

Weak versions of extended resolution

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Resolution

Refutes a propositional formula in conjunctive normal form (i.e., a set of clauses) by using the single rule

$$\frac{A \vee x \quad B \vee \neg x}{A \vee B}$$

to derive the empty clause.

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Throughout this talk, “proof” \equiv “refutation.”

Example: resolution proof

$$\Gamma = (\bar{x} \vee \bar{z}) \wedge (\bar{y} \vee z) \wedge (x \vee y \vee \bar{z}) \wedge (x \vee \bar{y}) \wedge (y \vee z)$$

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Tree-like:

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Sequence-like:

$$\bar{x} \vee \bar{z}, \bar{y} \vee z, x \vee y \vee \bar{z}, x \vee \bar{y}, y \vee z, z, x \vee y, x, \bar{x}, \perp$$

Extended resolution (ER)

At any step, derive

$$x \leftrightarrow p \wedge q,$$

where p , q are arbitrary literals and x is a *new variable*.

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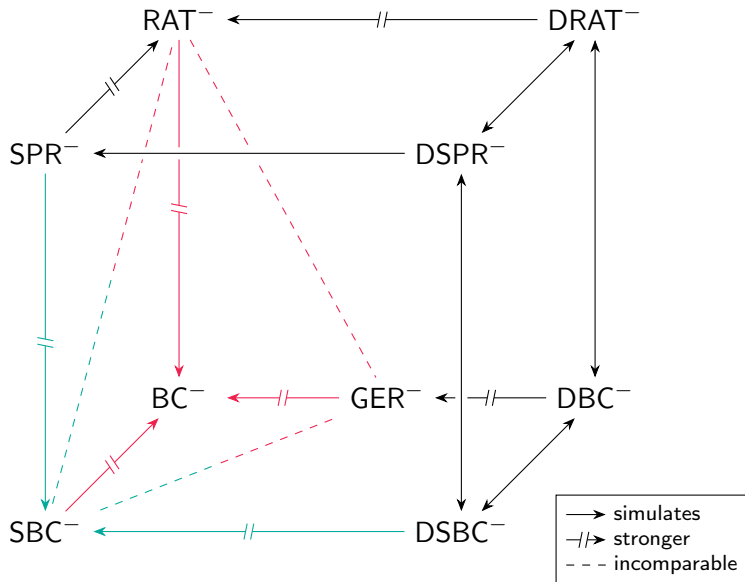
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This talk: Relative strengths of different weak versions of ER

Results



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- Parity principle
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- Tseitin tautologies
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Without loss of generality, pigeon $n + 1$ is mapped to hole n . . .

Redundancy

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Lemma

A clause C is redundant with respect to a formula Γ if and only if there exists a partial assignment τ such that

$$\Gamma \wedge \neg C \models (\Gamma \wedge C)|_{\tau}.$$

Syntactic criteria for redundancy

Definition (Blocked clause)

A clause $C = x \vee C'$ is *blocked* for x with respect to a formula Γ if, for every clause D of the form $\neg x \vee D'$ in Γ ,

$C' \vee D'$ is tautological.

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Example (Blocked clause)

$$C = x \vee y \vee \neg z$$

$$\Gamma = (\neg x \vee \neg y) \wedge (\neg x \vee z) \wedge (y \vee z)$$

C is blocked for x with respect to Γ .

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Claim. If assignment τ sets $x = 1$, then $\Gamma \wedge \neg C \models (\Gamma \wedge C)|_{\tau}$.

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Clauses of the form $\neg x \vee D'$ might be falsified by $\alpha \circ \tau$, but there is some $y \in C'$ such that $\neg y \in D'$ and $\alpha \circ \tau$ still sets y to $\alpha(y) = 0$.

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Definition (Resolution asymmetric tautology*)

A clause $C = x \vee C'$ is a *RAT* for x with respect to a formula Γ if, for every clause D of the form $\neg x \vee D'$ in Γ ,

$$C' \vee D' \text{ is tautological or subsumed by } \Gamma.$$

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Definition (Set-blocked clause)

A clause $C = L \vee C'$ is an *SBC* for L with respect to a formula Γ if, for every clause D in Γ such that $D \cap \neg L \neq \emptyset$ and $D \cap L = \emptyset$,

$$(C \setminus L) \vee (D \setminus \neg L) \text{ is tautological.}$$

The proof systems

- ER resolution + extension
- BC resolution + blocked clause addition
- BC⁻ BC without new variables
- DBC BC with deletion
- DBC⁻ BC with deletion and without new variables
- ⋮
- SPR resolution + “SBC × RAT” addition
- ⋮

Effectively*, BC^- simulates ER

Lemma

Suppose that a formula Γ has an ER proof of size m and that $X = (y \vee x_1 \vee \cdots \vee x_m) \wedge y$ has no variables in common with Γ . Then $\Gamma \wedge X$ has a BC^- proof of size $O(m)$.

