A Lower Bound for k-DNF Resolution on Random CNF Formulas via Expansion

Anastasia Sofronova, Dmitry Sokolov

EPFL

Proofs and their complexity

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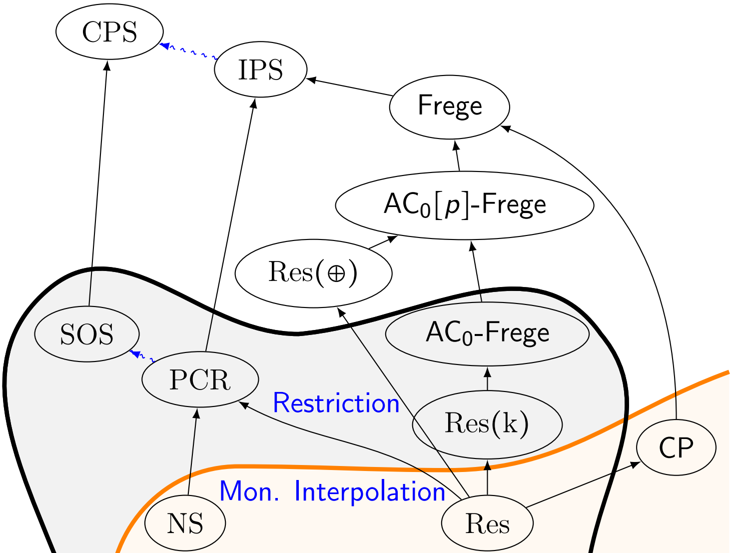
Example: Resolution

$$\textit{Weakening: } \frac{F}{F \vee \ell};$$
$$\textit{Resolution rule: } \frac{F \vee \ell_i, G \vee \neg \ell_i}{F \vee G}.$$

Where we're at



Res(k)

Weakening: $\frac{F}{F \vee \ell}$;

AND-*introduction:* $\frac{F \vee \ell_1, \ldots, F \vee \ell_w}{F \vee (\bigwedge_{i=0}^{w} \ell_i)}$;

AND-*elimination:* $\frac{F \vee (\bigwedge_{i=0}^{w} \ell_i)}{F \vee \ell_i}$;

Cut: $\frac{F \vee (\bigwedge_{i=0}^{w} \ell_i), G \vee (\bigvee_{i=0}^{w} \neg \ell_i)}{F \vee G}$.

A subsystem of AC_0-Frege.

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From $k = \log^{1+\varepsilon} n$ would follow for all k.

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Density $\frac{m}{n}$ threshold for SAT/UNSAT.
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Underlying graph is a bipartite expander.

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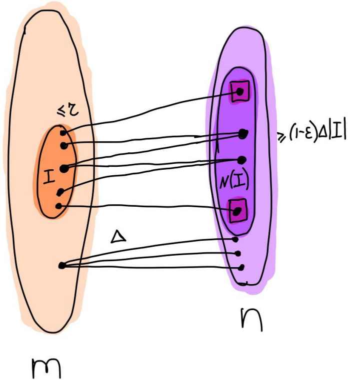
Any small subset of vertices has a lot of (unique) neighbours.

$(r, \Delta, (1-\varepsilon)\Delta)$-*(boundary)* expander: $(1-\varepsilon)\Delta|I|$ *(unique)* neighbours for $I \subseteq L, |I| \leq r$.



Random Δ-CNFs: what is known?

$\mathfrak{D} \coloneqq \frac{m}{n}$, clause density.

$\mathfrak{D} = \mathcal{O}(1)$	$\Delta \geq 3$	$k = \mathcal{O}\left(\sqrt{\frac{\log n}{\log \log n}}\right)$	[Ale11]
$\mathfrak{D} = n^{1/6}$	$\Delta = \mathcal{O}\left(k^2\right)$	$k = \mathcal{O}(1)$	[SBI04]

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$\mathfrak{D} = \mathcal{O}(1)$	$\Delta = \mathcal{O}(1)$	$k = \mathcal{O}(\sqrt{\log n})$
$\mathfrak{D} = \mathrm{poly}(n)$	$\Delta = \mathcal{O}(1)$, ind of k	k = const

Main result

Theorem

φ is a Δ-CNF and its dependency graph G is an $(r, \Delta, 0.95\Delta)$-boundary expander. Then for $\delta > 0$ if:

$$n^{\delta}\left(\frac{n}{0.4r}\right)^{20k^2} = o(r/k)$$

then $\mathrm{Res(k)}$ proof of φ has size $\geq 2^{n^{\delta}}$.

