TFNP

TBD

Proof Complexity, Circuit Complexity, and TFNP

Noah Fleming
Memorial University

Based on work with Sam Buss and Russell Impagliazzo

Monotone Circuit Complexity

Task

Object

Model

Monotone
Circuit Model M

Monotone Circuit Complexity

Task

Object

Model

Monotone Circuit Model M

Monotone Circuit Complexity

Task

Object

Model

Monotone Function f

Monotone Circuit Model M

Monotone Circuit Complexity

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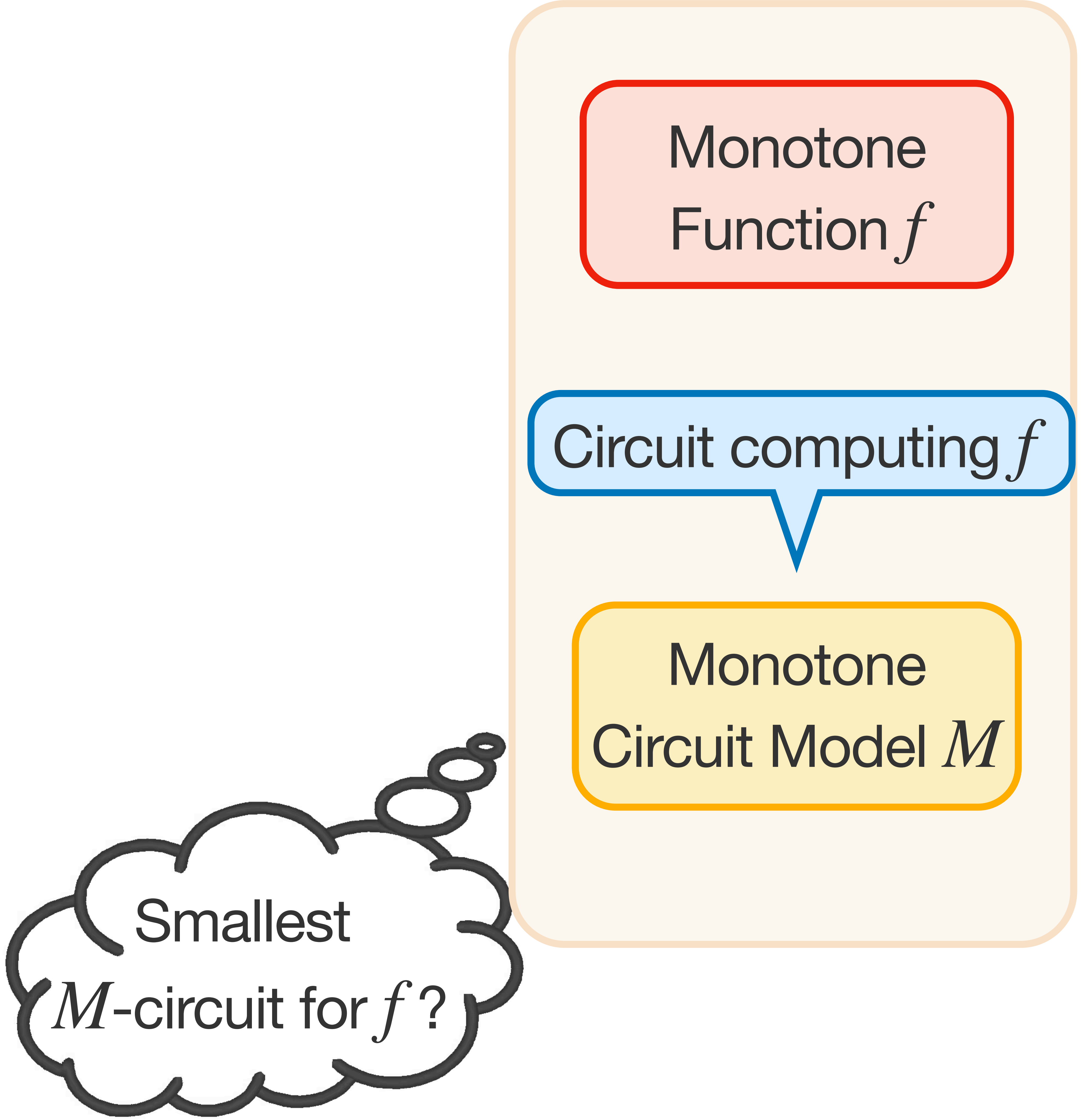
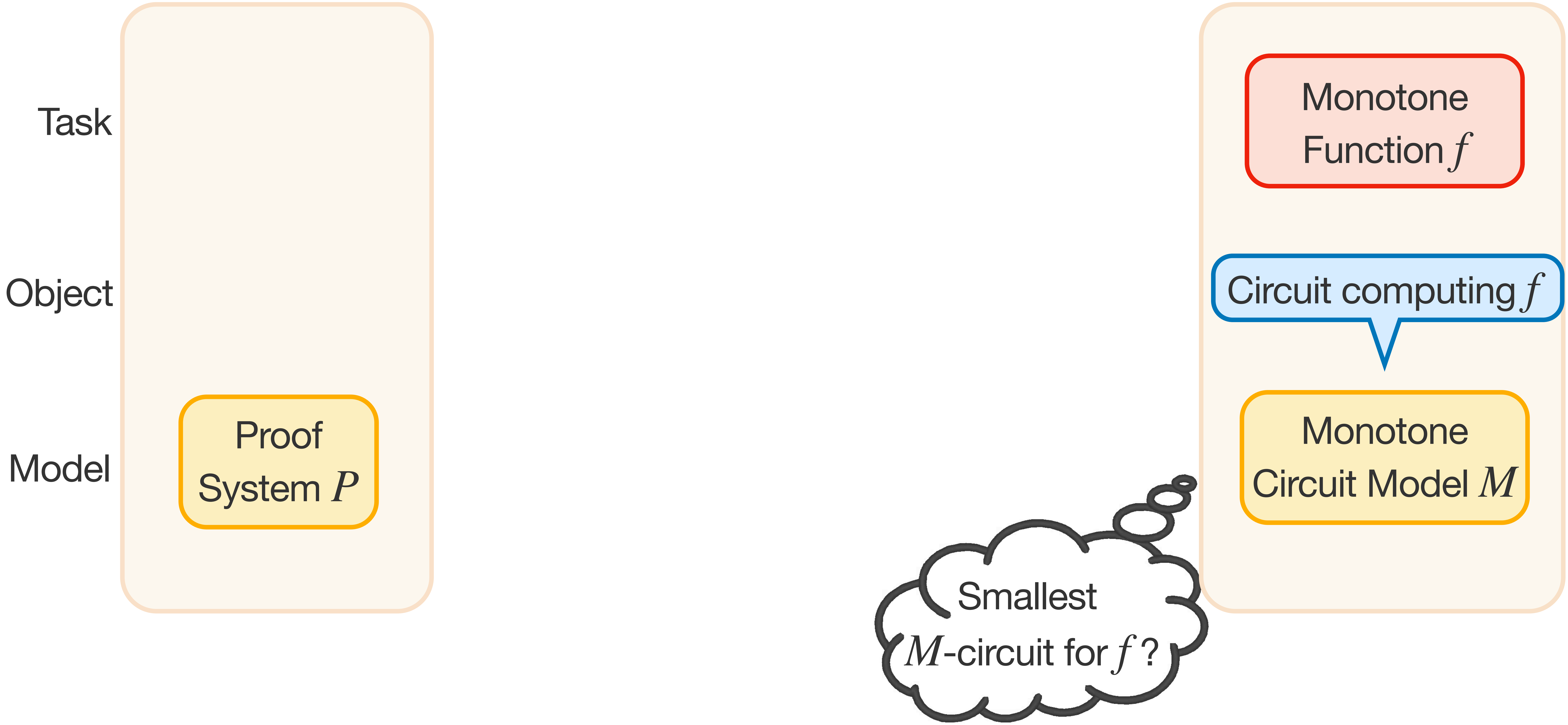


Proof Complexity



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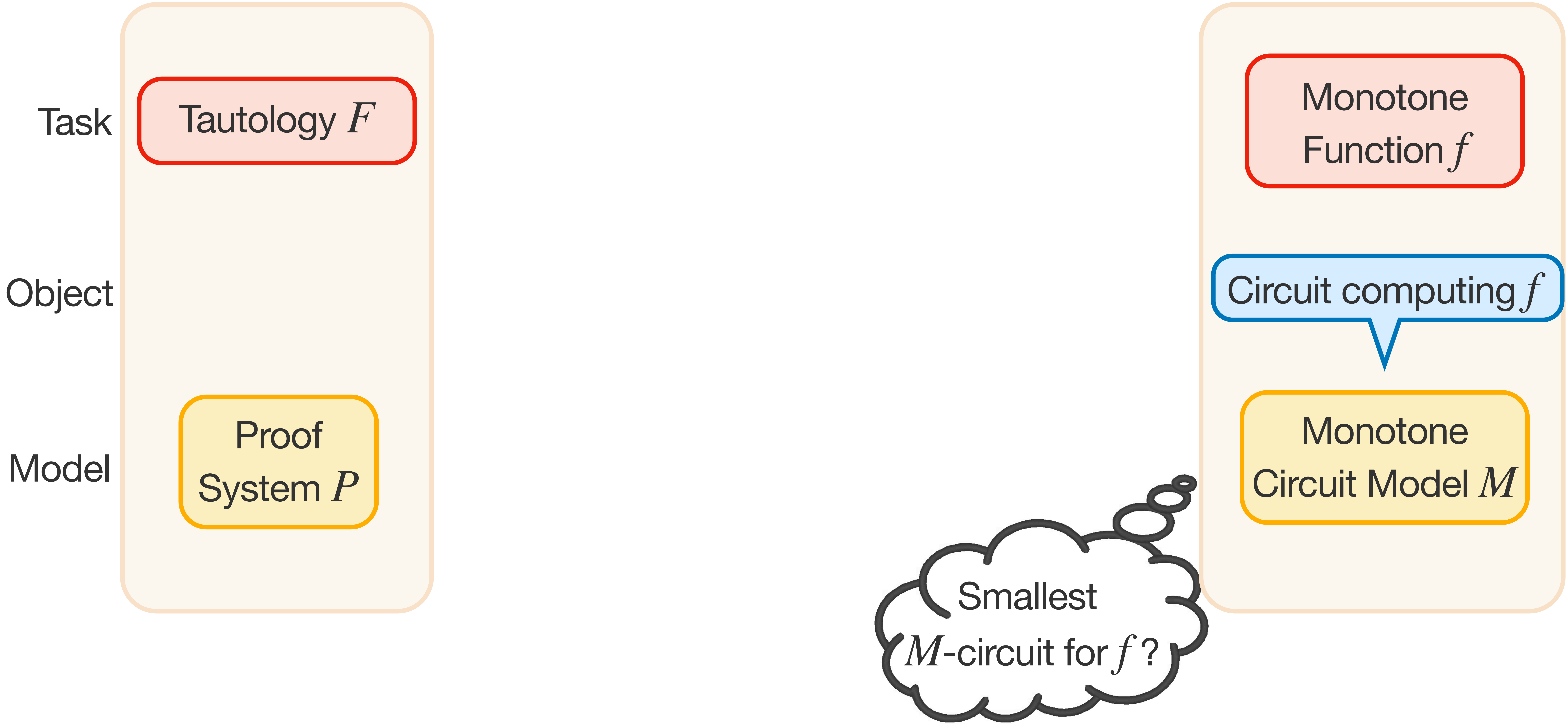

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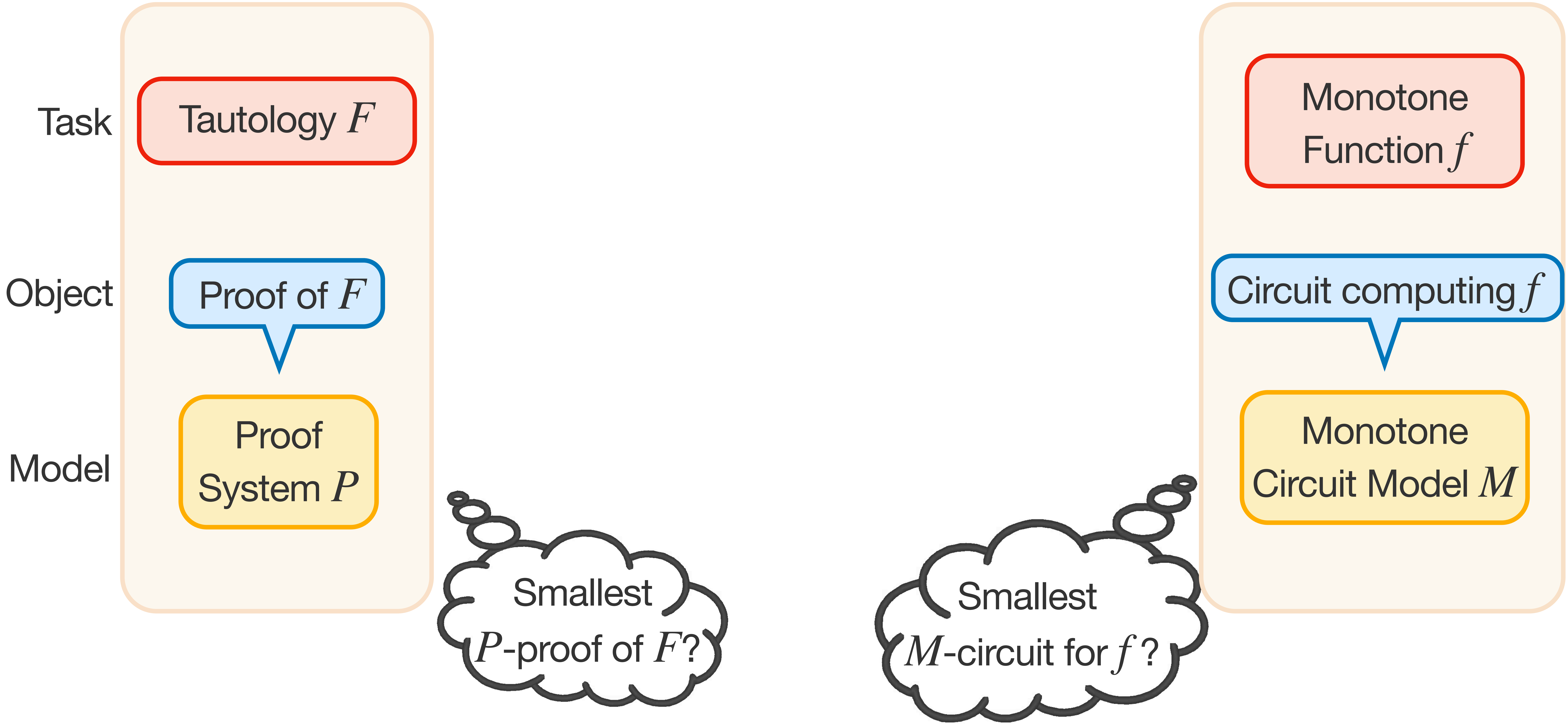

Interplay

Major breakthroughs resulted from uncovering deep connections between these areas!

