Towards P≠NP from Extended Frege lower bounds

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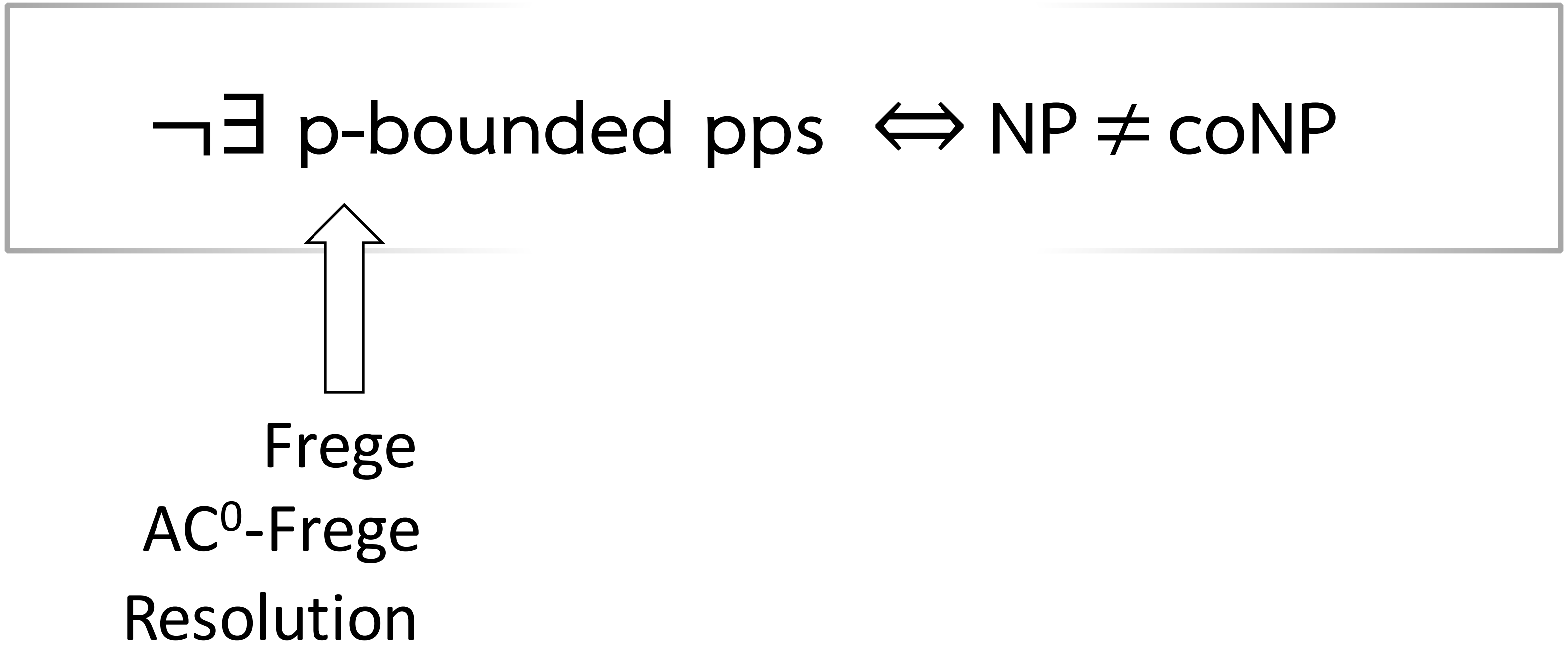
joint work with **Rahul Santhanam**

Proof complexity

$\neg\exists$ p-bounded pps $\Leftrightarrow$ NP $\neq$ coNP

Proof complexity



$\neg\exists$ p-bounded pps $\Leftrightarrow$ NP $\neq$ coNP

Frege

AC^0-Frege

Resolution

Proof complexity

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$\Uparrow$

Frege

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Cook-Reckhow program

Proof complexity

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⇑

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Resolution

EF lower bounds $\overset{?}{\Rightarrow}$ P $\neq$ NP

Cook-Reckhow program

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Resolution

lifting $\Rightarrow$ monotone P/poly lbs

IPS lb $\Rightarrow$ VP$\neq$VNP

R lb $\Rightarrow$ P$\neq$NP T_R -consistent

EF lower bounds $\overset{?}{\Rightarrow}$ P $\neq$ NP

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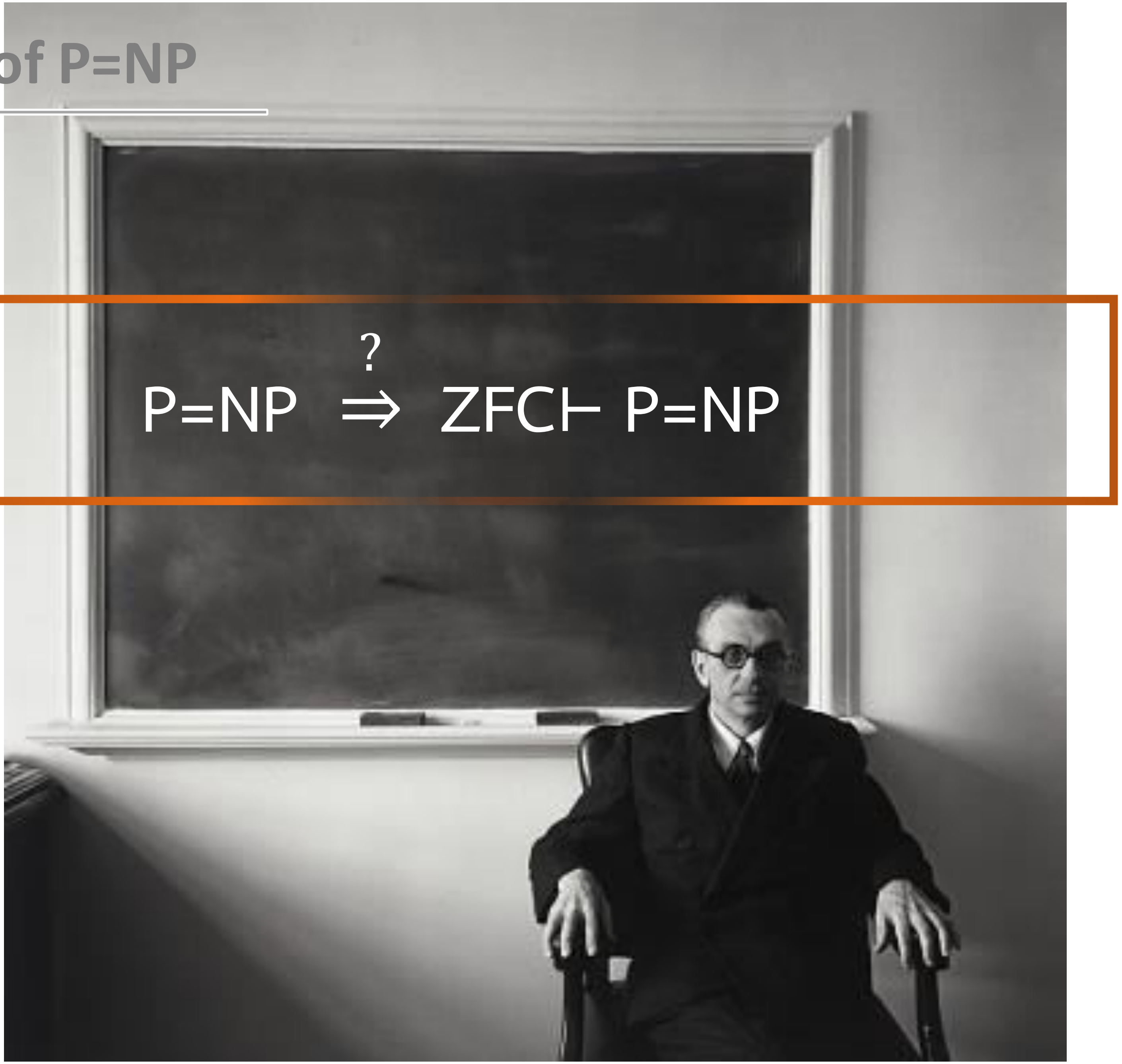
Impagliazzo’s worlds shortly before collision