Expanders from proof complexity point of view

Applying restriction:

- preserve the structure of the formula;
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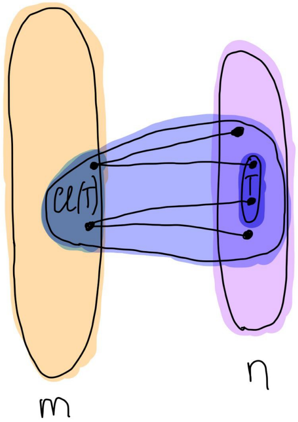


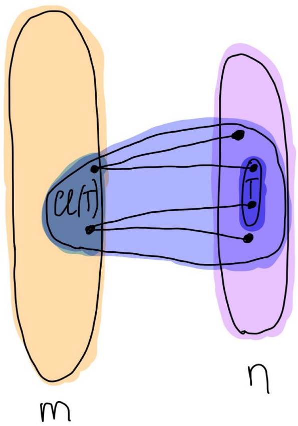
Closure: delete small part of the graph T, then delete something else to make it expander again.

Widely used to prove lower bounds in Res, PCR, SOS, etc.

Closure

Can do it differently:

1. [AR03, Ale+04] Delete the set that violates expansion.



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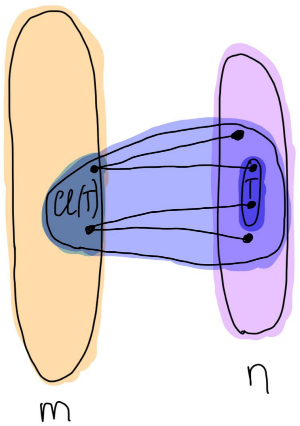

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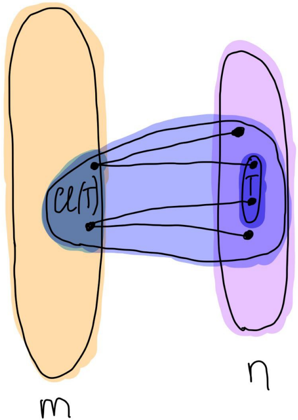

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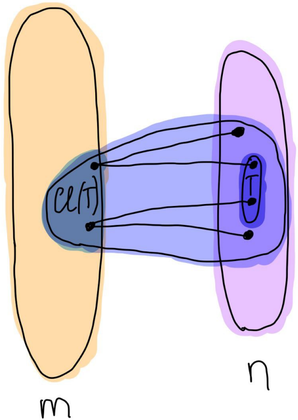

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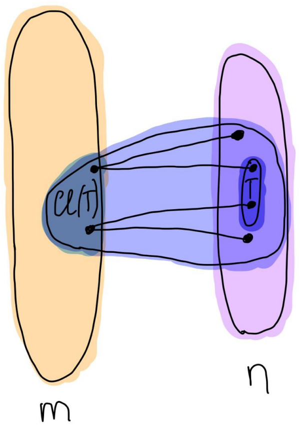

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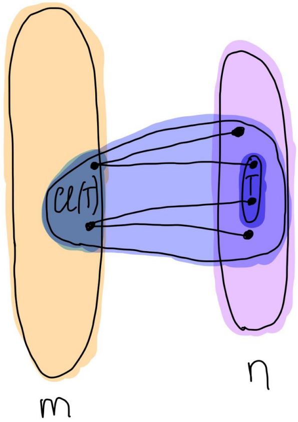
   $(r - \mathcal{O}(|T|), \Delta, (1-2\varepsilon)\Delta)$, is uniquely defined.