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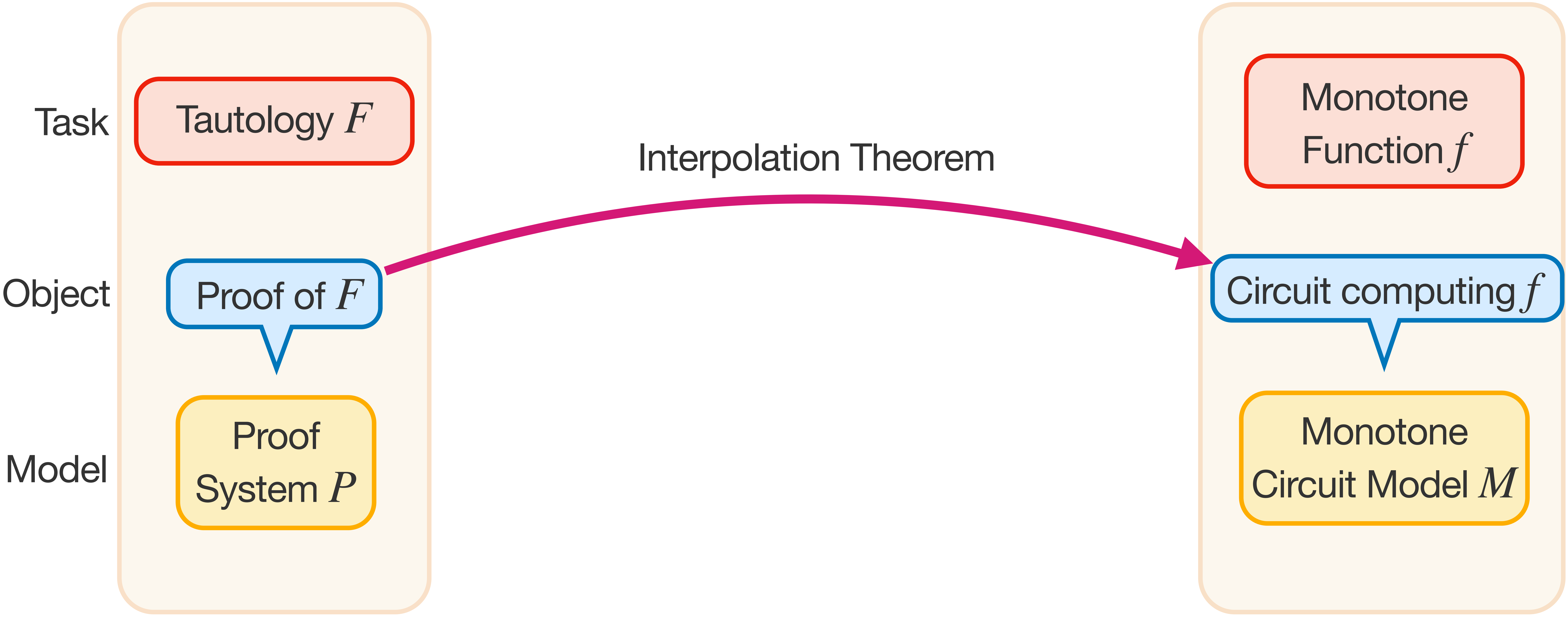

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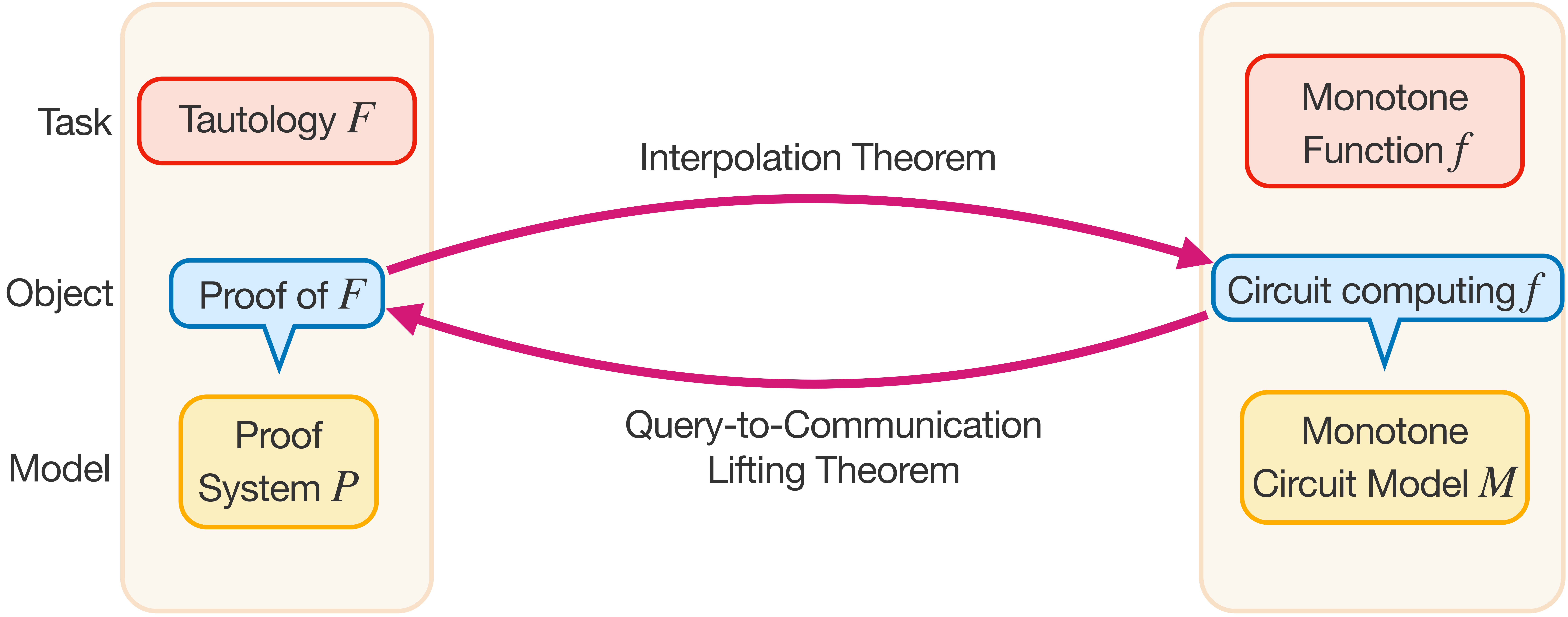

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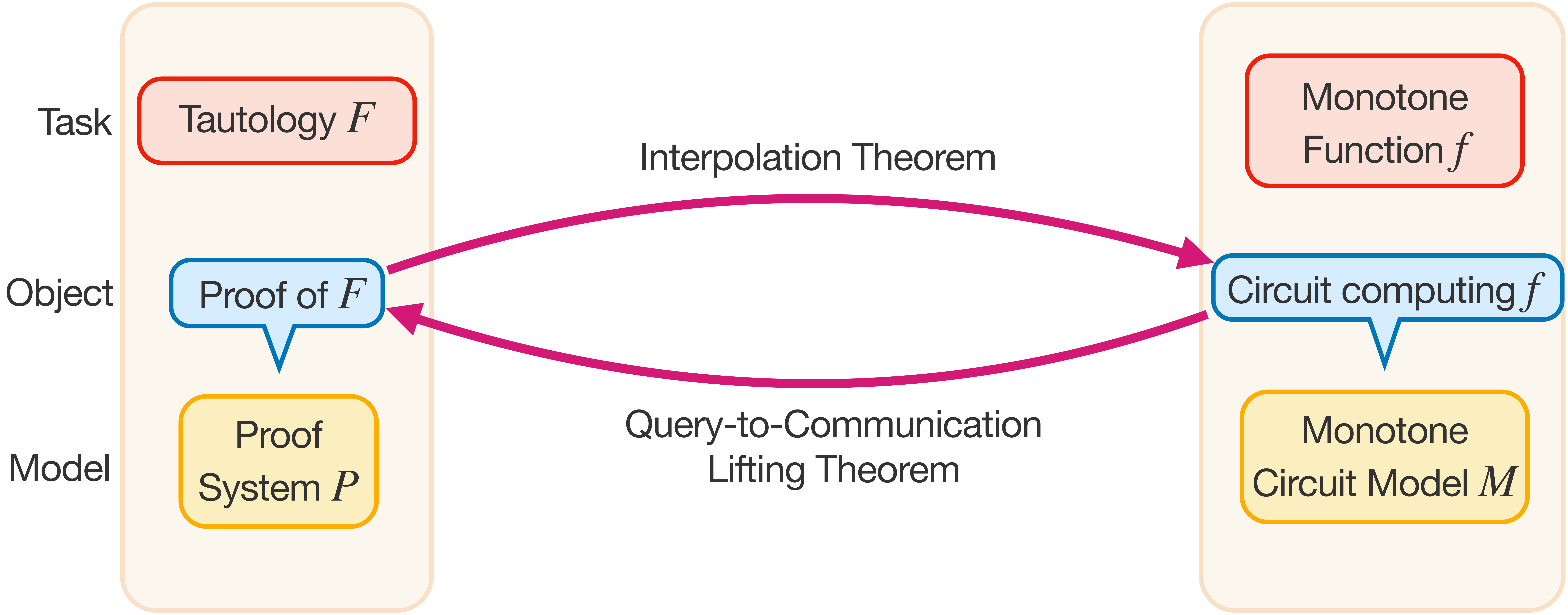

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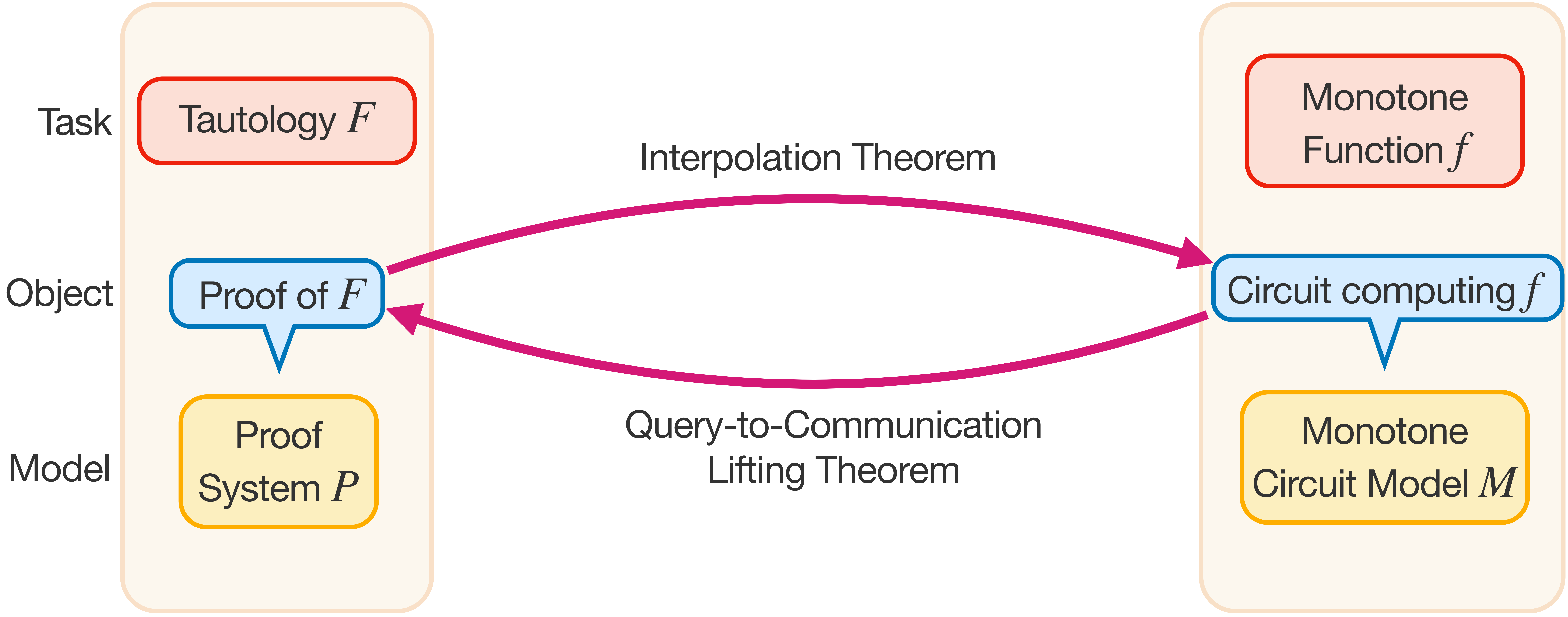
Upshot: Tools from one area can be applied to the other!

Interplay

Q. When and why do these connections occur?

TFNP has emerged as a roadmap for interpolation and lifting theorems

Characterizations by Total Search Problems



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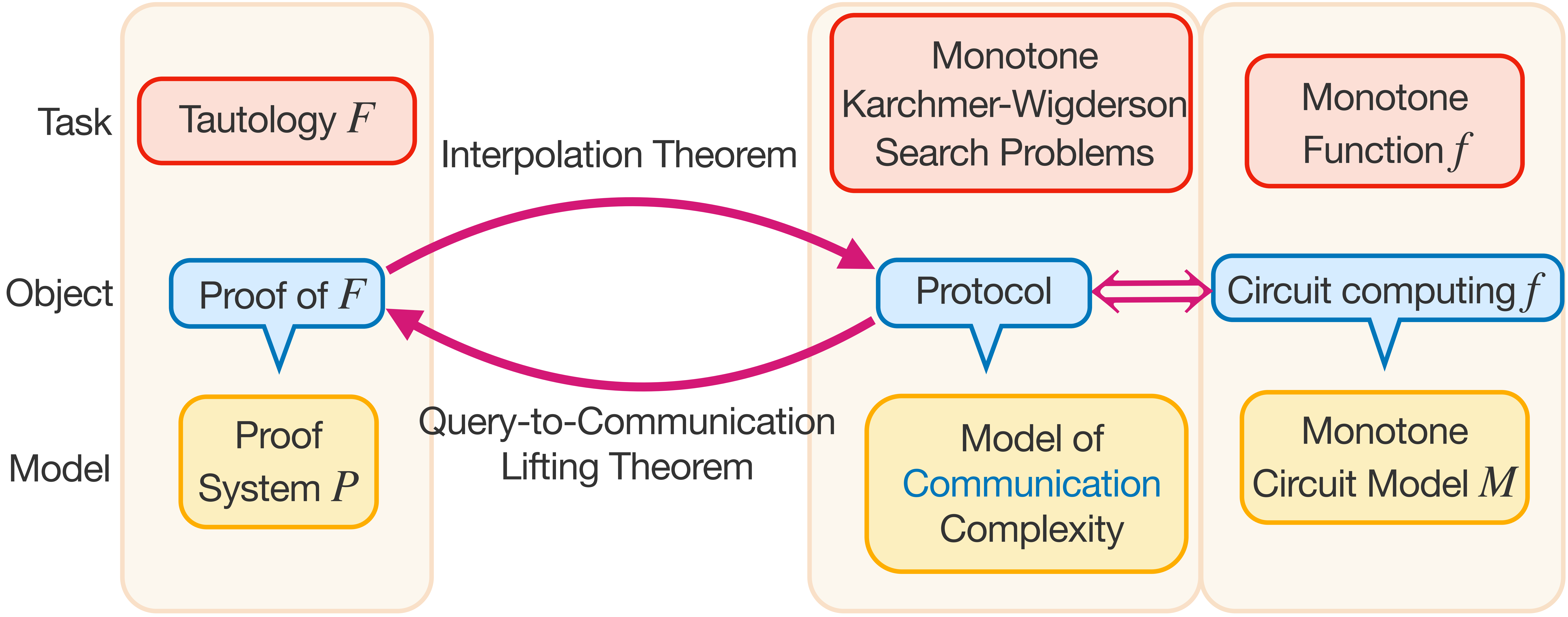
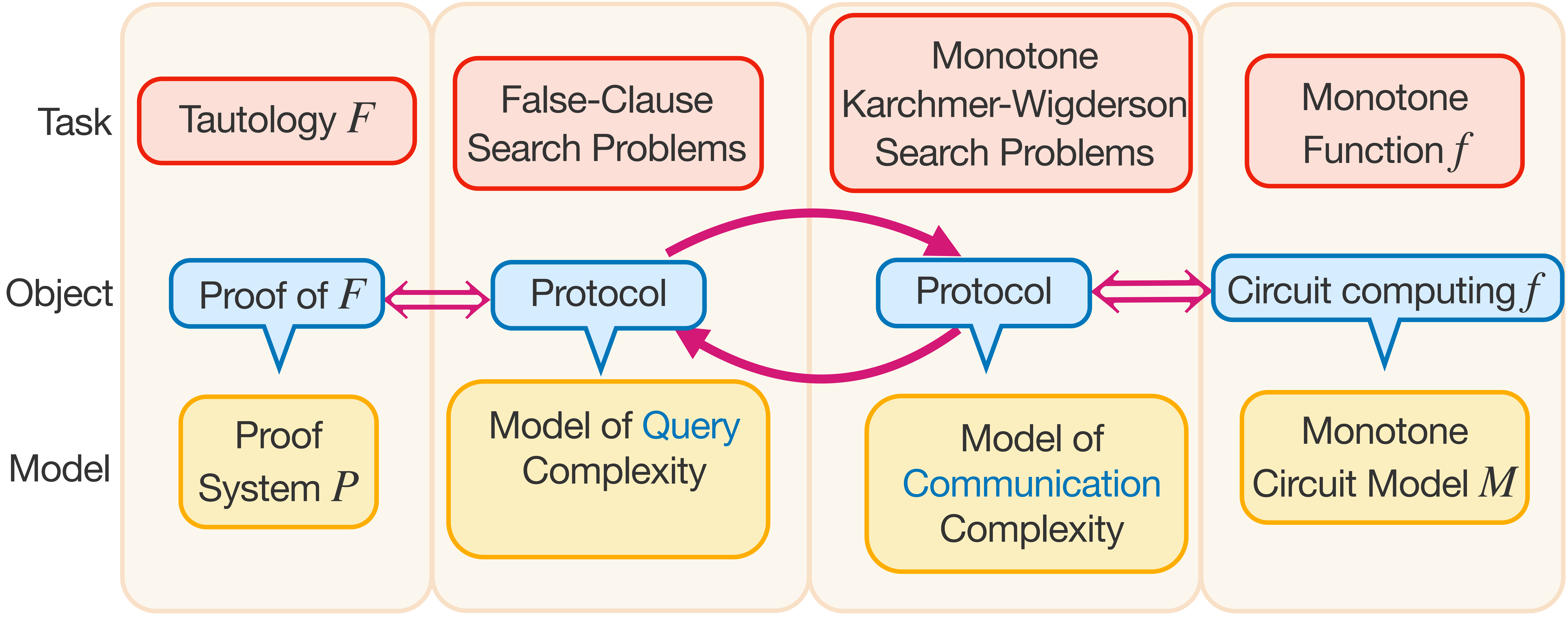


mKW_f: Given $(x, y) \in f^{-1}(1) \times f^{-1}(0)$ output $i \in [n]$ such that $x_i \neq y_i$

Characterizations by Total Search Problems



$Search_F$: Given $x \in \{0,1\}^n$ output the index of a clause of F falsified by x