Proof complexity

Impagliazzo’s worlds shortly before collision

Self-provability of P=NP



$$P=NP \overset{?}{\Rightarrow} ZFC\vdash P=NP$$

Self-provability of P=NP

$\mathsf{SAT}_n(x, y) \equiv$ “formula x satisfied by assignment y”

Witnessing P $\neq$ NP

$$\mathsf{SAT}_n \notin \mathsf{Circuit}[n^{10k}] \quad \overset{?}{\Rightarrow} \quad \begin{array}{l} \exists \text{ p-time } f \text{ s.t. } \forall C \in \mathsf{Circuit}[n^k] \\ \mathsf{SAT}_n(f_1(C), f_2(C)) \wedge \neg \mathsf{SAT}_n(f_1(C), C(f_1(C))) \end{array}$$

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random

⇩

$$h \text{ is one-way} \Rightarrow \text{“}h(x) = h(a)\text{” is a hard } \mathsf{SAT}\text{-instance}$$

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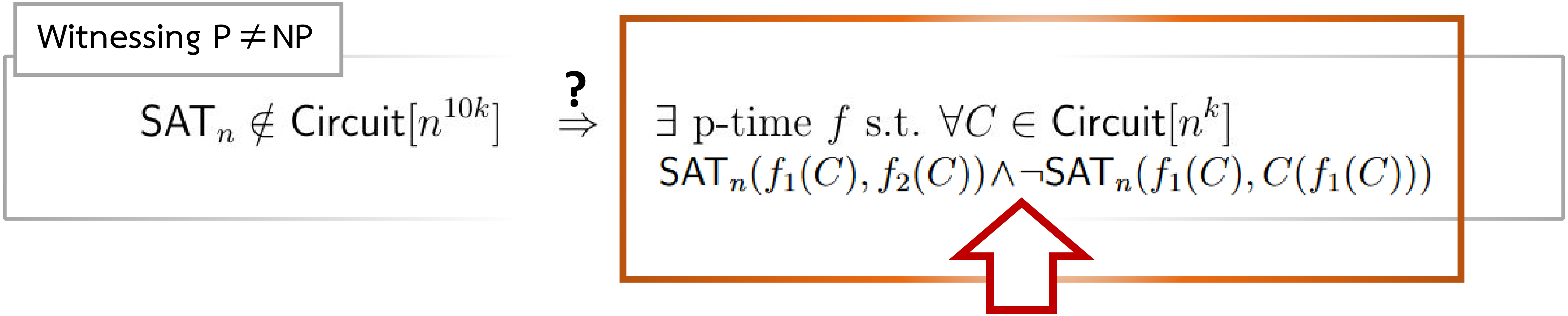

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E hard for subexponential-size circuits

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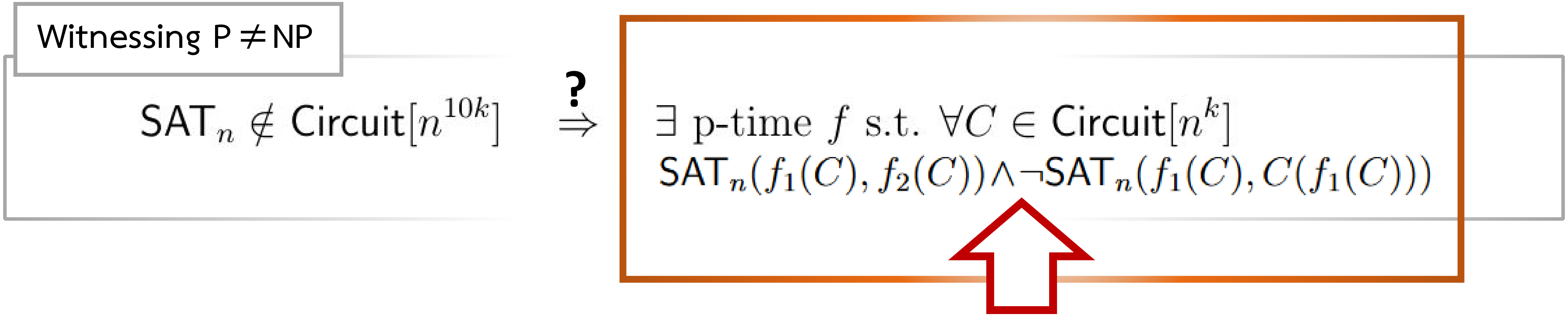

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E hard for subexponential-size circuits

[Gutfreund Shaltiel Ta-Shma]-style constructions in uniform setting

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$$w_n^k(f) := [\mathsf{SAT}_n(x,y) \rightarrow \mathsf{SAT}_n(x, C(x))] \vee [\mathsf{SAT}_n(f_1(C), f_2(C)) \wedge \neg \mathsf{SAT}_n(f_1(C), C(f_1(C)))]$$

variables: x, y, C

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$\Uparrow$

encodes n^k-size circuits

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⇧

encodes n^k-size circuits

$w_n^k(f) \in \mathsf{TAUT}$

⇩

$\mathsf{EF} + w^k(f)$

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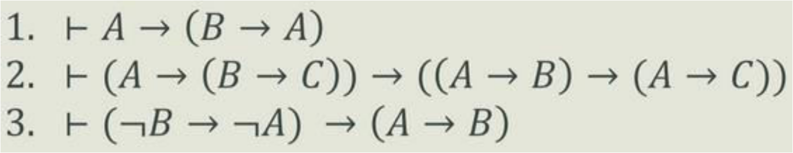
encodes n^k-size circuits

$$w_n^k(f) \in \mathsf{TAUT}$$

$$\Downarrow$$

$$\mathsf{EF} + w^k(f)$$



1. $\vdash A \to (B \to A)$
2. $\vdash (A \to (B \to C)) \to ((A \to B) \to (A \to C))$
3. $\vdash (\neg B \to \neg A) \to (A \to B)$

Self-provability of P=NP

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encodes n^k-size circuits

$$w_n^k(f) \in \mathsf{TAUT}$$
$$\Downarrow$$
$$\mathsf{SAT}_n \in \mathsf{Circuit}[n^{k/10}] \quad \Rightarrow \quad \mathsf{EF} + w^k(f) \vdash \text{“}\mathsf{SAT}_n \in \mathsf{Circuit}[n^k]\text{”}$$