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Proof. Consider a use of the extension rule in the ER proof that introduces $x_i \leftrightarrow p \wedge q$. WLOG, the literals p and q are not new.

Add the clauses

$$\neg x_i \vee \neg y \vee p \qquad \neg x_i \vee \neg y \vee q \qquad x_i \vee \neg p \vee \neg q$$

in sequence as blocked clauses. Resolve against y . □

Guarded extension variables

Let Γ be a formula with an ER proof of size $m = |\Gamma|^{O(1)}$.

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Idea: Guard the variables by clauses to make them hard to access.
 P will somehow use the included variables to simulate the ER proof.
 Q will be unable to achieve any speedup using the included variables.

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Example

With respect to the formula $(\neg x \vee y) \wedge (\neg x \vee \neg y)$, any clause blocked for x has to include both $\neg y$ and y .

Example: lower bound

Lemma

$f(\Gamma) := \Gamma \wedge \bigwedge_{i=1}^m [(x_i \vee \Gamma) \wedge (\neg x_i \vee \Gamma)]$ is no easier than Γ for BC^- .

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4. For the assignment $\alpha(x_i) = 1$, we have $(f(\Gamma) \wedge \Delta)|_\alpha = \Gamma \wedge \Delta'$, where Δ' is possible to derive from Γ in BC^- .

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5. Resolution is closed under restrictions, which implies that $f(\Gamma) \wedge \Delta$ is at least as hard for resolution as $\Gamma \wedge \Delta'$.

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5. Resolution is closed under restrictions, which implies that $f(\Gamma) \wedge \Delta$ is at least as hard for resolution as $\Gamma \wedge \Delta'$.
6. If $\Gamma \wedge \Delta'$ is easy for resolution, then Γ is easy for BC^- . □

Separating constructions

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$$f(\Gamma) := \Gamma \wedge \bigwedge_{i=1}^m [(x_i \vee \Gamma) \wedge (\neg x_i \vee \Gamma)]$$

$f(\Gamma)$ is easy for RAT^- (regardless of whether Γ is).

$f(\Gamma)$ is at least as hard as Γ for GER^- and SBC^- .

Separating constructions

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$\text{RAT}^- \not\leq \text{GER}^-$

$\text{RAT}^- \not\leq \text{SBC}^-$

$$g(\Gamma) := \Gamma \wedge \bigwedge_{i=1}^m [(\neg x_i \vee y_i) \wedge (x_i \vee \neg y_i)]$$

$g(\Gamma)$ is easy for both GER^- and SBC^- (for different reasons).

$g(\Gamma)$ is at least as hard as Γ for RAT^- .

Separating constructions

Let Γ be a formula with an ER proof of size $m = |\Gamma|^{O(1)}$.

$SBC^- \not\leq GER^-$

$$h_s(\Gamma) := \Gamma \wedge \bigwedge_{i=1}^m \bigwedge_{j=1}^s [(x_i \vee y_j \vee \neg z_j) \wedge (\neg x_i \vee y_j \vee \neg z_j)] \\ \wedge \bigwedge_{j=1}^s [(\neg y_j \vee z_j) \wedge (y_j \vee \Gamma) \wedge (z_j \vee \Gamma) \wedge (\neg z_j \vee \Gamma)]$$

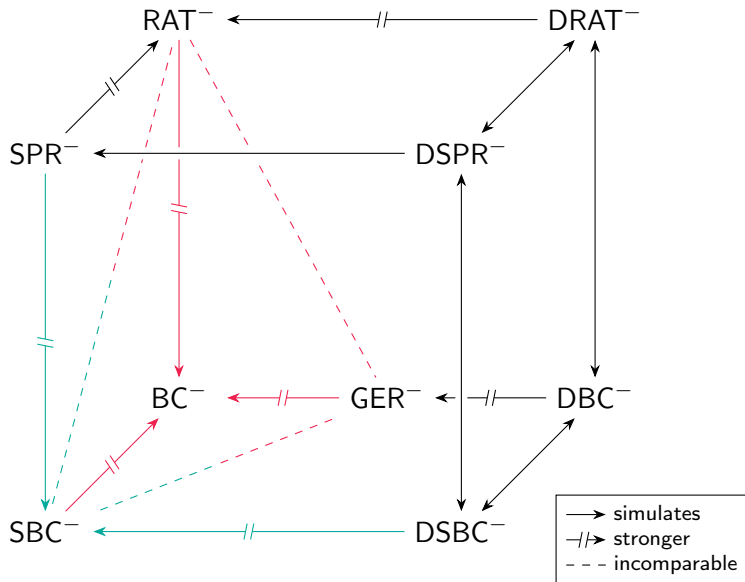
$h_s(\Gamma)$ is easy for GER^- (regardless of whether Γ is).

$h_s(\Gamma)$ is at least as hard as Γ for SBC^- with suitable choice of s .

Open questions

- Lower bounds for SPR^- and above
- Separations using “natural” principles
- Any subsystem of Frege above resolution that DBC^- simulates
- Other uses of the high-level idea in proof complexity

Results



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