TFNP

Studies the complexity of computing total search problems

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→Organizes them into a variety of classes with complete problems

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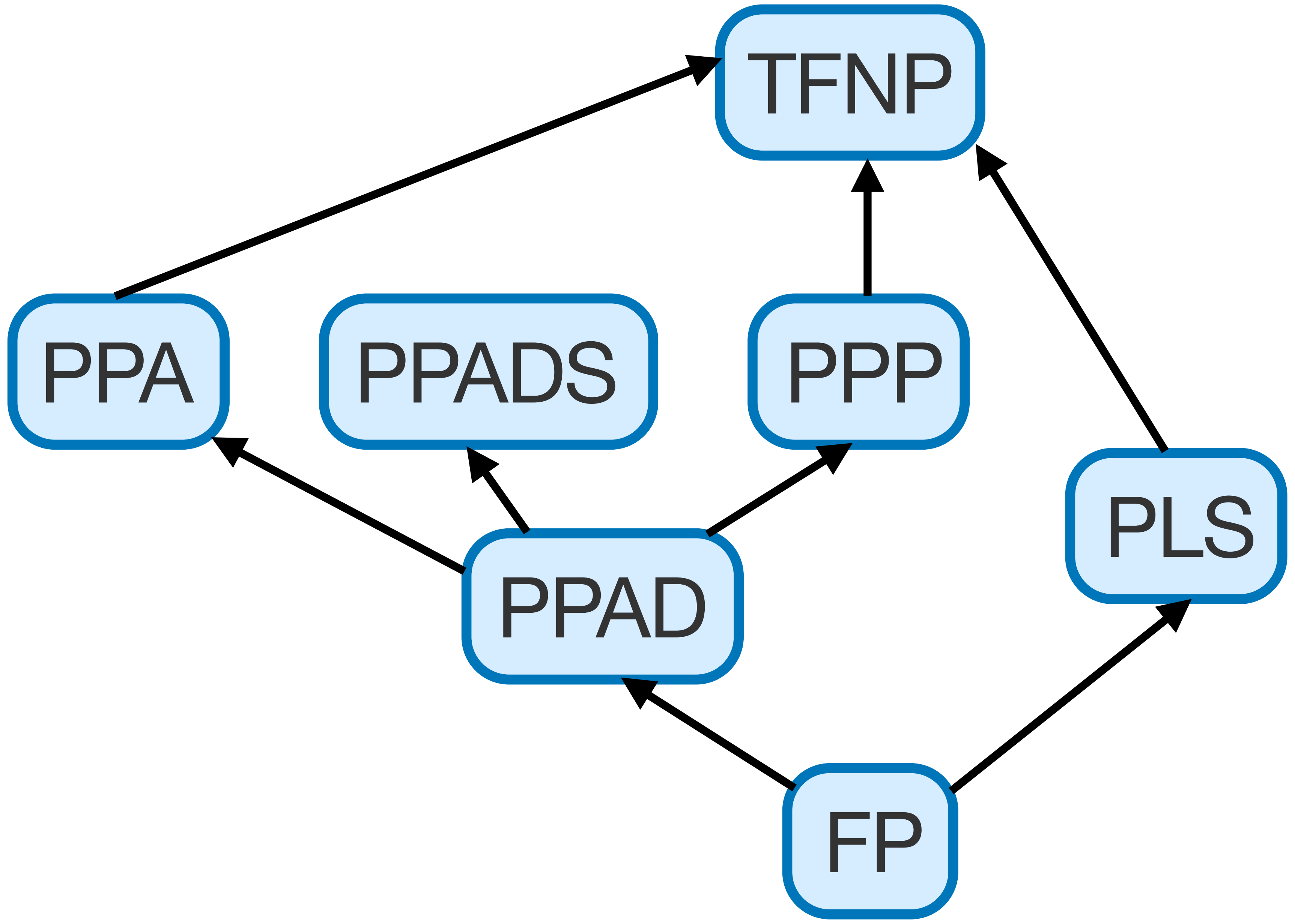

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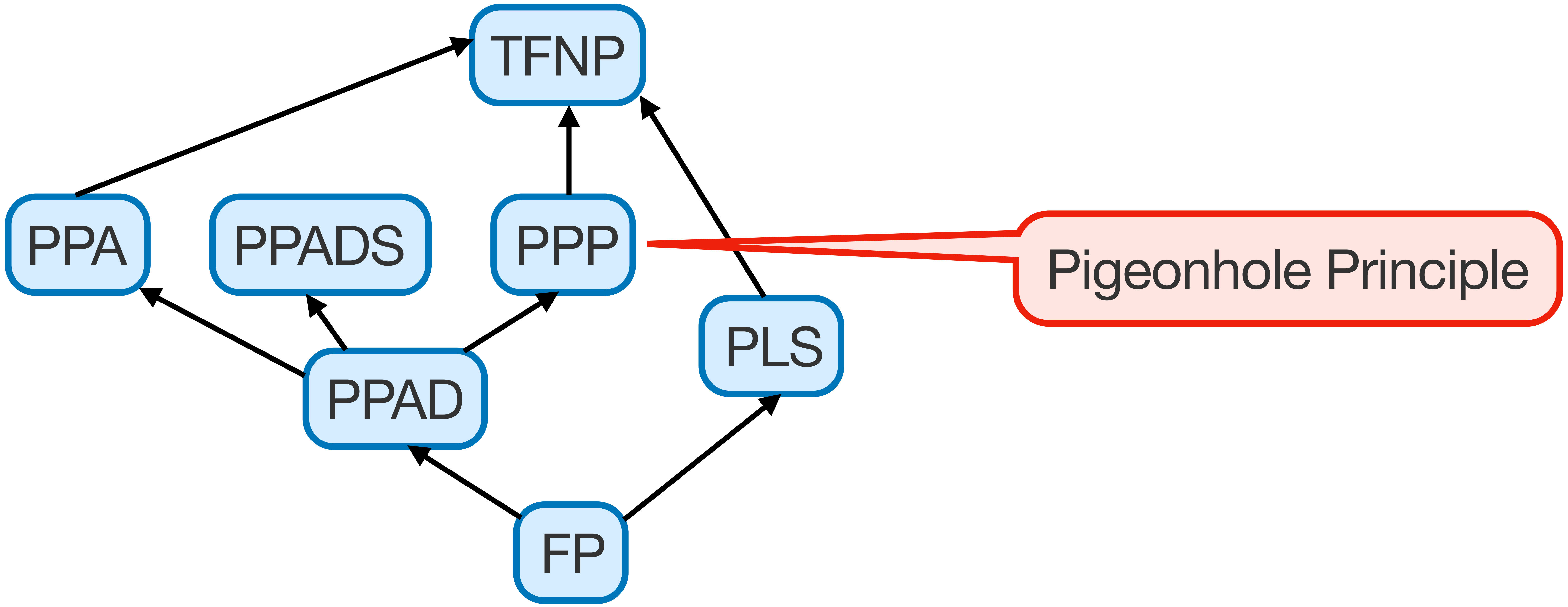

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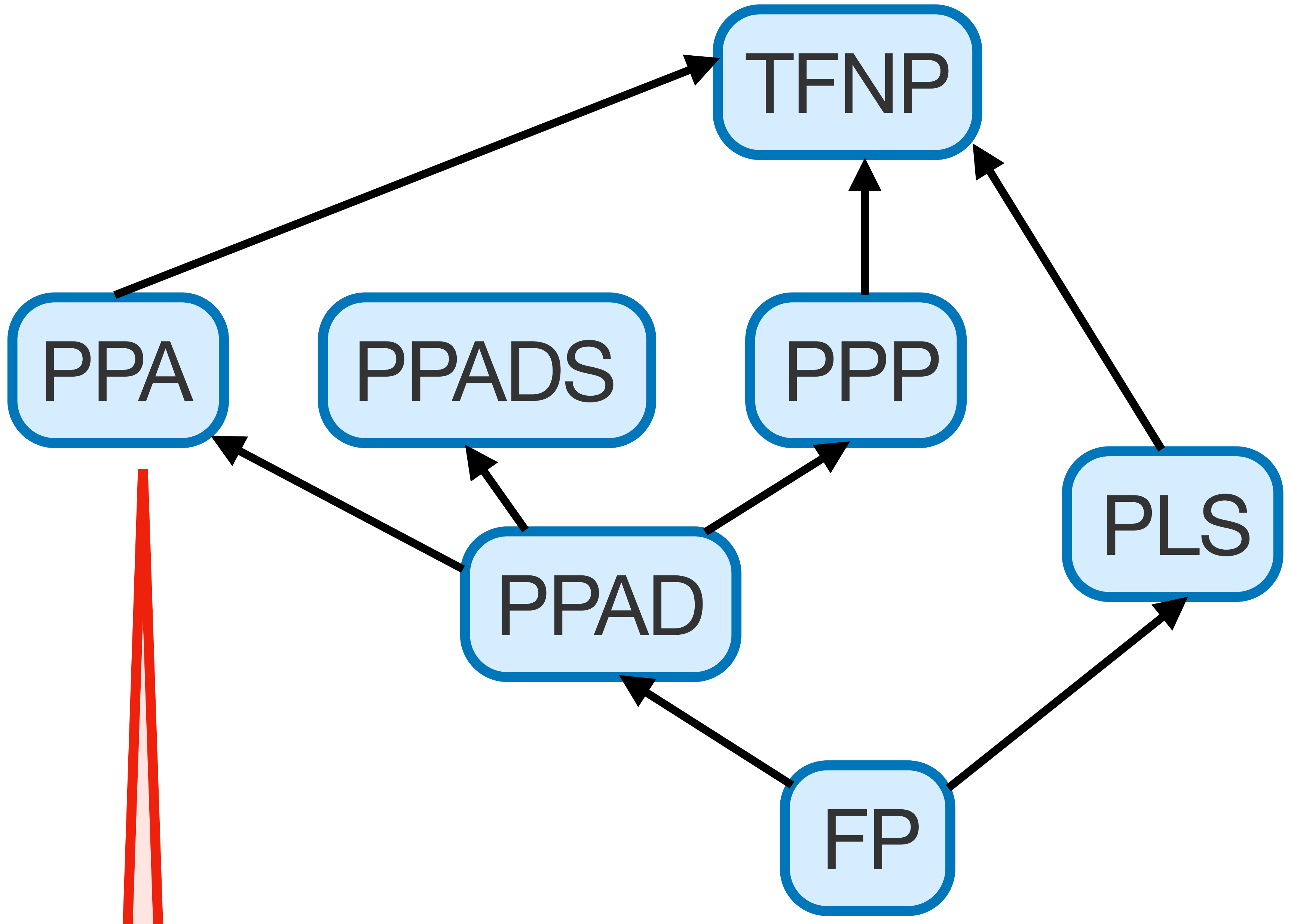

Every odd degree vertex has another

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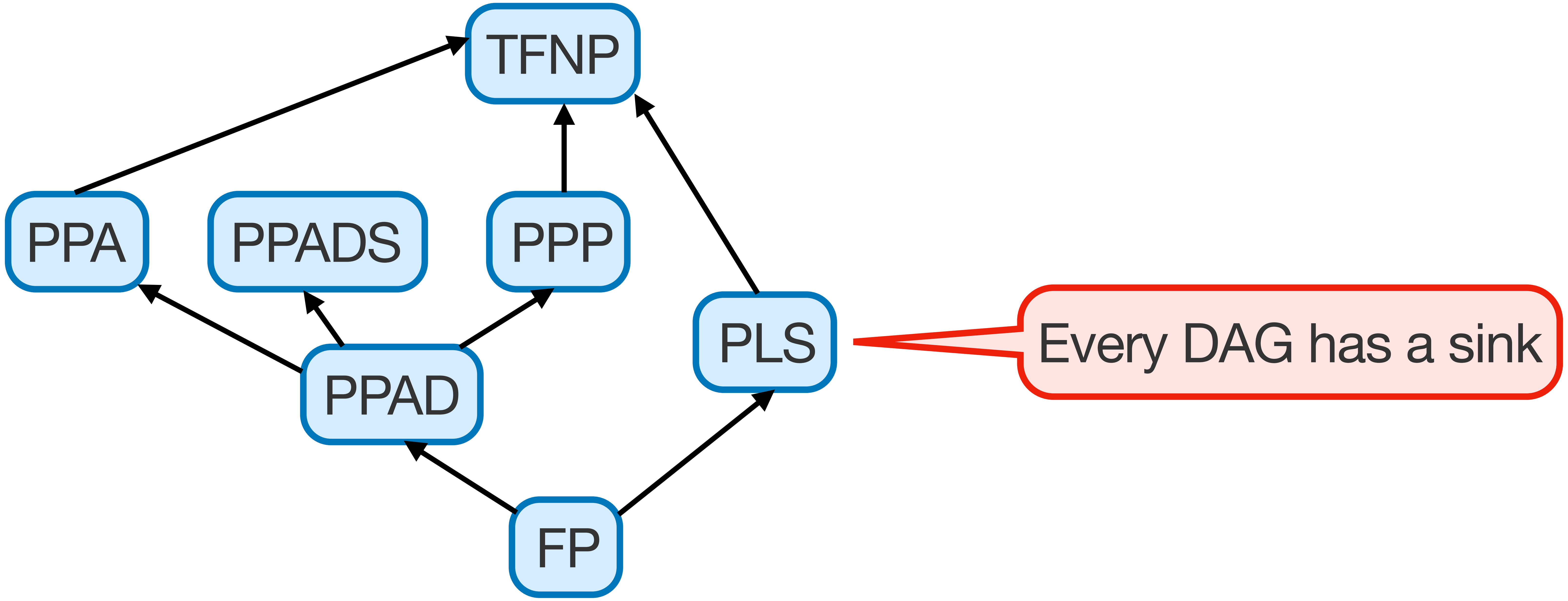

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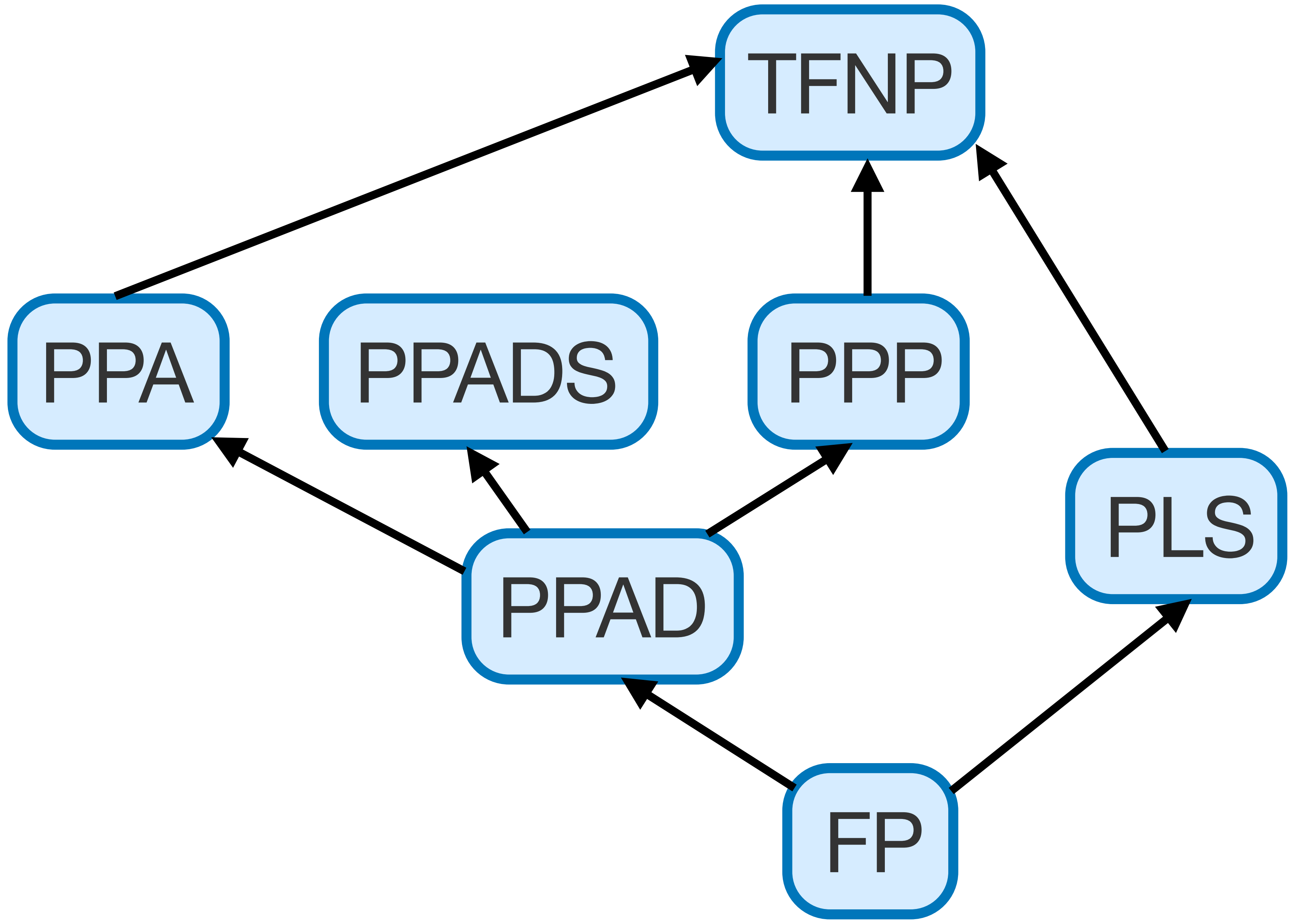