Self-provability of P=NP

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$$\Rightarrow \quad \mathsf{EF} + w^k(f) \text{ is p-bounded}$$

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$(\phi \in \mathsf{TAUT} \Rightarrow \mathsf{EF} \vdash \neg \mathsf{SAT}(\neg\phi, C(\neg\phi)) \Rightarrow \mathsf{EF} + w^k(f) \vdash \neg \mathsf{SAT}(\neg\phi, y) \Rightarrow \mathsf{EF} + w^k(f) \vdash \phi)$

Circuit complexity ⇐ proof complexity & witnessing of P≠NP

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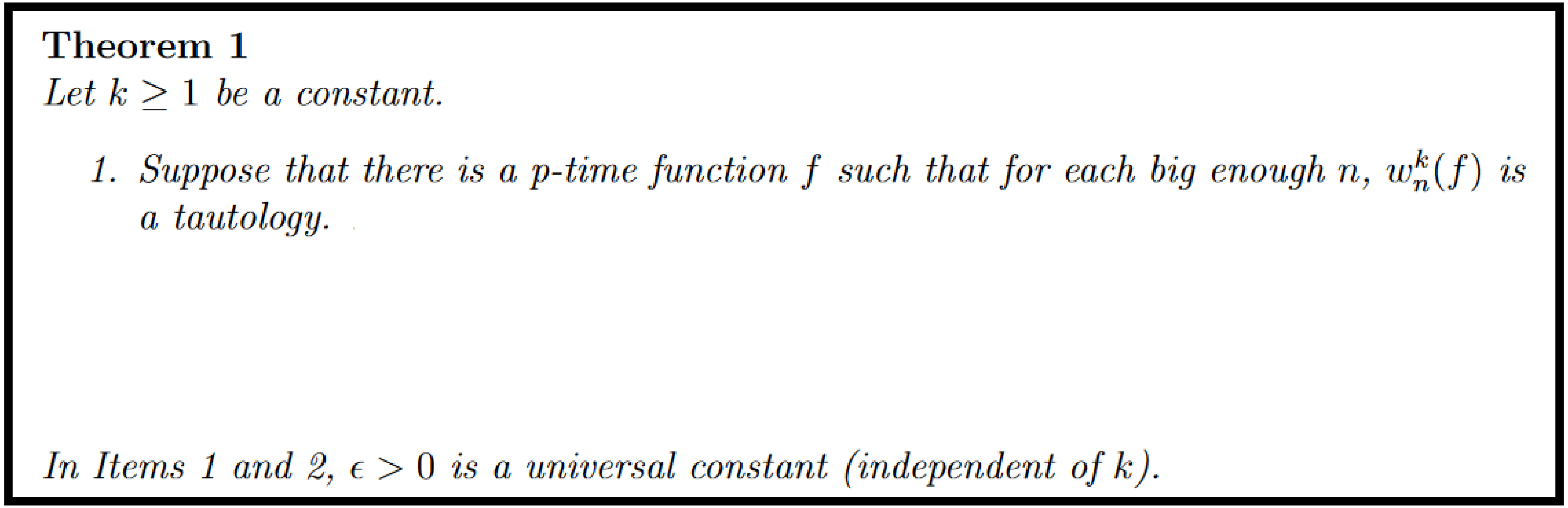
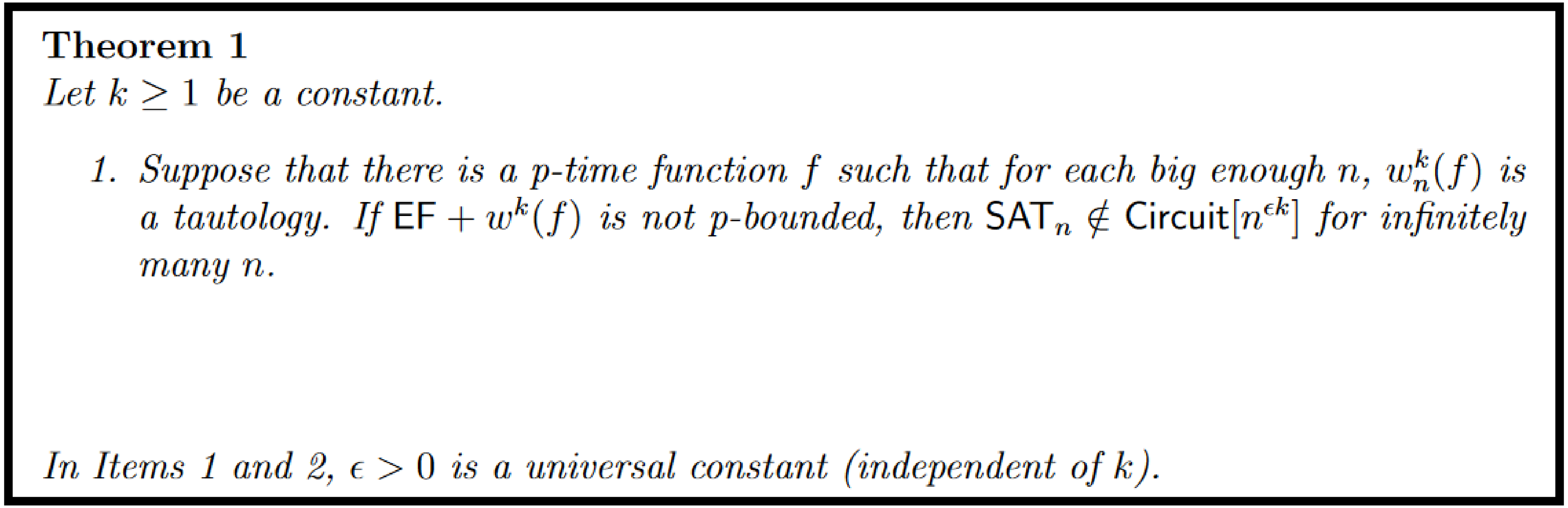


Theorem 1
Let $k \geq 1$ be a constant.

1. *Suppose that there is a p-time function f such that for each big enough n, $w_n^k(f)$ is a tautology.*

In Items 1 and 2, $\epsilon > 0$ is a universal constant (independent of k).

