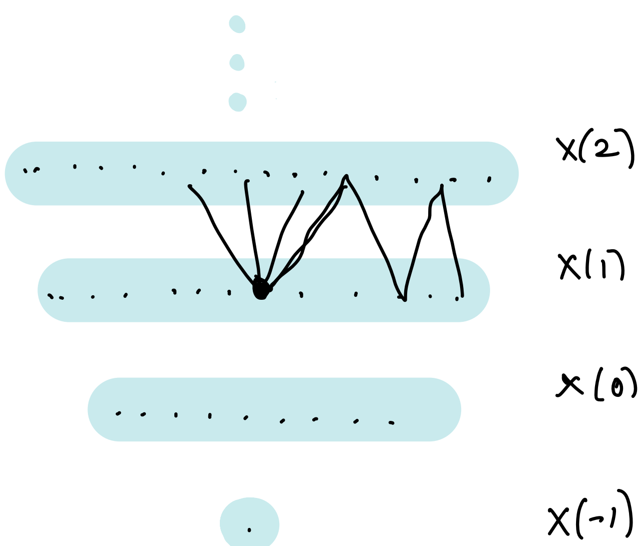
① connectivity

② expansion in links

POSETS

Simplicial complexes are not the most general

- subspaces
- cubical complexes
- ...



Can define U, D, M^+,

& expansion relating M^+ to UD

& can find a good analytic basis for C^i.

co-boundary & co-systolic expansion

co-boundary $\delta : C^i \to C^{i+1}$ $\qquad \delta f(s) := \sum_{t<s} f(t) \pmod 2$

boundary $\partial : C^i \to C^{i-1}$ $\qquad \partial f(s) := \sum_{t>s} f(t) \pmod 2$

$$C^i = \mathbb{F}_2^{X(i)} = \{ f : X(i) \to \mathbb{F}_2 \}$$

using the incidence structure to define linear maps

δ, ∂ can be viewed as linear encoding maps
& parity check maps

from here we get LTCs, quantum LDPCs

Boundary operator $\partial_i : C_i \to C_{i-1}$ "down"

Coboundary operator $\delta_i : C_i \longrightarrow C_{i+1}$ "up"

$$\delta_i f(s) = \sum_{t<s} f(t) \pmod 2$$

$$\delta_i \circ \delta_{i-1} = 0$$

$$C_{i-1} \xrightarrow{\delta_{i-1}} C_i \xrightarrow{\delta_i} C_{i+1}$$

$$\underset{\substack{\| \\ B^i \\ \text{coboundaries}}}{\operatorname{Im} \delta_{i-1}} \subseteq \underset{\substack{\| \\ Z^i \\ \text{cocycles}}}{\operatorname{Ker} \delta_i}$$

$$B^i \subseteq Z^i \subseteq C_i$$

Cohomology $H^i = Z^i / B^i$

Example:

$$\underset{\substack{\| \\ \mathbb{F}_2^{\{\phi\}} \\ \| \\ \mathbb{F}_2}}{C_{-1}} \xrightarrow{\delta_{-1}} C_0 \xrightarrow{\delta_0} C_1 \xrightarrow{\delta_1} C_2$$

1. $f \in C_0$ $f = 1_S$
   $\delta_0 f = \mathbb{1}_{E(S,\bar{S})}$
   $B^1 = \{\text{all cuts}\}$

2. $f \in C_{-1}$ $\delta_{-1} f = \bar{0}$ or $\bar{1}$
   so $B^0 = \{\bar{0}, \bar{1}\}$.

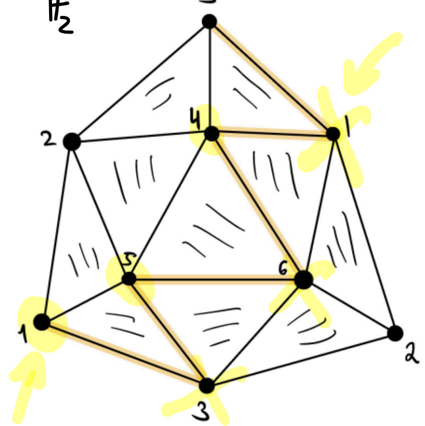
3. $f \in Z^0 = \operatorname{Ker} \delta_0$ so $\forall uv$ $\underset{\substack{\| \\ f(u)+f(v)}}{\delta f(uv)} = 0$

   $\rightsquigarrow$ f is constant on conn. comp.
   $Z^0 = B^0$ iff Graph is connected.
   $H^0 = Z^0 / B^0 = \{0\}$

4. $f \in Z^1 = \operatorname{Ker} \delta_1$
   is $f \in B^1$? ...NO

$$B^1 \underset{\neq}{\subset} Z^1$$

$\mathbb{F}_2^{X(0)}$



Edge expansion $\longleftrightarrow$ coboundary expansion

Edge expansion $\quad h(G) = \min_{\emptyset \neq S \subsetneq V} \frac{|E(S, \bar{S})|}{\min(|S|, |V \setminus S|)}$

or $\quad h(G) = \min_{f \notin B^0} \frac{wt(\delta f)}{dist(f, B^0)} \qquad B^0 = \{\bar{0}, \bar{1}\}$

more generally. $\quad h_i(X) = \min_{f \notin B^i} \frac{wt(\delta f)}{dist(f, B^i)} \qquad$ coboundary expansion

even more generally. $\quad h_i(X) = \min_{f \notin \underbrace{Z^i}_{Ker\, \delta_i}} \frac{wt(\delta f)}{dist(f, \underbrace{Z^i}_{Ker\, \delta_i})} \qquad$ cosystolic expansion

$\delta f \equiv$ a test for the property $f \in Z^i$?

$$C_{i-1} \xrightarrow{\delta_{i-1}} C_i \xrightarrow{\delta_i} C_{i+1}$$