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SinkOfDag

Vertices: $1, \ldots, n$

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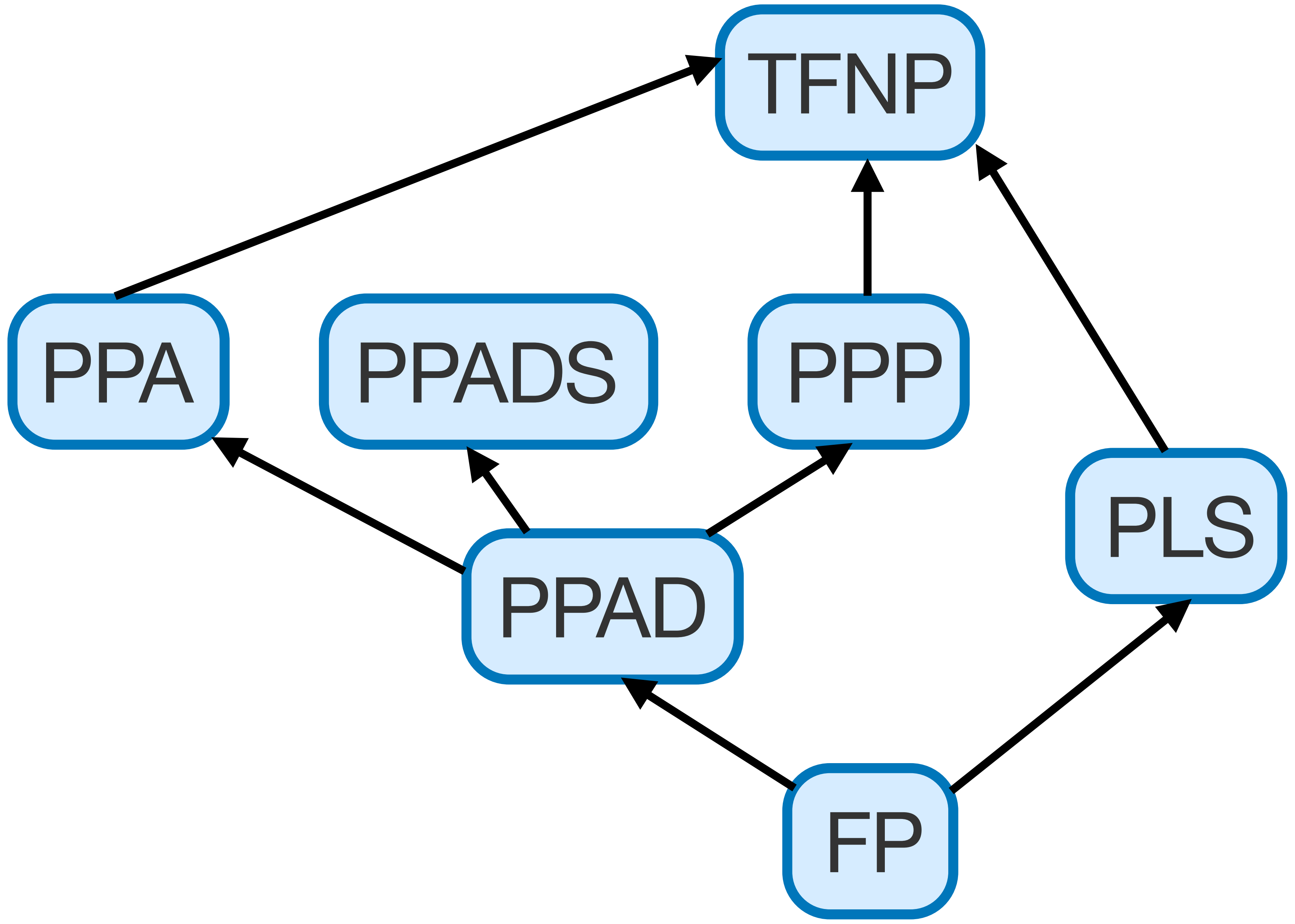

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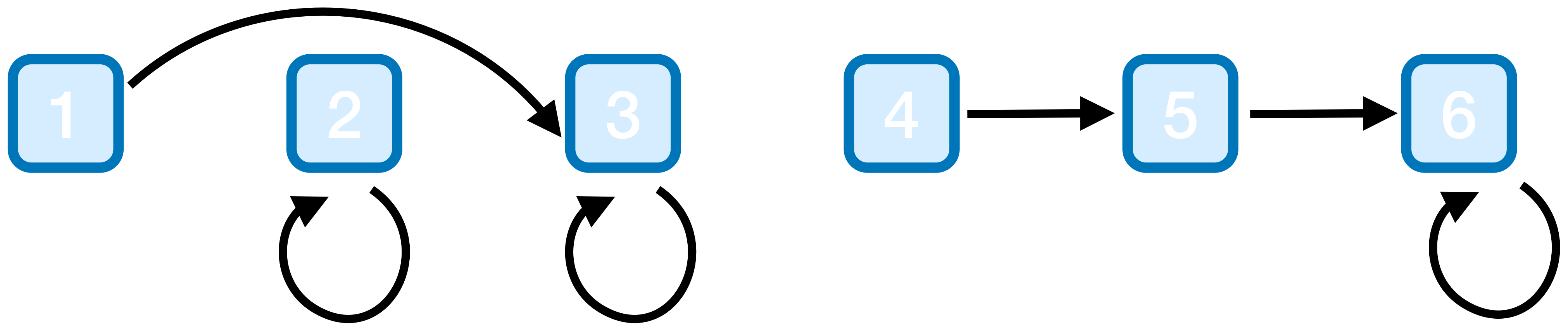

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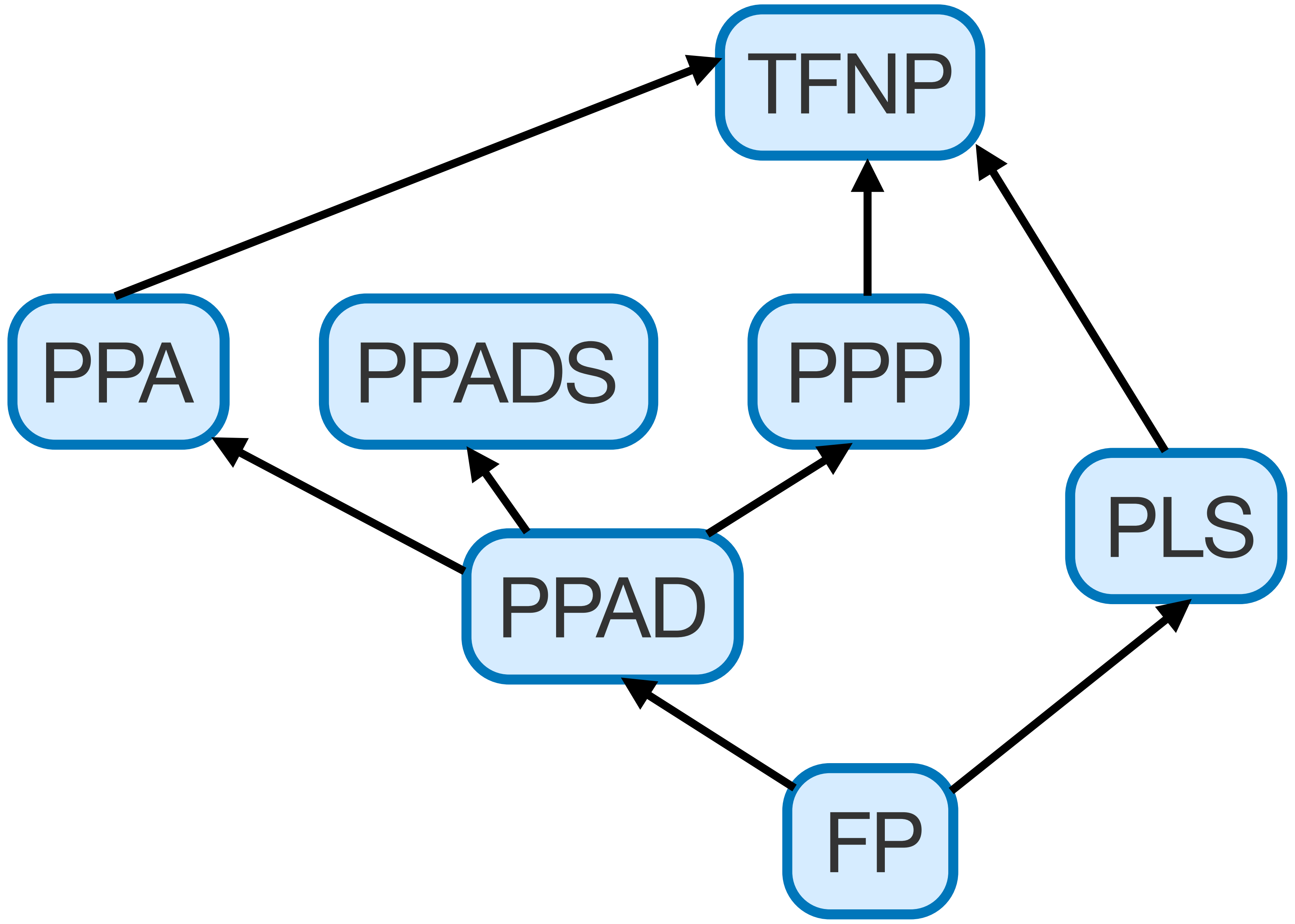

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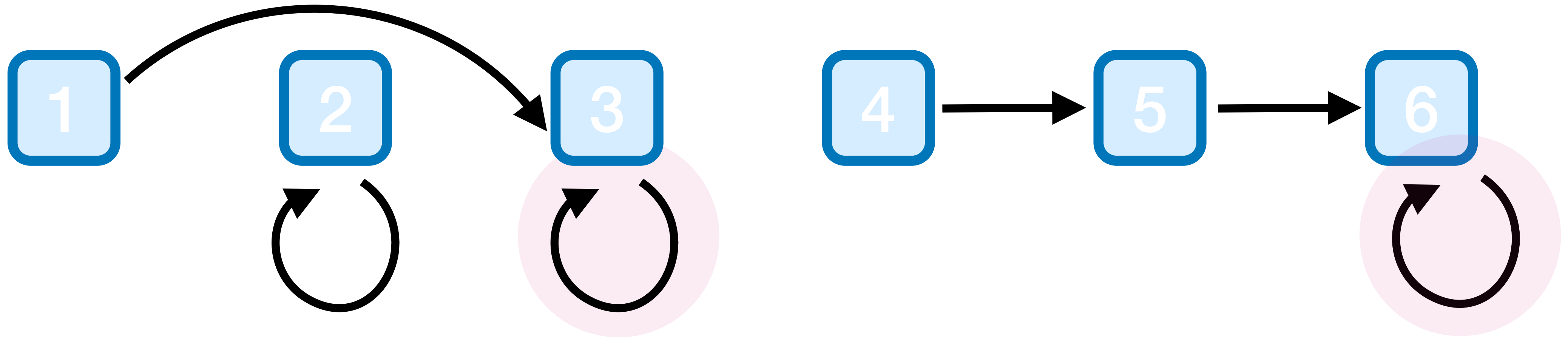

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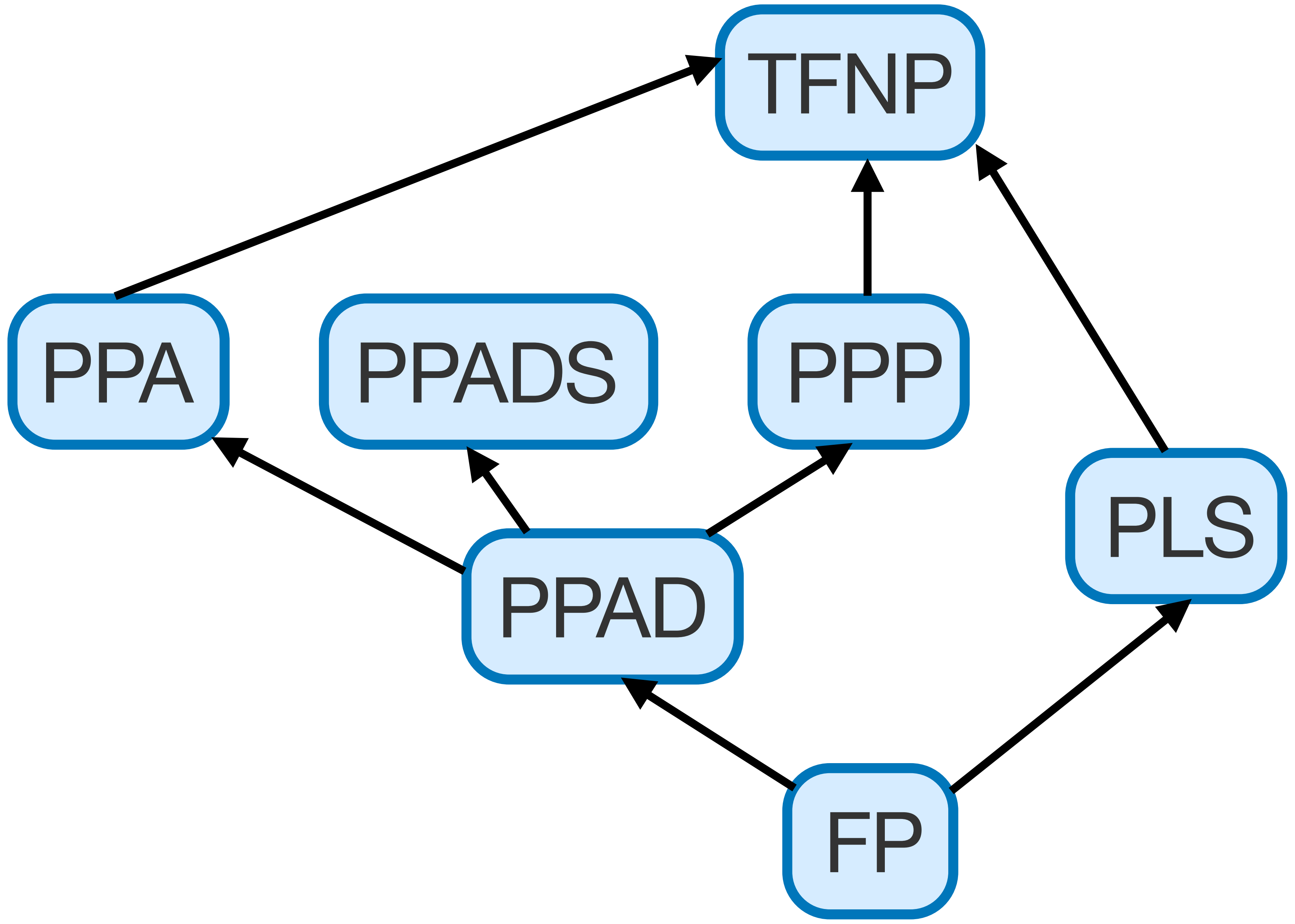
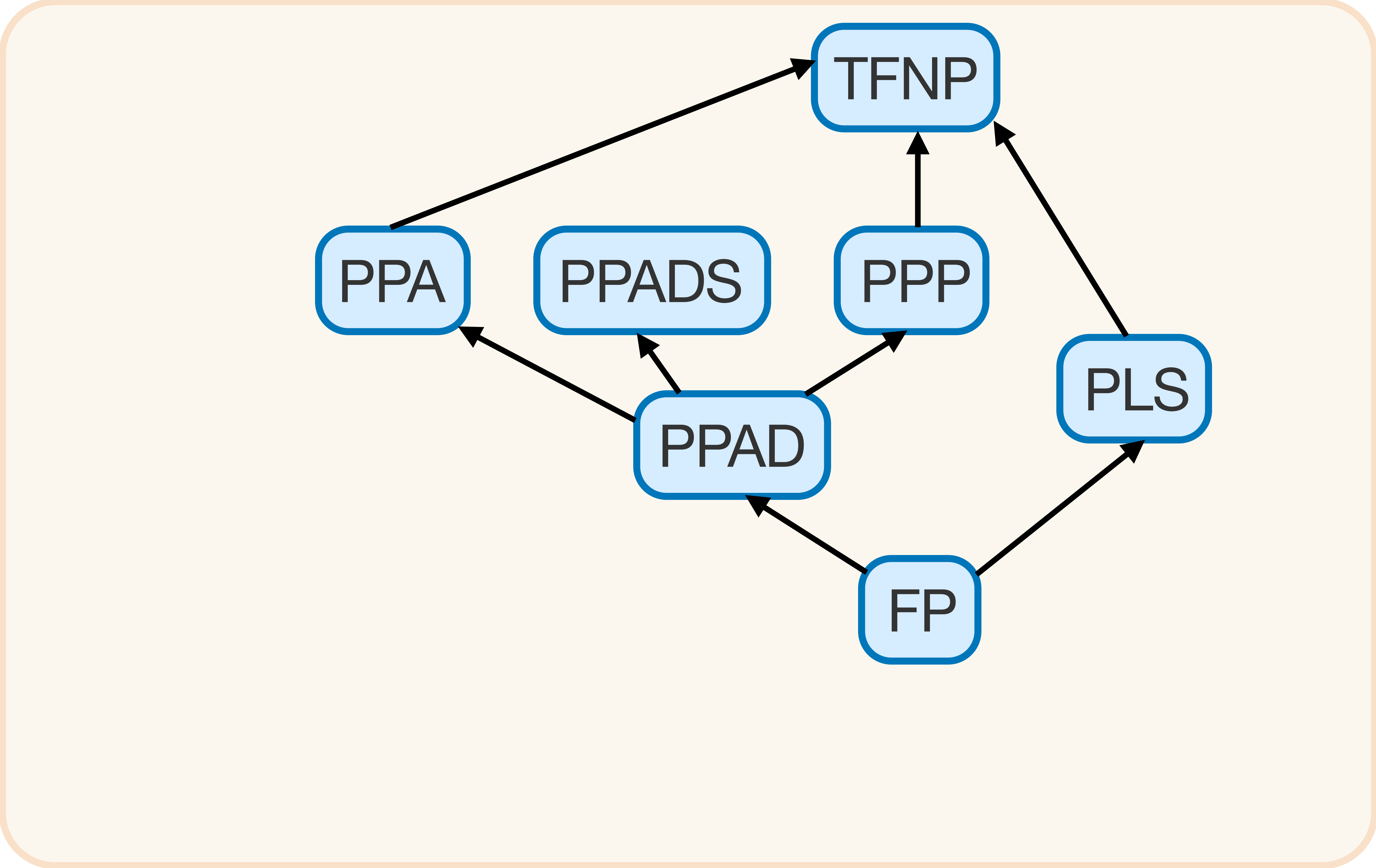