Circuit complexity ⇐ proof complexity & witnessing of P≠NP



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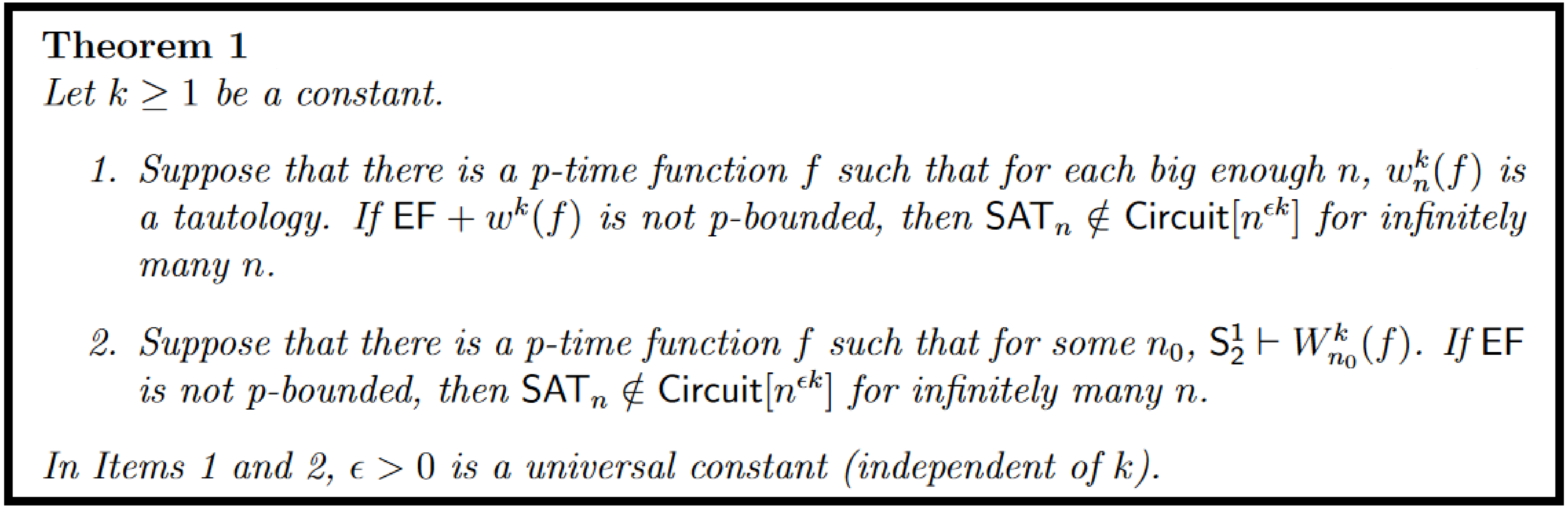

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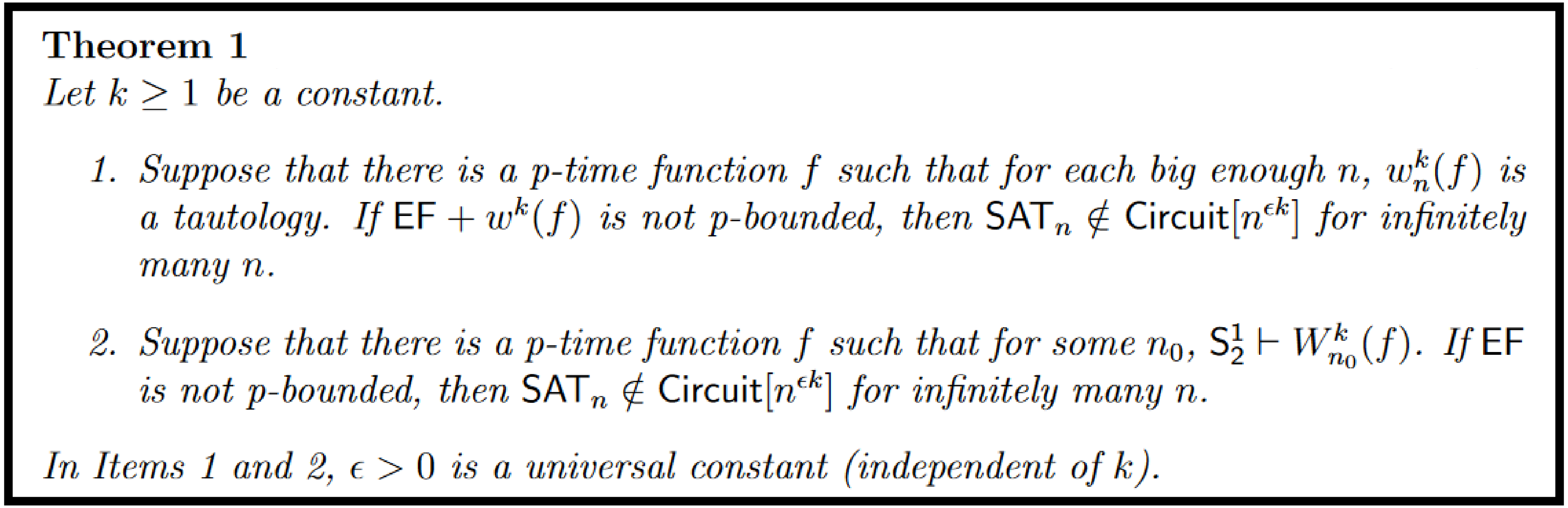

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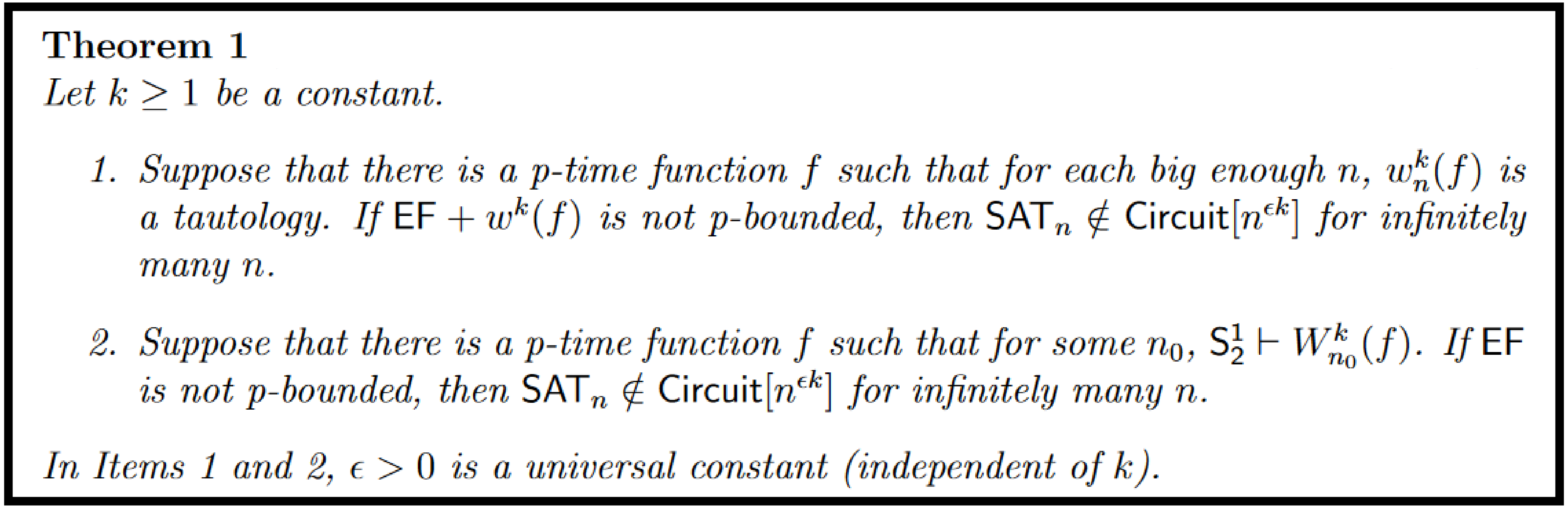
- Generalizes to stronger systems

Circuit complexity ⇐ proof complexity & witnessing of P≠NP



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Open problem: $w_n^k(f) \in \mathsf{TAUT}$?