k-DNFs and coverings

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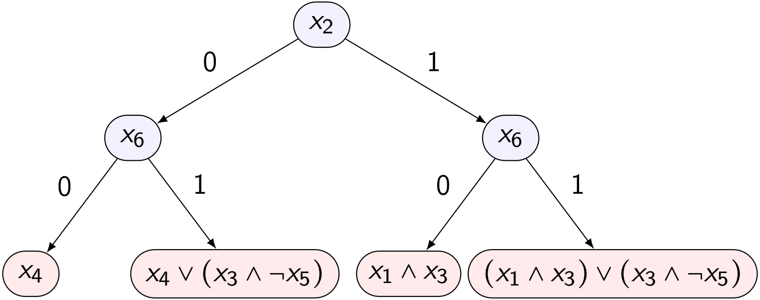
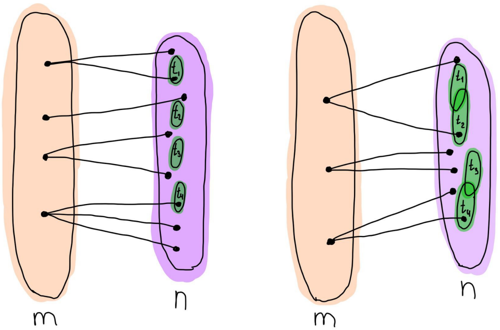
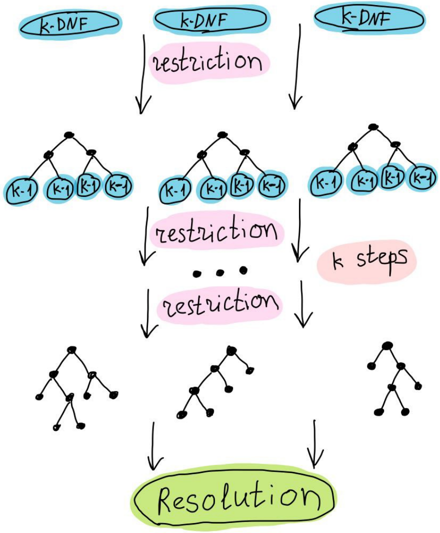


k-DNFs under random restrictions

Ideas from [Segerlind, Buss, Impagliazzo '04; Alekhnovich '11]:

- Big covering number → a lot of "independent terms";
- otherwise equivalent to a decision tree + small collection of $(k-1)$-DNFs;
- iterate that k times, what's left is a Resolution proof.




k-DNF
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restriction
k-1
k-1
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restriction
k steps
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Resolution

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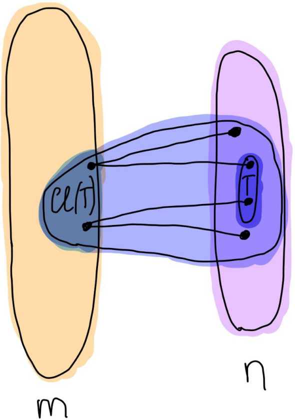
What part of the graph actually depends on a term? Closure!



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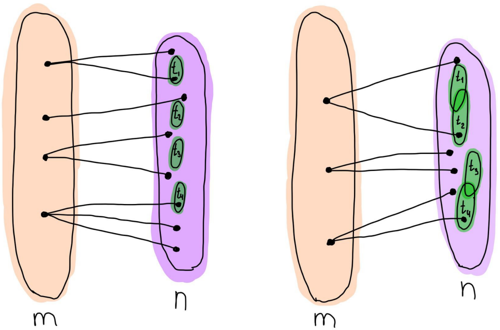
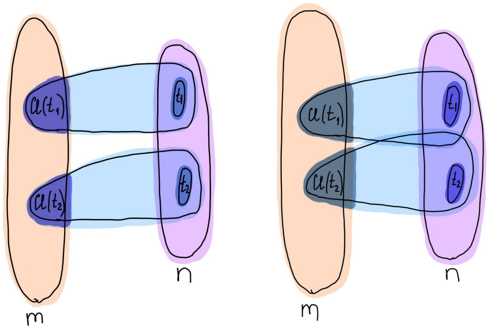
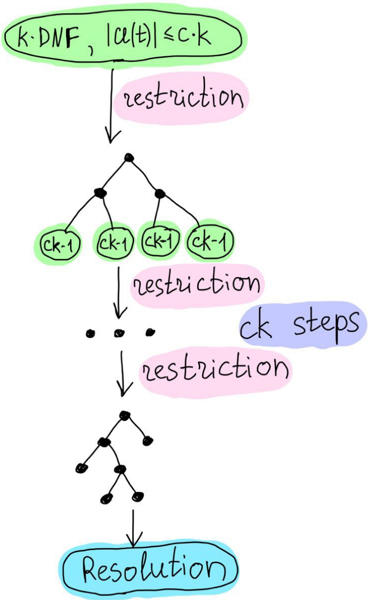


k-DNFs under random restrictions

- Big closure covering number → a lot of "closure independent terms";
- otherwise equivalent to a decision tree + small collection of DNFs where terms have smaller closure;
- iterate $\mathcal{O}(k)$ times, what's left is a Resolution proof.





Important property of individual closure: subgraph-preserving.

Open problems

- Lower bounds for larger k.
- WPHP for $k = 2$.