Typically study the Turing Machine complexity of total search problems

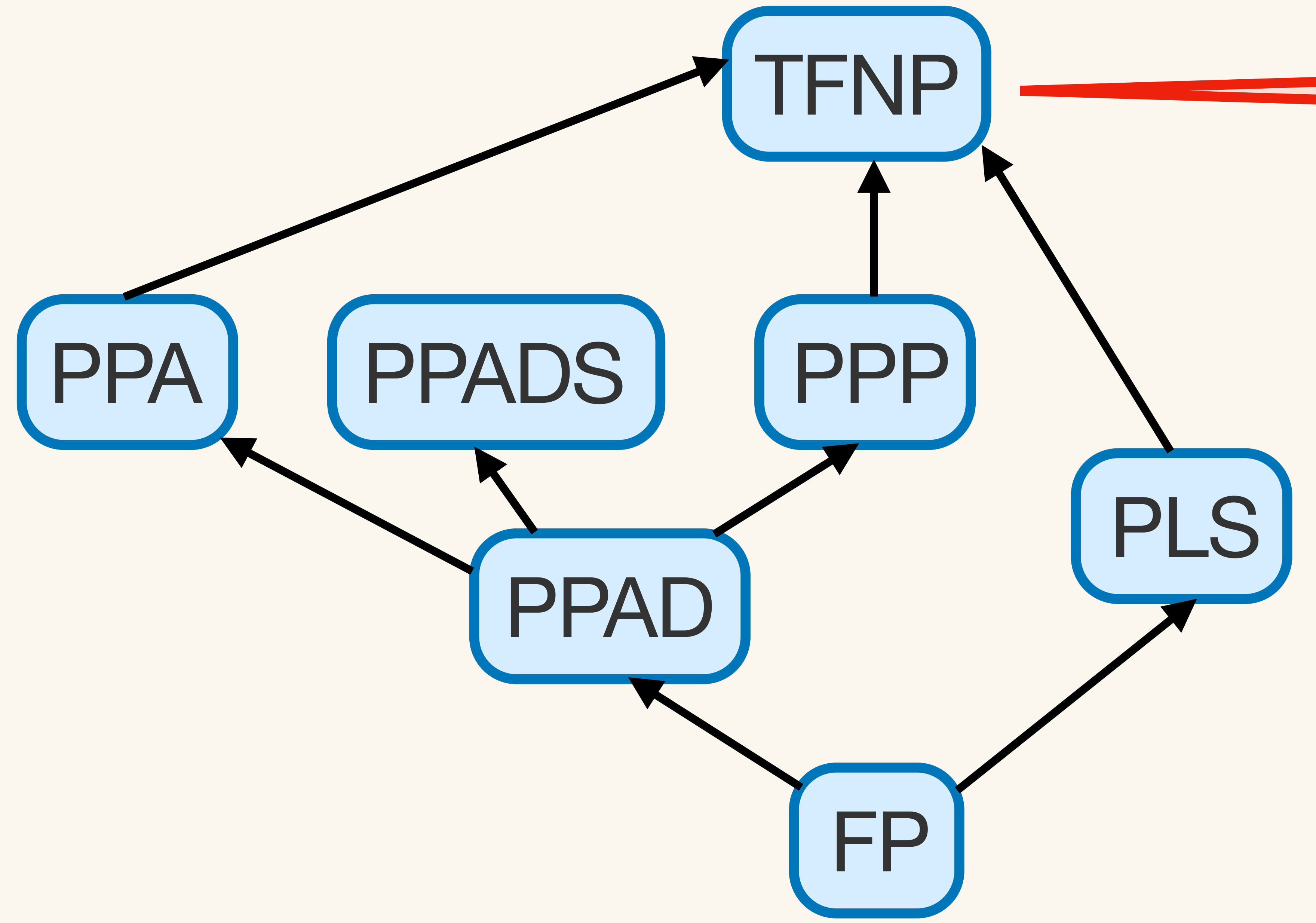
However, useful to consider other models of computation

TFNP



Model of Computation: Decision Trees

TFNP



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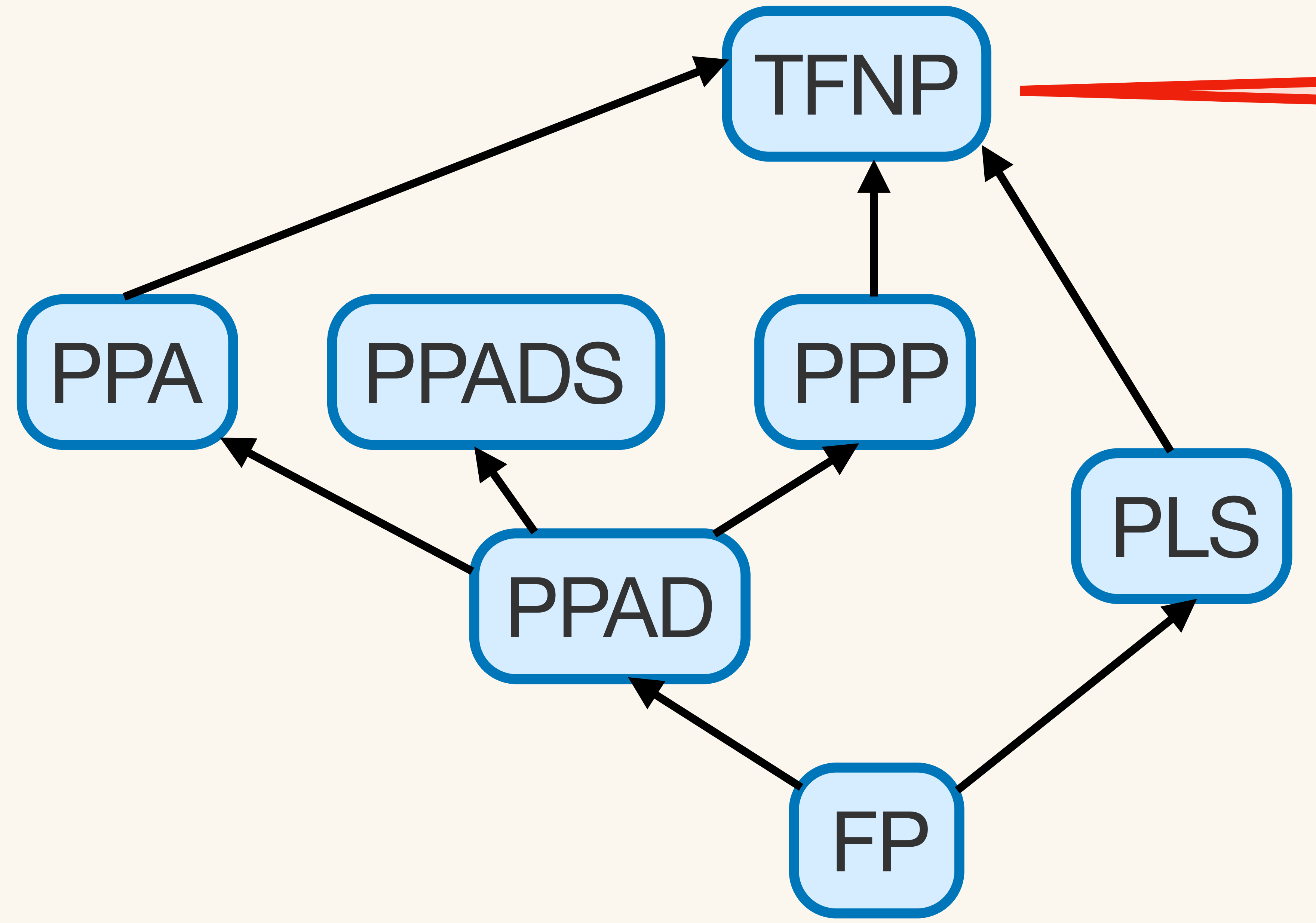

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$\forall \ell \in \mathcal{O}$ there is $polylog(n)$ -depth T_ℓ such that $(x, \ell) \in S \iff T_\ell(x) = 1$

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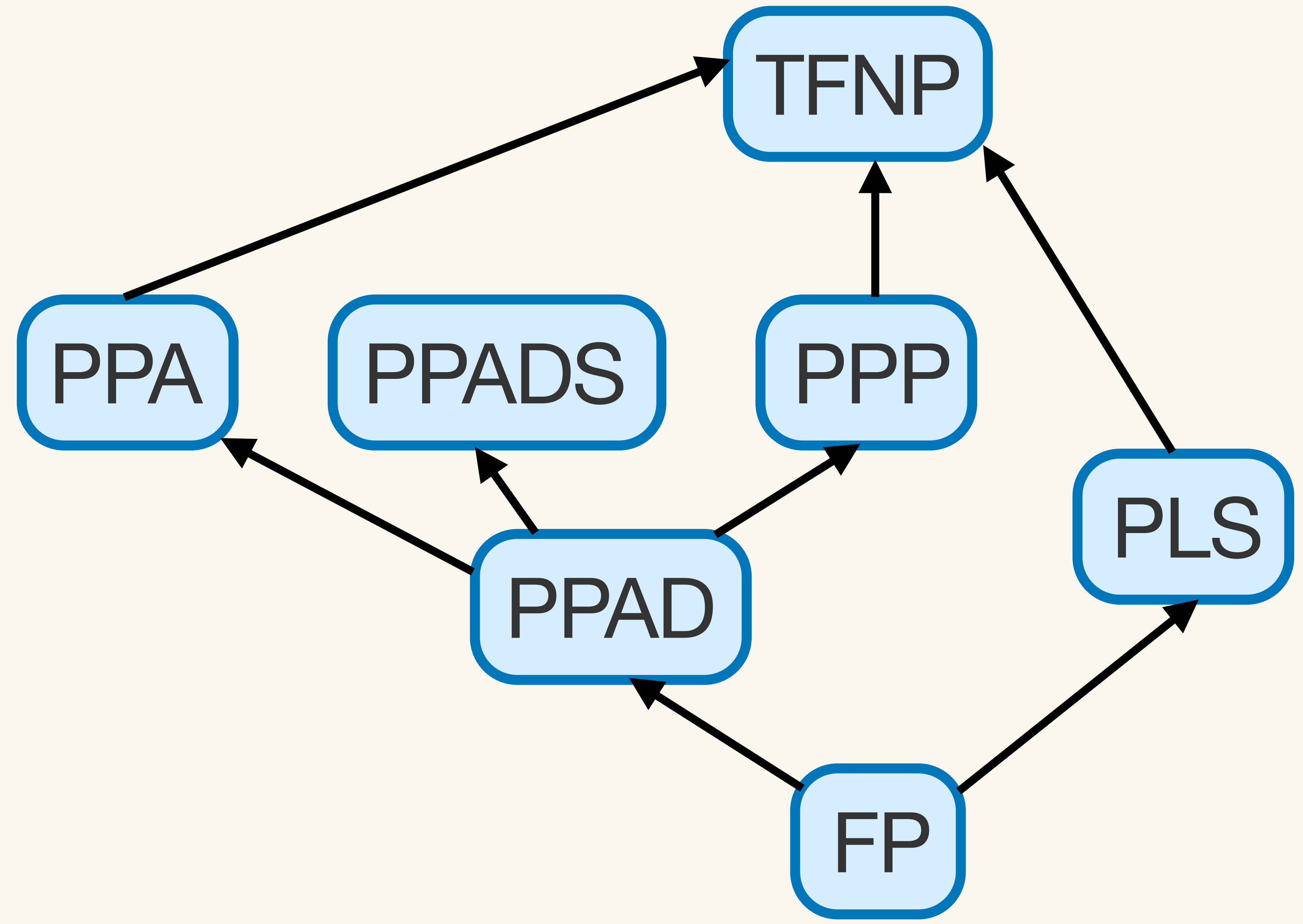
[BCEIP98] Separations imply black-box / generic oracle separations



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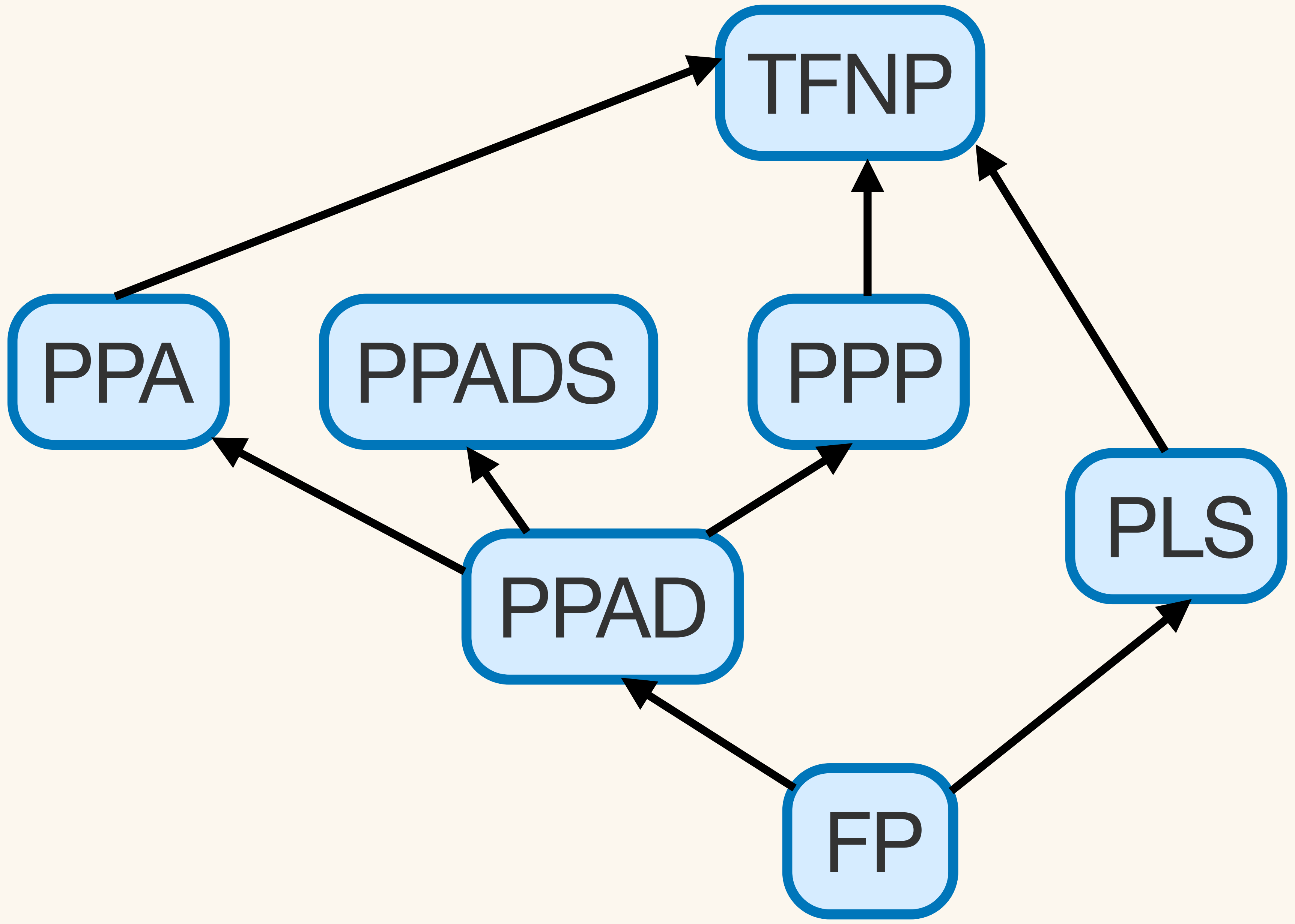
[GKRS18] Certain proof systems are **equivalent** to decision tree TFNP classes!