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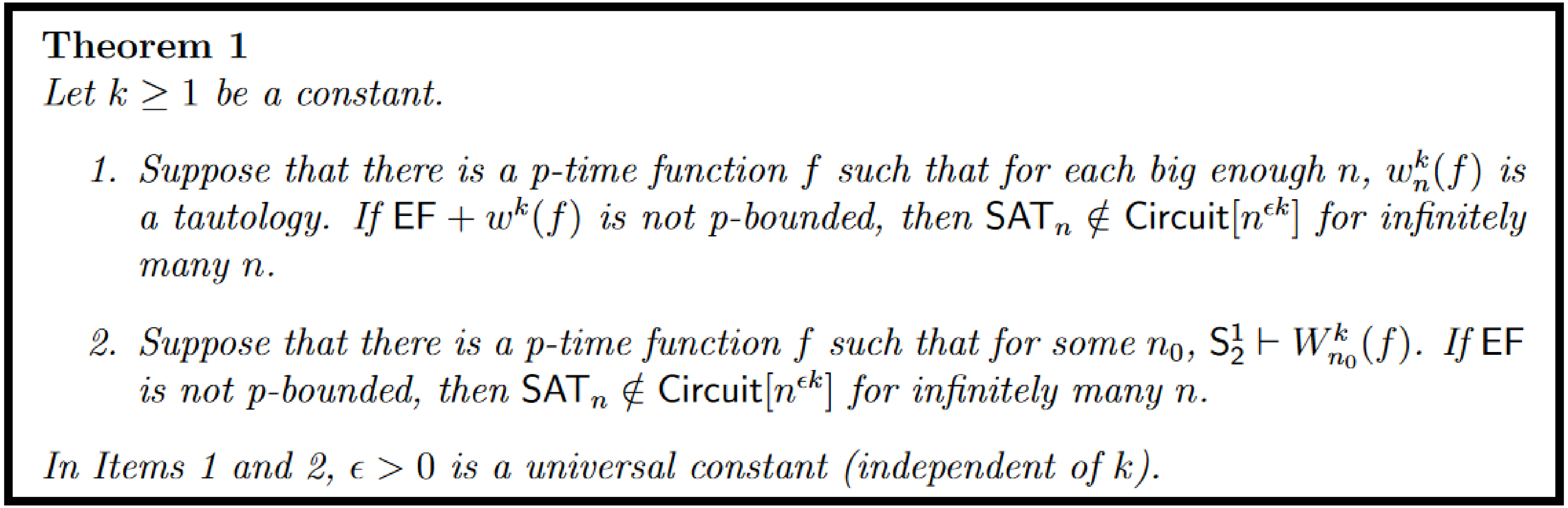

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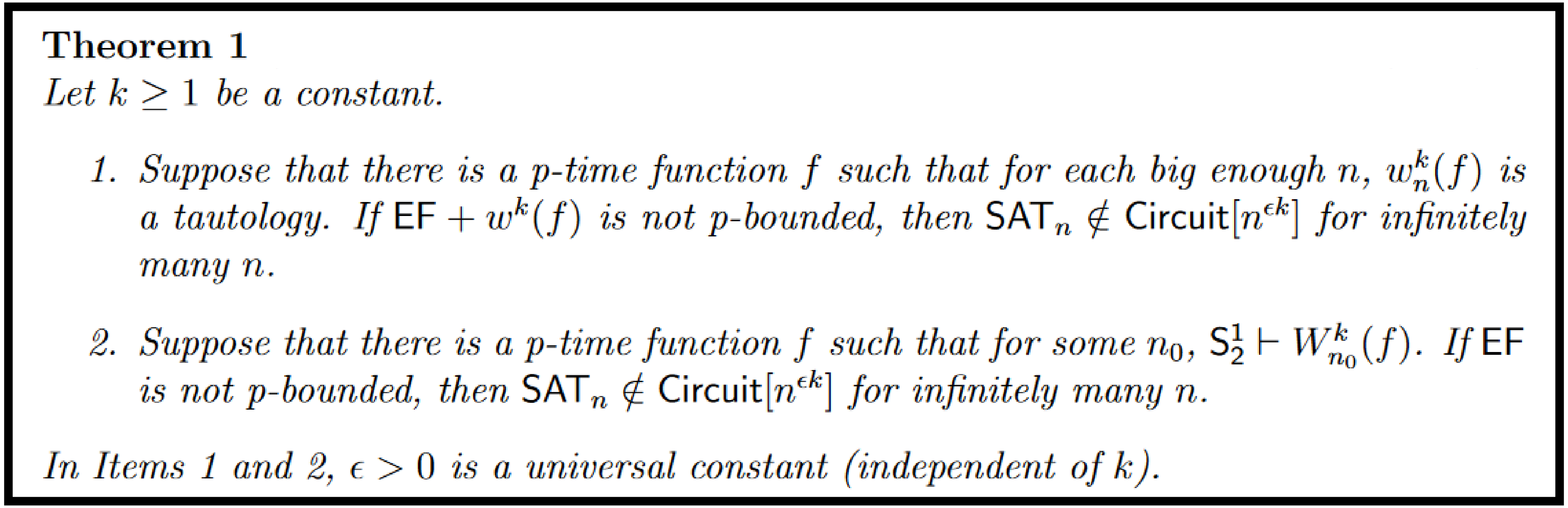
For each p-time f some circuit looks like it solves SAT?

Circuit complexity ⇐ proof complexity & witnessing of P≠NP



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In Items 1 and 2, $\epsilon > 0$ is a universal constant (independent of k).

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$$\forall k \exists f, w^k_n(f) \in \mathsf{TAUT} \quad \Rightarrow \quad \mathsf{NEXP} \not\subseteq \mathsf{P/poly}$$

Nonuniform witnessing

$$\alpha_n^s := \big(\mathsf{SAT}_n(x,y) \to \mathsf{SAT}_n(x,B(x))\big) \vee \Big(\bigvee_{z\in A} C(z) \neq \mathsf{SAT}_n(z)\Big)$$

Nonuniform witnessing

fixed p-size circuit

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fixed p-size set

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fixed p-size set

$\exists\ poly(s)\text{-size } A \qquad \mathsf{SAT}_n \notin \mathsf{Circuit}[s^3] \quad \Rightarrow \quad \forall s\text{-size } C, \bigvee_{x \in A} C(x) \neq \mathsf{SAT}_n(x)$

anti-checkers

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fixed p-size set

$\exists\ s^3$-size B' $\quad \mathsf{SAT}_n \in \mathsf{Circuit}[s^3] \Leftrightarrow \forall x \in \{0,1\}^n, B'(x) = \mathsf{SAT}_n(x)$

$\exists\ poly(s)$-size A $\quad \mathsf{SAT}_n \notin \mathsf{Circuit}[s^3] \quad \Rightarrow \quad \forall s\text{-size } C, \bigvee_{x \in A} C(x) \neq \mathsf{SAT}_n(x)$

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Theorem 2 (Circuit complexity from nonuniform proof complexity)**.**
Let $k \geq 3$ be a constant. If there are tautologies without p-size EF*-derivations from substitutional instances of tautologies $\alpha_n^{n^k}$, then* $\mathsf{SAT}_n \notin \mathsf{Circuit}[n^k]$ *for infinitely many n.*