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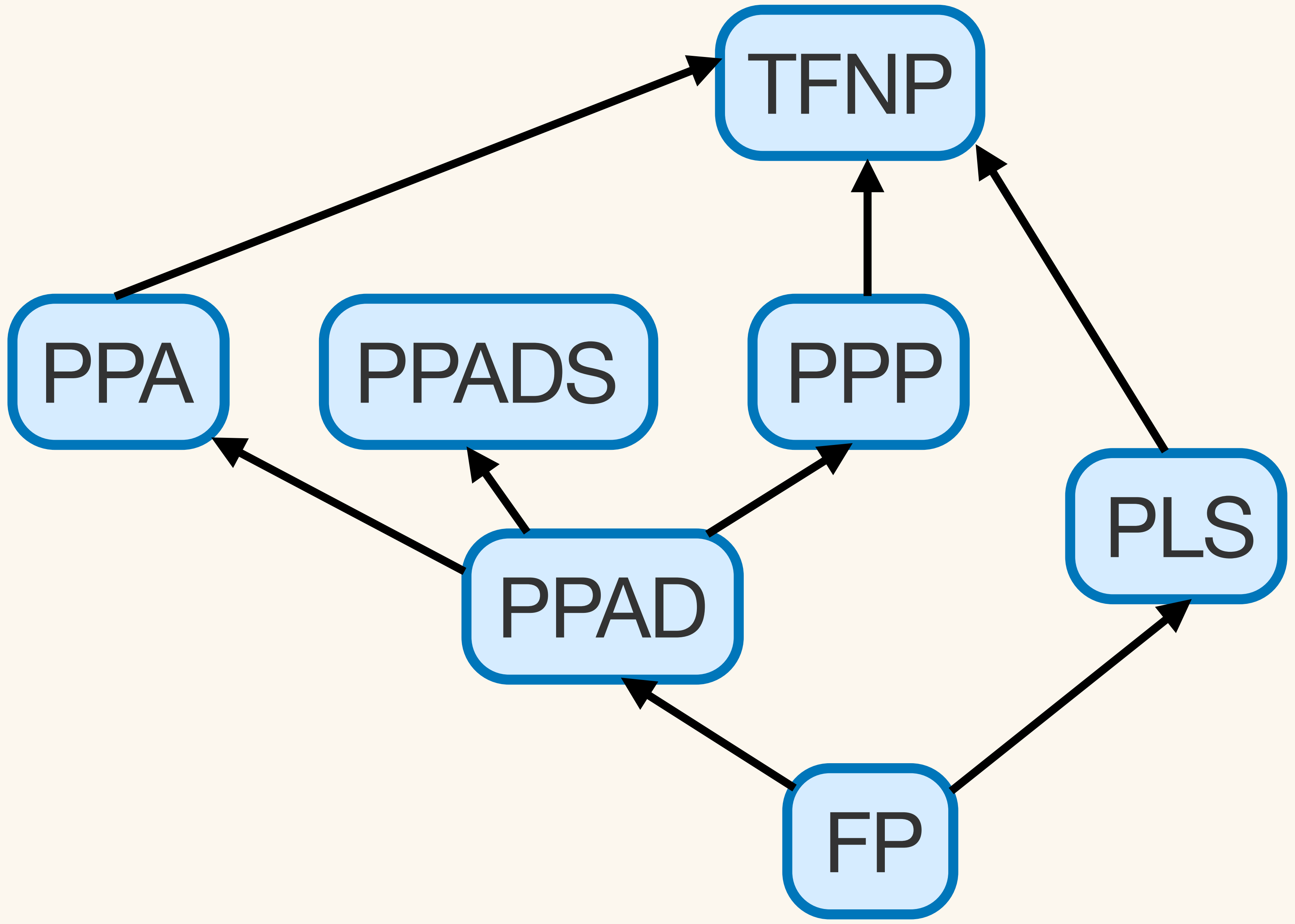
Say that these proof systems are characterized by the TFNP class



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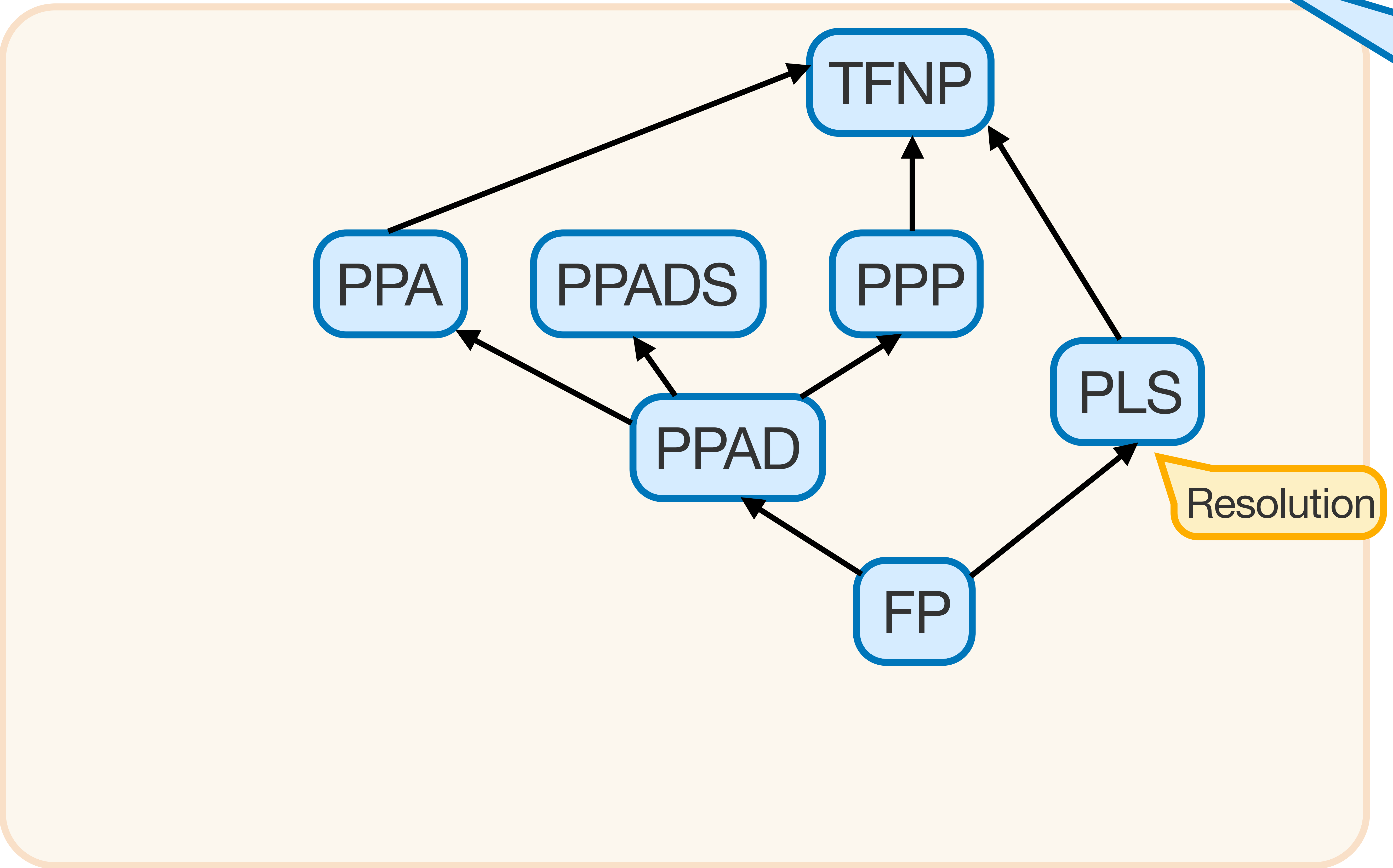

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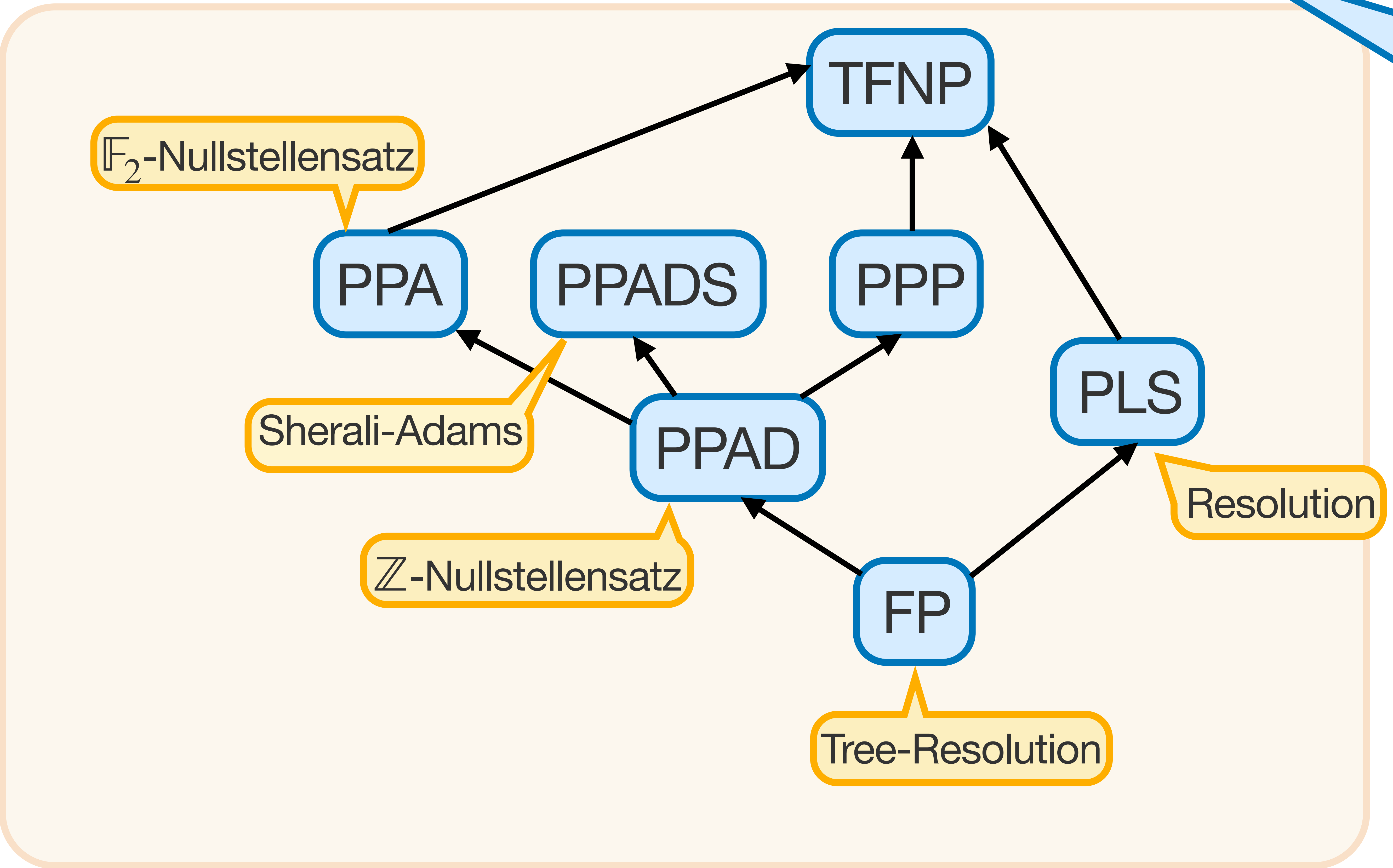

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$$F = \bigwedge_{\ell \in \mathcal{O}} \neg DNF(T_\ell)$$

Expresses that R is not total:

A clause of $\neg DNF(T_\ell)$ is false under $x \iff (x, \ell) \in R$

Resolution is PLS

TFNP subclasses defined as everything $polylog(n)$-reducible to a particular search problem

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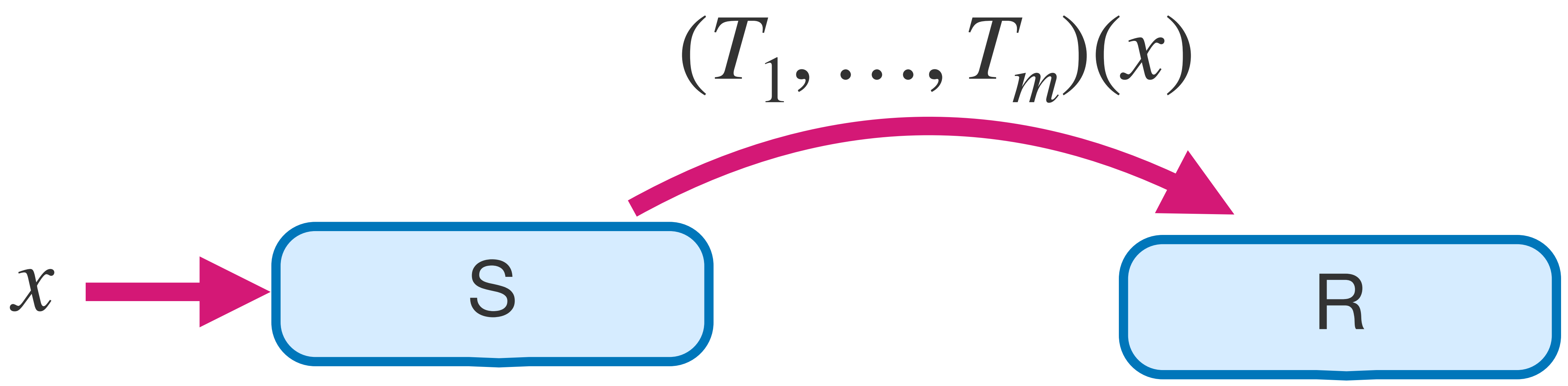
- $T_1, \ldots, T_m$ turning inputs to S into inputs to R



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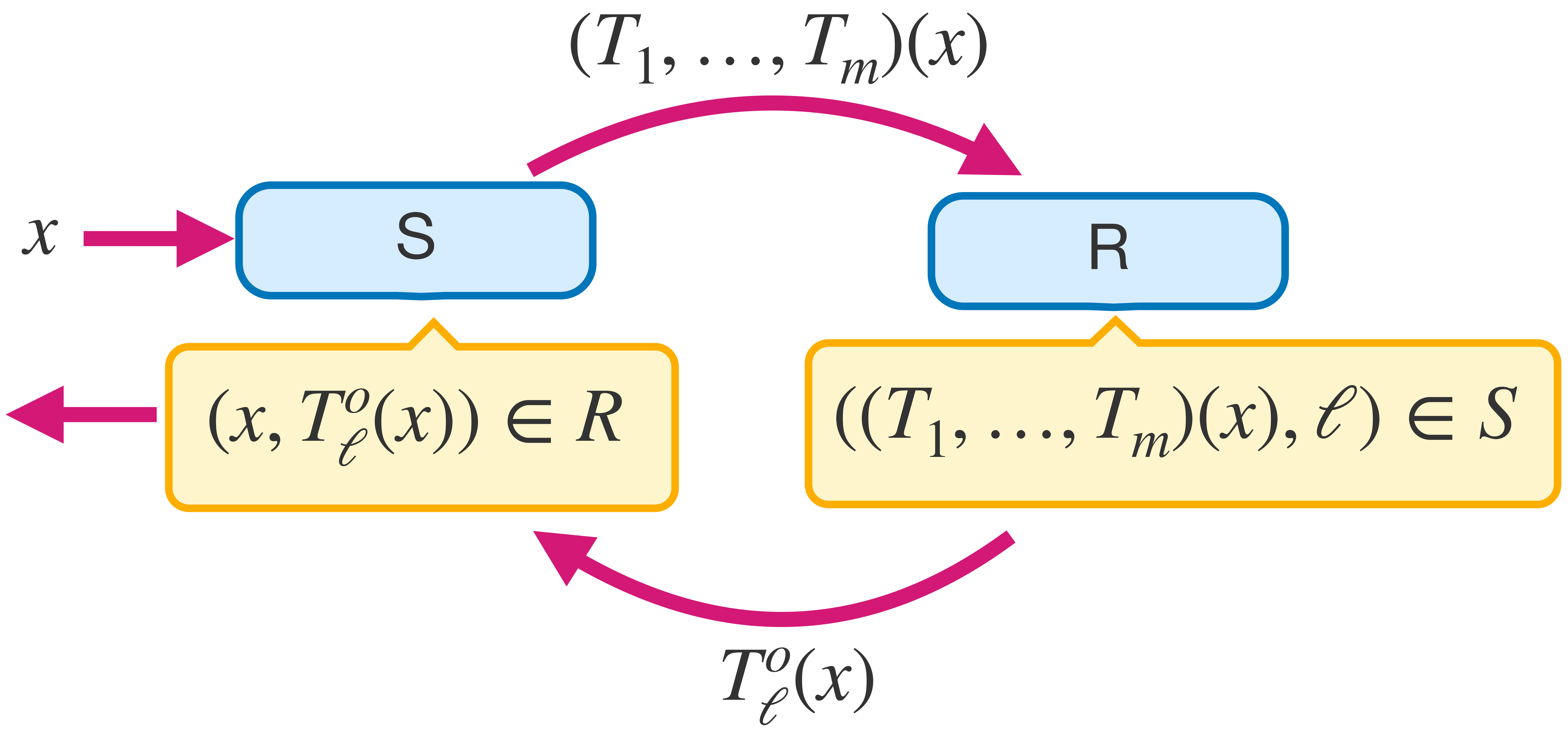
- $T_1^o, \ldots, T_{|\mathcal{Q}|}^o$ translating solutions to R into solutions to S



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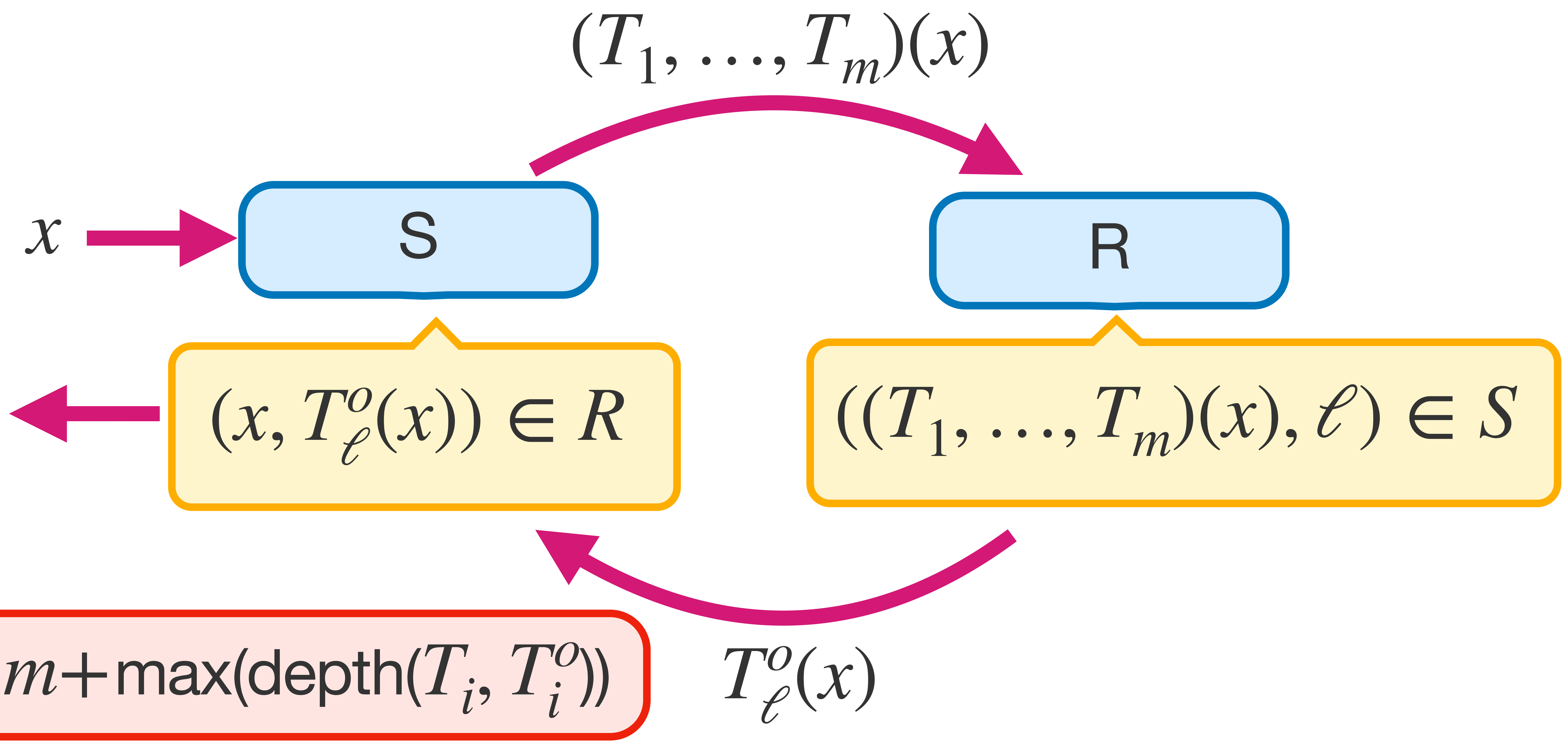

Complexity: $\log m + \max(\text{depth}(T_i, T_i^o))$

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Vertices: $1, \ldots, n$
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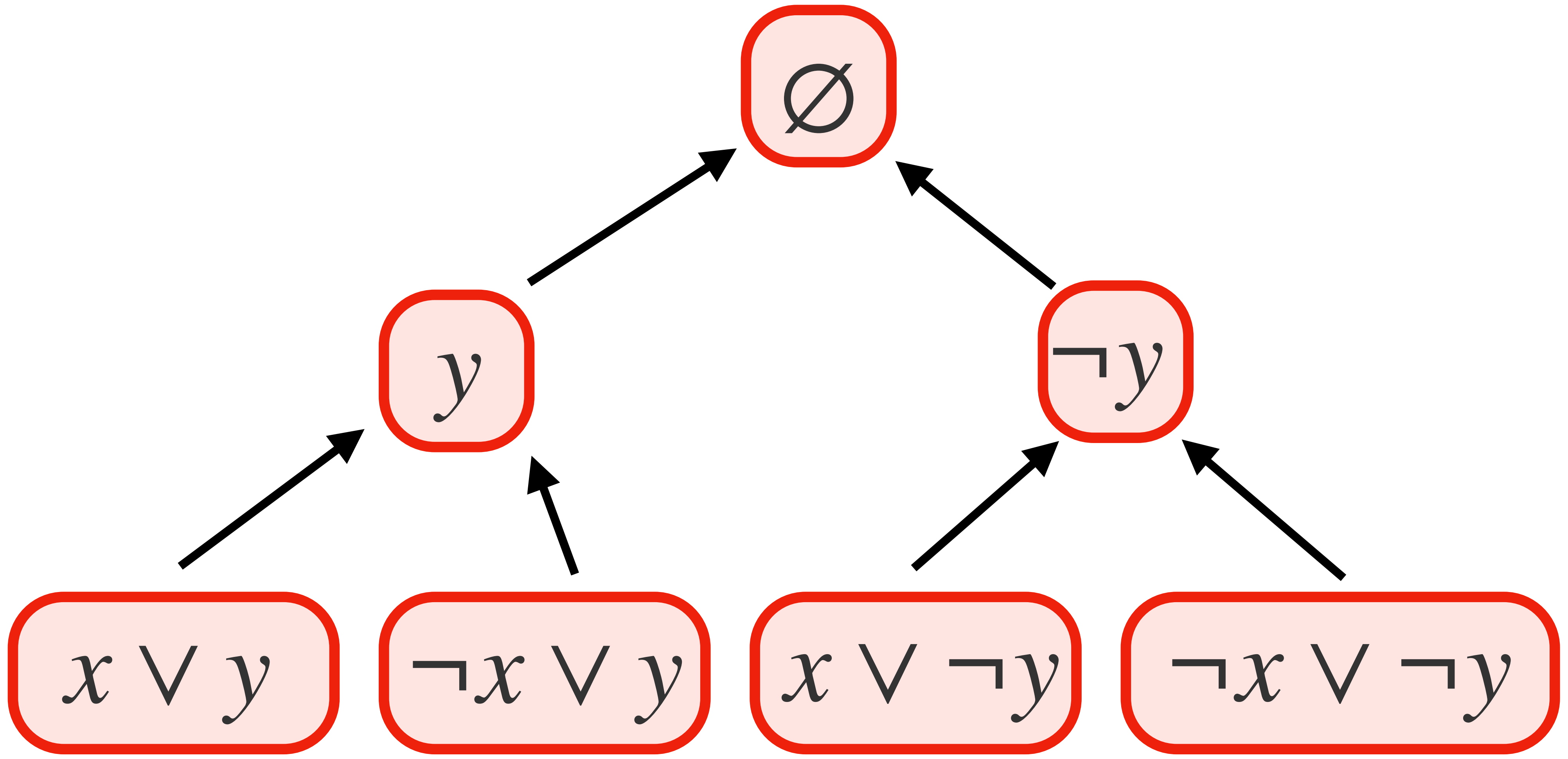

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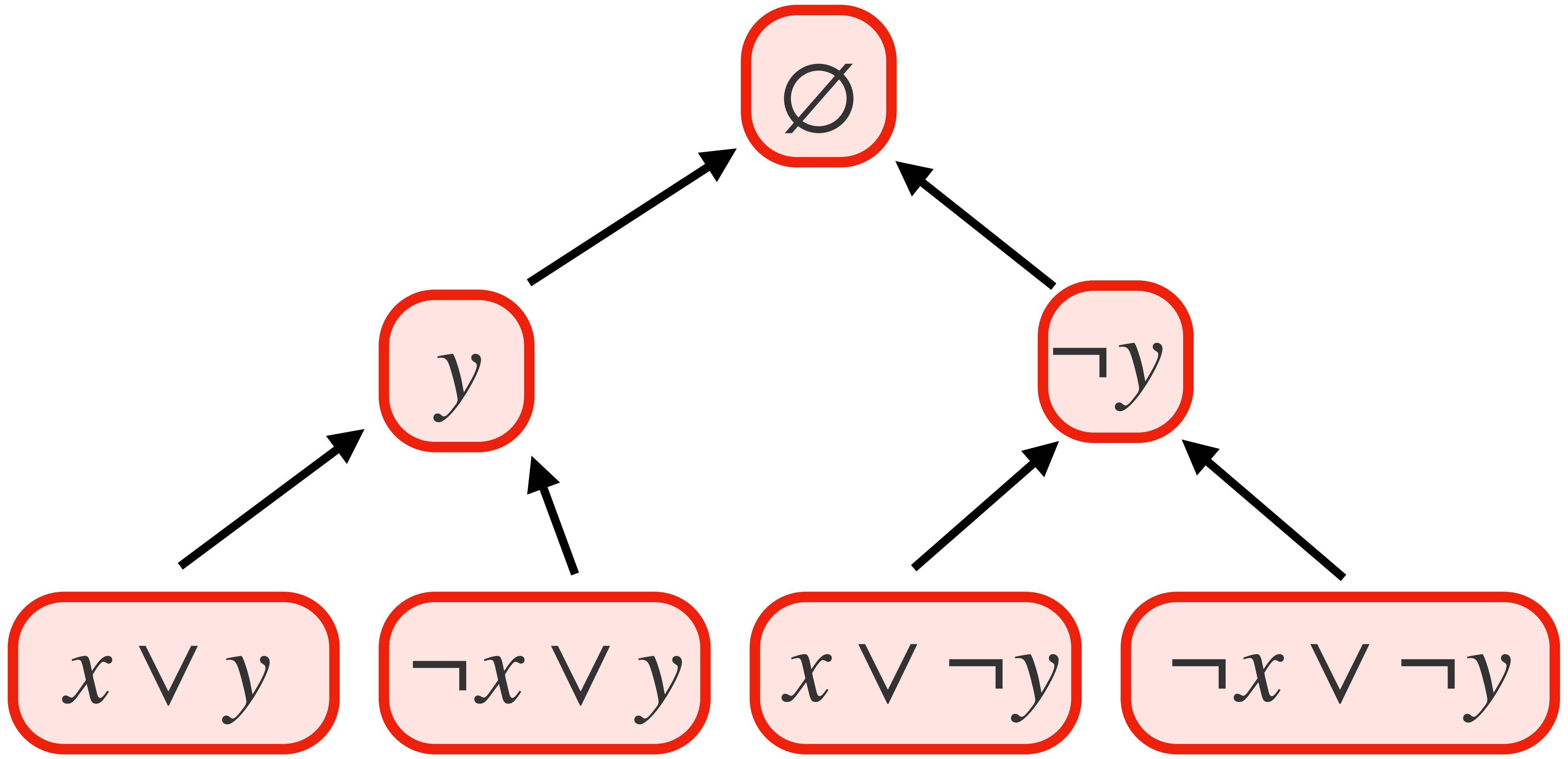

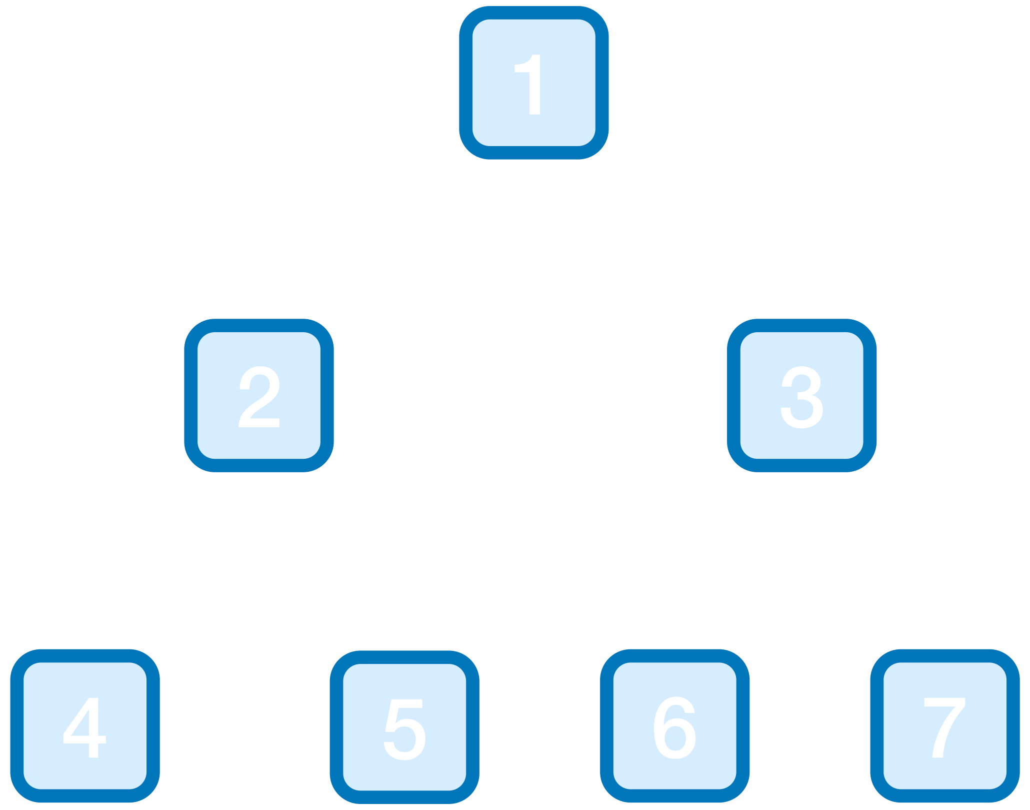

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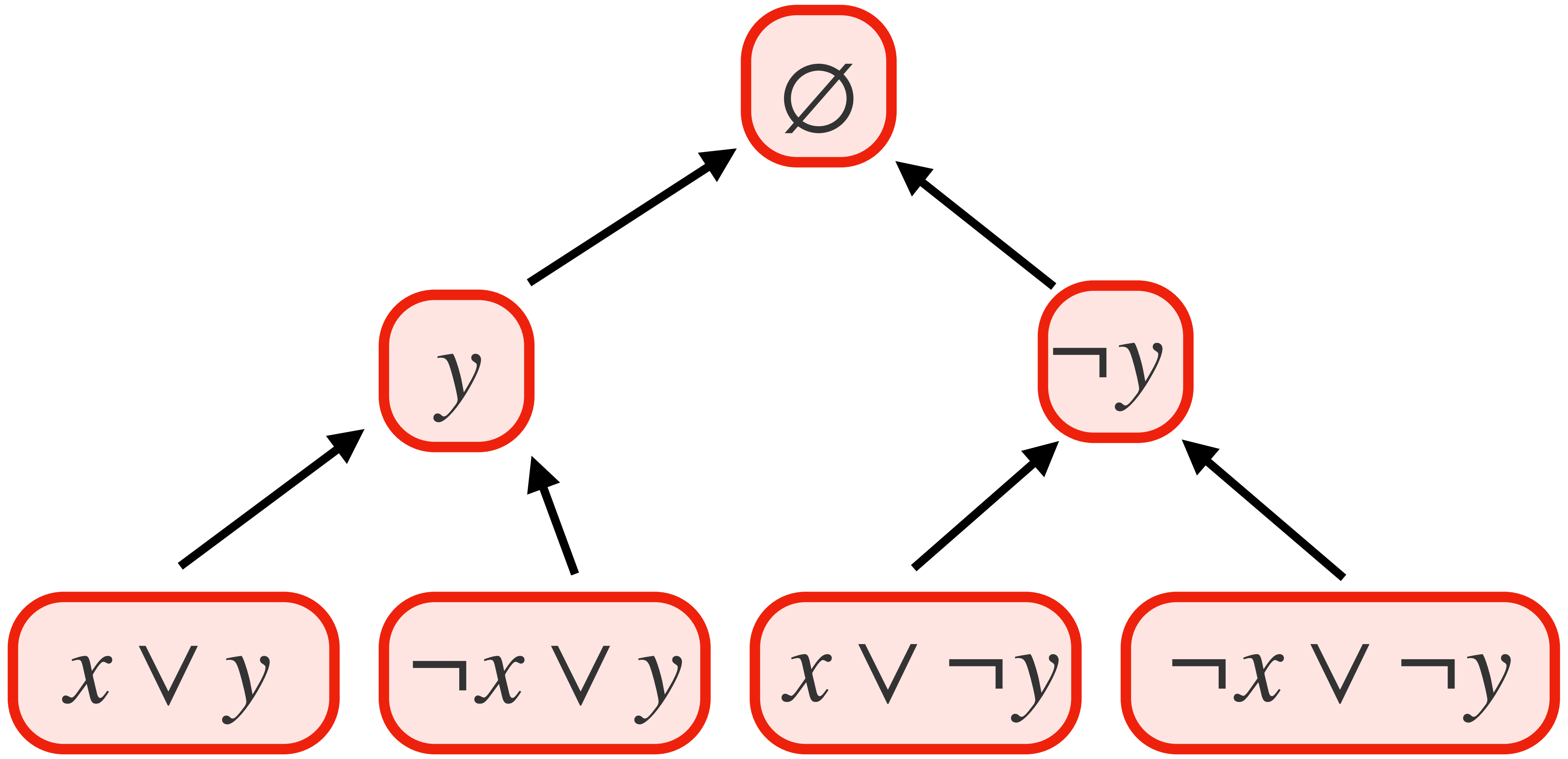