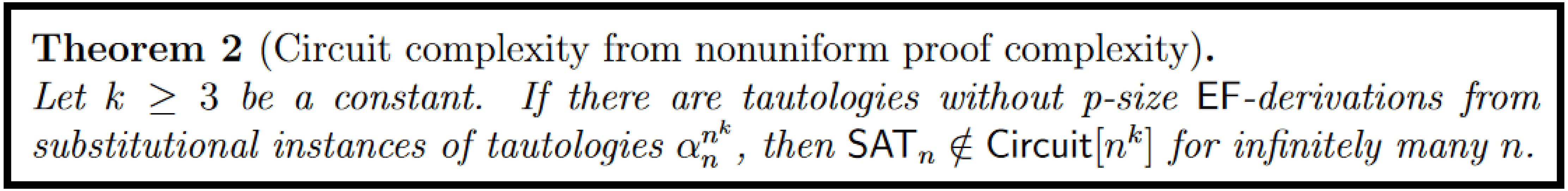
Nonuniform witnessing

fixed p-size circuit

$$\alpha_n^s := \left(\mathsf{SAT}_n(x,y) \to \mathsf{SAT}_n(x, B(x))\right) \vee \Big(\bigvee_{z \in A} C(z) \neq \mathsf{SAT}_n(z) \Big)$$

fixed p-size set

Open problem: Feasible MinMax?



Theorem 2 (Circuit complexity from nonuniform proof complexity)**.** *Let $k \geq 3$ be a constant. If there are tautologies without p-size* EF*-derivations from substitutional instances of tautologies $\alpha_n^{n^k}$, then $\mathsf{SAT}_n \notin \mathsf{Circuit}[n^k]$ for infinitely many n.*

Collapsing Impagliazzo’s worlds

OWF ⇐ P≠NP

Proof complexity collapse from "OWF $\Leftarrow$ P$\neq$NP" & hardness of E

Theorem

$S^1_2 \vdash$ **E** hard on average for subexponential-size circuits

&

$S^1_2 \vdash$ OWF $\Leftarrow$ P$\neq$NP

$\Longrightarrow$

EF not p-bounded $\Rightarrow$ P$\neq$NP

Proof complexity collapse from “OWF $\Leftarrow$ P$\neq$NP” & hardness of E

Theorem

$S^1_2 \vdash$ **E** hard on average for subexponential-size circuits

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$S^1_2 \vdash$ OWF $\Leftarrow$ P$\neq$NP

$$\Longrightarrow$$

EF not p-bounded $\Rightarrow$ P$\neq$NP

- No need for the **provability** of “E is hard” if EF replaced by EF+“**E** is hard”

Proof complexity collapse from “OWF $\Leftarrow$ P$\neq$NP” & hardness of E

Theorem

$S^1_2 \vdash$ **E** hard on average for subexponential-size circuits

&

$S^1_2 \vdash$ OWF $\Leftarrow$ P$\neq$NP

$$\Longrightarrow$$

EF not p-bounded $\Rightarrow$ P$\neq$NP

- No need for the **provability** of “E is hard” if EF replaced by EF+“**E** is hard”
- Generalizes to stronger systems, e.g. **ZFC**

Proof complexity collapse from “OWF ⇐ P≠NP” & hardness of E

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EF not p-bounded ⇒ P≠NP

- No need for the **provability** of “E is hard” if EF replaced by EF+“**E** is hard”
- Generalizes to stronger systems, e.g. **ZFC**
- Requires **p-time reductions** witnessing that OWF ⇐ P≠NP

Proof

$$h \text{ is one-way} \Rightarrow \text{“}h(x) = h(\overset{\text{random}}{\overset{\Downarrow}{a}})\text{”} \text{ is a hard } \mathsf{SAT}\text{-instance}$$

E hard on average for subexponential-size circuits

$$\Longrightarrow$$

$$\exists \text{ p-time } f \text{ s.t. } \forall C \in \mathsf{Circuit}[n^k]$$

$$\mathsf{SAT}_n(f_1(C), f_2(C)) \wedge \neg \mathsf{SAT}_n(f_1(C), C(f_1(C)))$$

Proof

$\mathsf{S}^1_2 \vdash$

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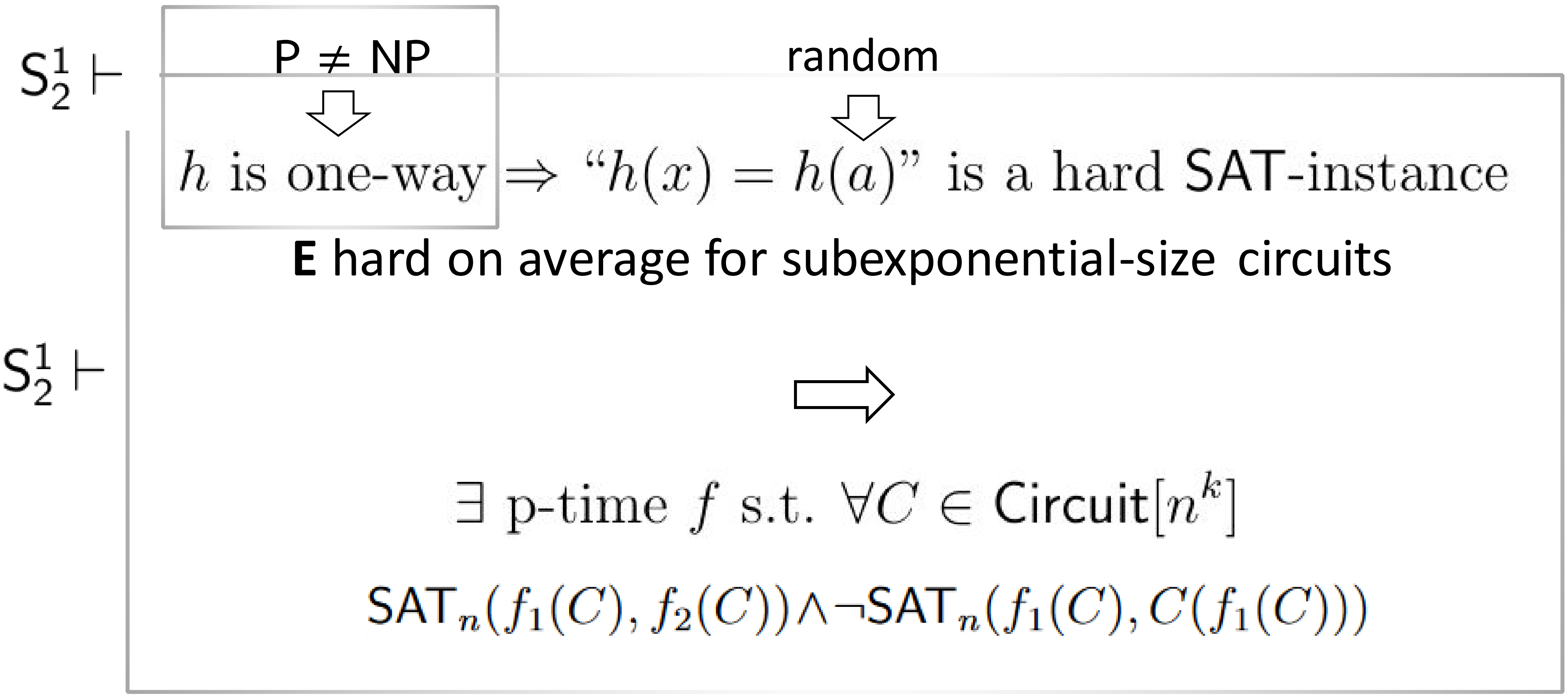

$\mathsf{S}^1_2 \vdash$

P ≠ NP

random

h is one-way $\Rightarrow$ "$h(x) = h(a)$" is a hard SAT-instance

E hard on average for subexponential-size circuits

$\mathsf{S}^1_2 \vdash$

$\Longrightarrow$

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Proof

$\mathsf{S}^1_2 \vdash$ P ≠ NP random

⇓ ⇓

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E hard on average for subexponential-size circuits

$\mathsf{S}^1_2 \vdash$

$\Longrightarrow$

$\mathsf{S}^1_2 \vdash$

P = NP or

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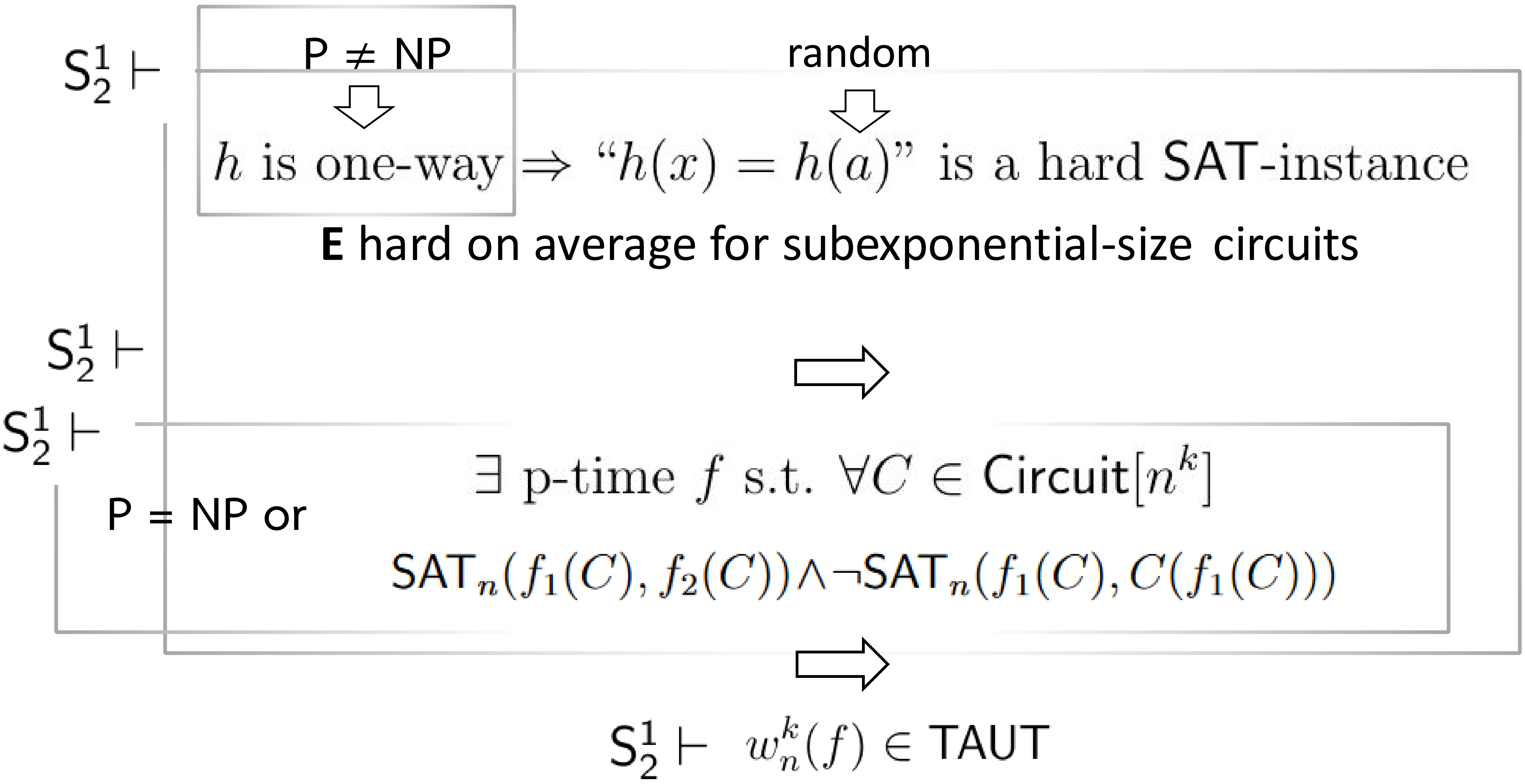

$\mathsf{S}^1_2 \vdash$

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$\Longrightarrow$

$\mathsf{S}^1_2 \vdash \ w^k_n(f) \in \mathsf{TAUT}$

Learning or Crypto

Theorem

$S^1_2 \vdash$ **E** hard on average for subexponential-size circuits

&

$S^1_2 \vdash$ OWF $\Leftarrow$ P$\neq$NP

$$\Longrightarrow$$

EF not p-bounded $\Rightarrow$ P$\neq$NP

- Can replace "OWF $\Leftarrow$ P$\neq$NP" by "**Learning or Crypto**"
  if EF lower bounds replaced by EF lower bounds for tautologies
  expressing circuit lower bounds

Learning or Crypto

Theorem

$S^1_2 \vdash$ **E** hard on average for subexponential-size circuits

&

$S^1_2 \vdash$ **OWF or Learning P/poly**

$$\Longrightarrow$$

EF $\nvdash$ circuit lower bound $\Rightarrow$ P$\neq$NP

- Can replace “OWF $\Leftarrow$ P$\neq$NP” by “**Learning or Crypto**”
  if EF lower bounds replaced by EF lower bounds for tautologies expressing circuit lower bounds

Automatability or OWF

Theorem

$S^1_2 \vdash$ **E** hard on average for subexponential-size circuits

&

$S^1_2 \vdash$ **OWF or EF automatable**

⇨

EF $\nvdash$ circuit lower bound $\Rightarrow$ P$\neq$NP

- Can replace “OWF $\Leftarrow$ P$\neq$NP” by “**Automatability or OWF**”
  if EF lower bounds replaced by EF lower bounds for tautologies expressing circuit lower bounds

Concluding remarks

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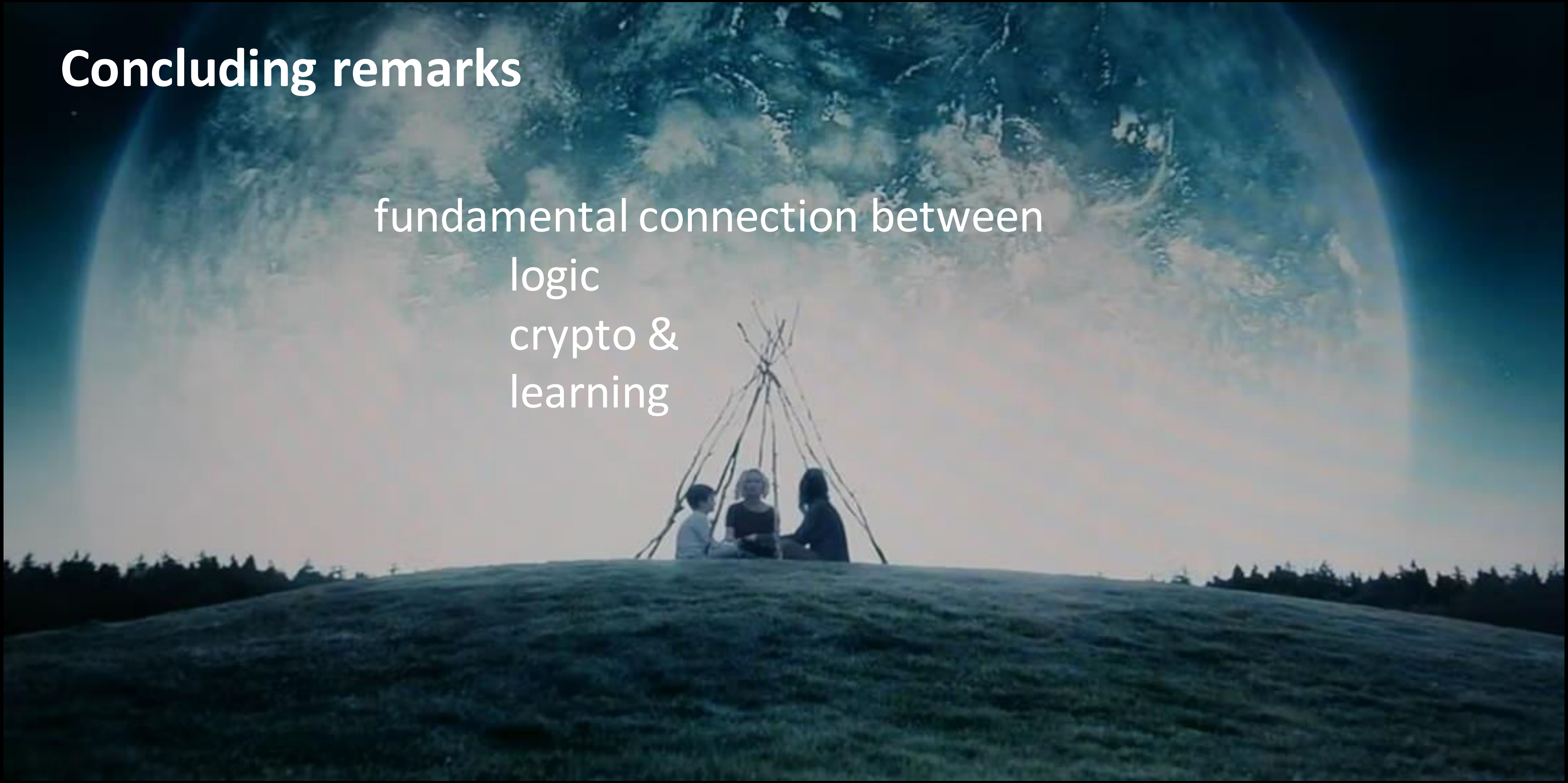
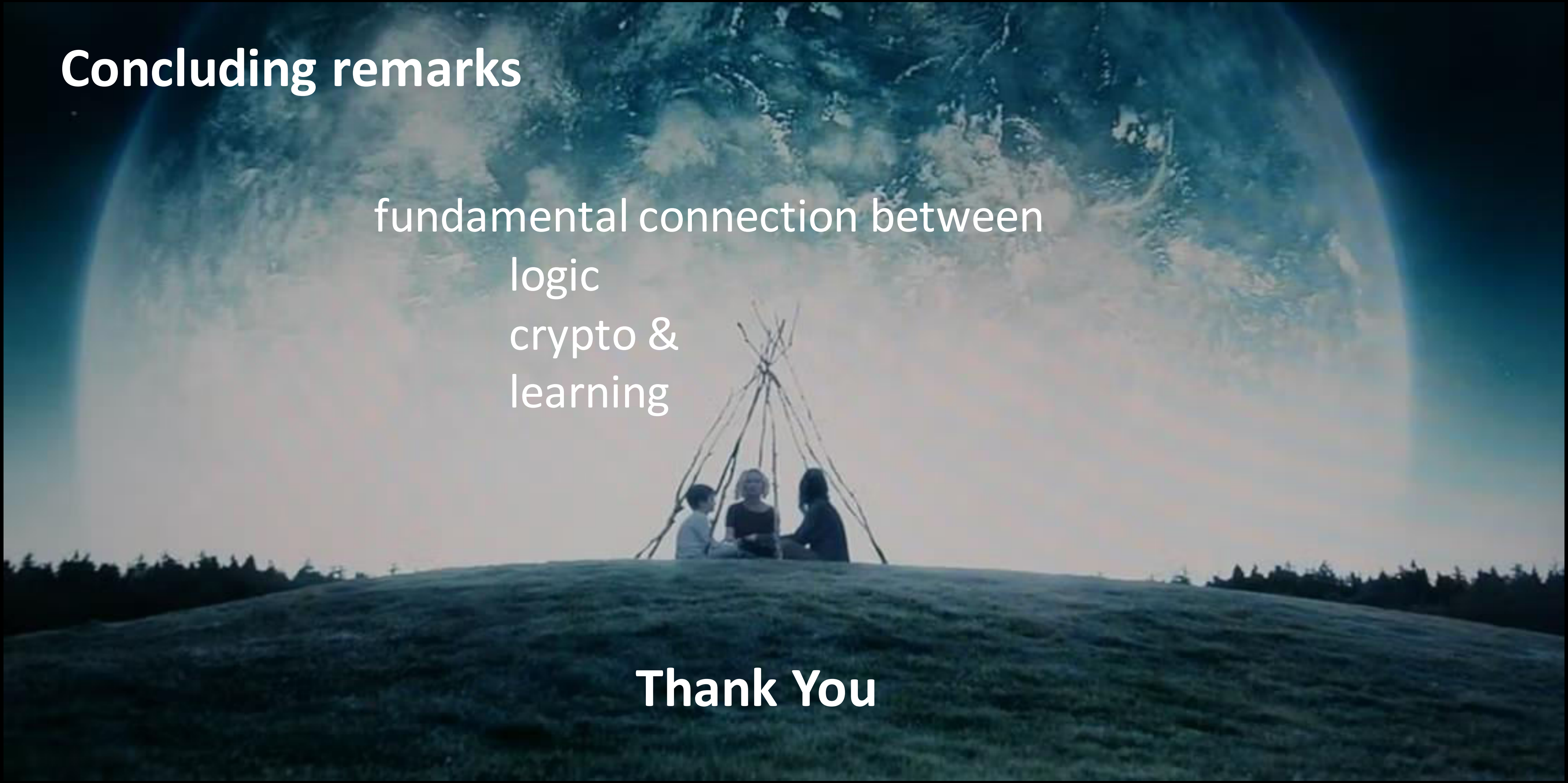


fundamental connection between

logic

crypto &

learning



Concluding remarks

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Thank You