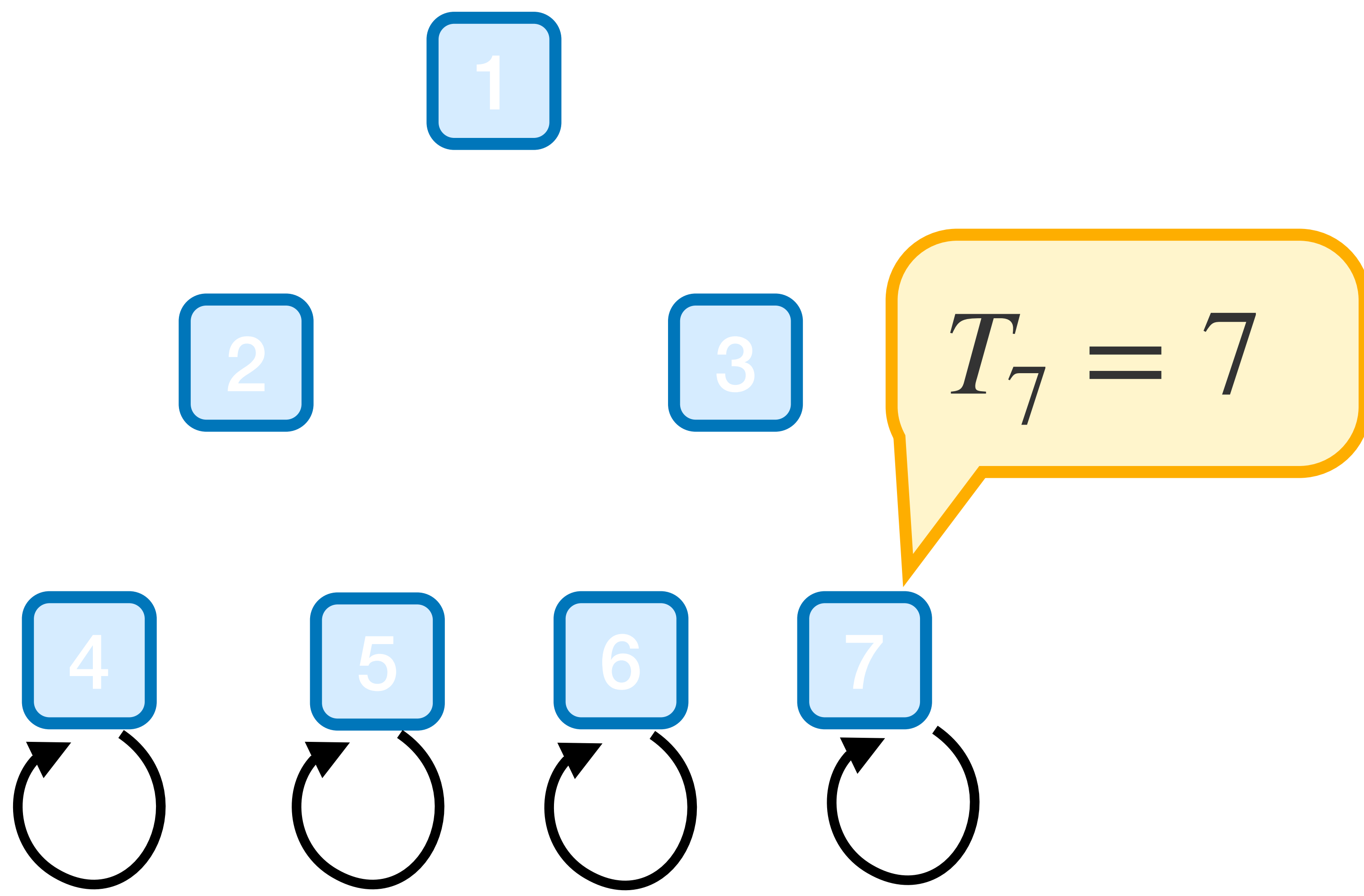

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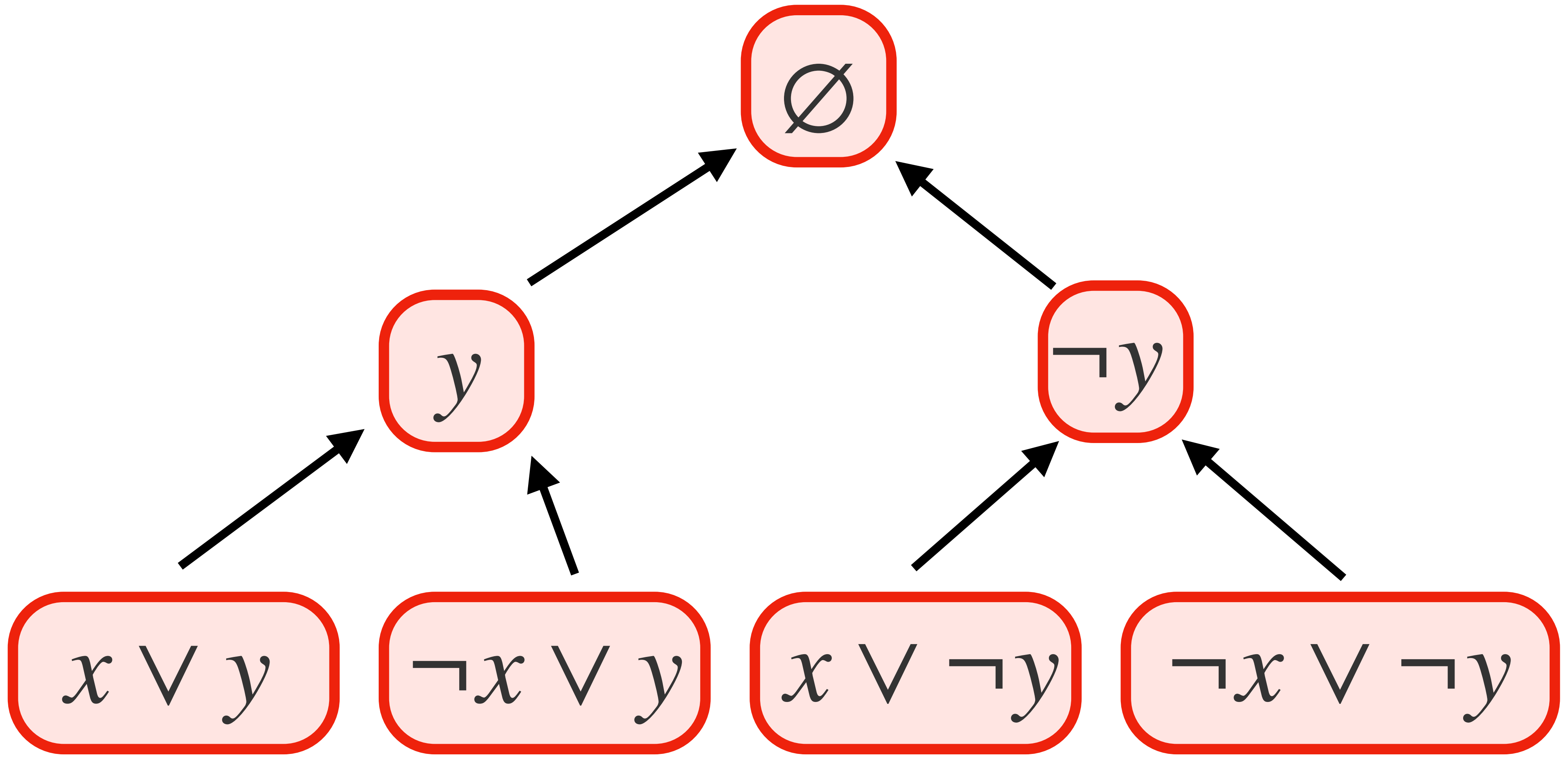

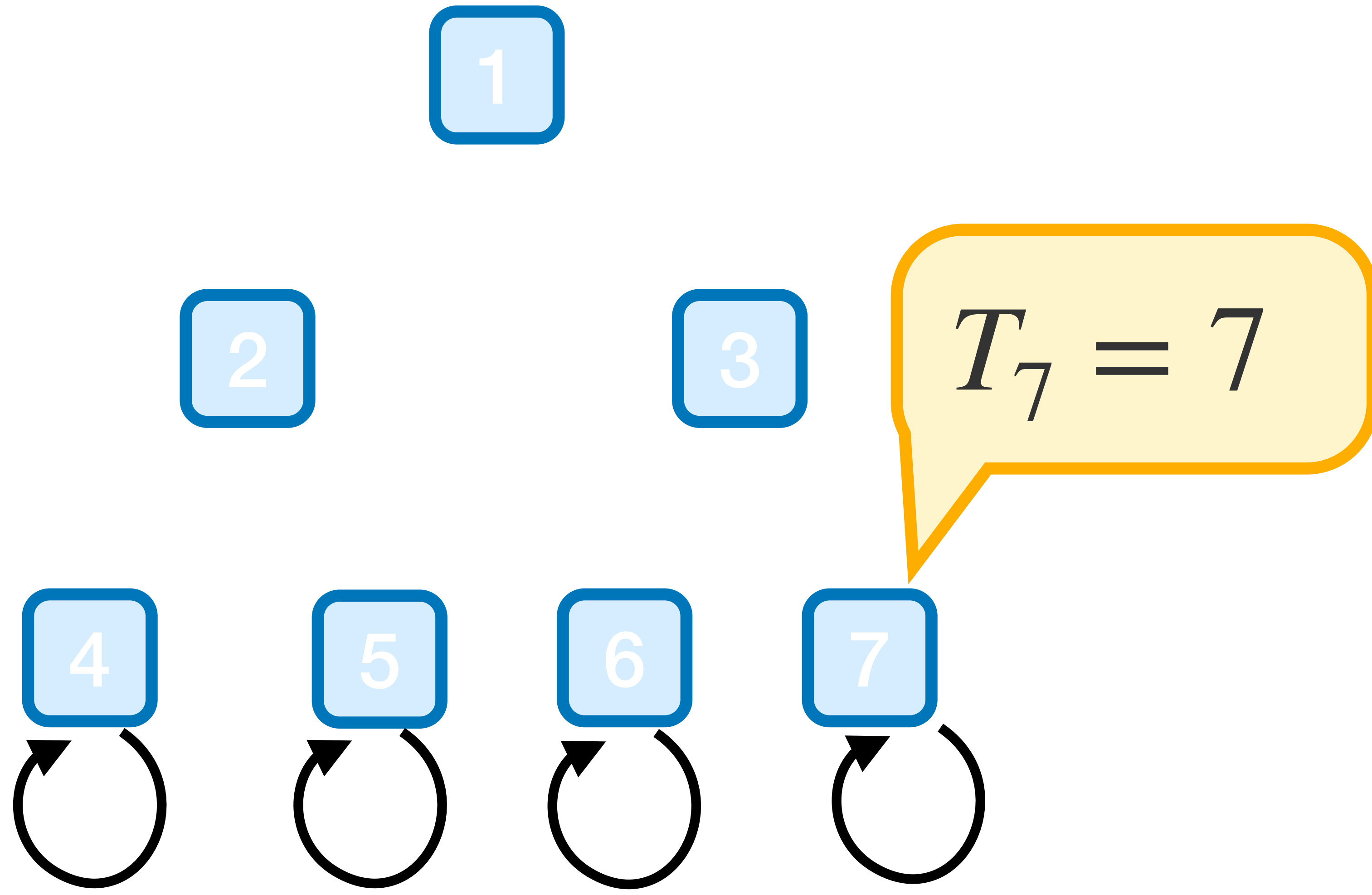

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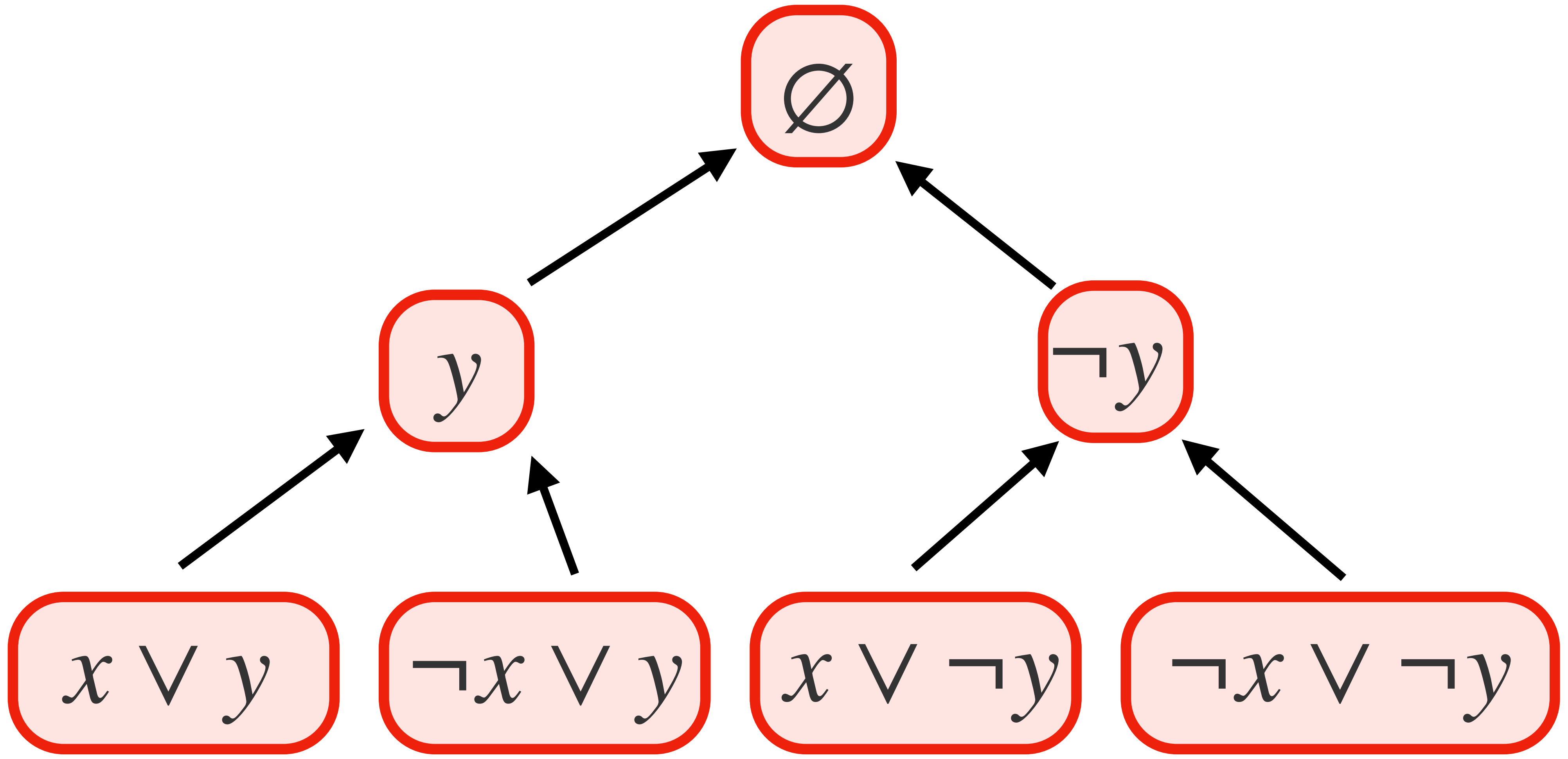
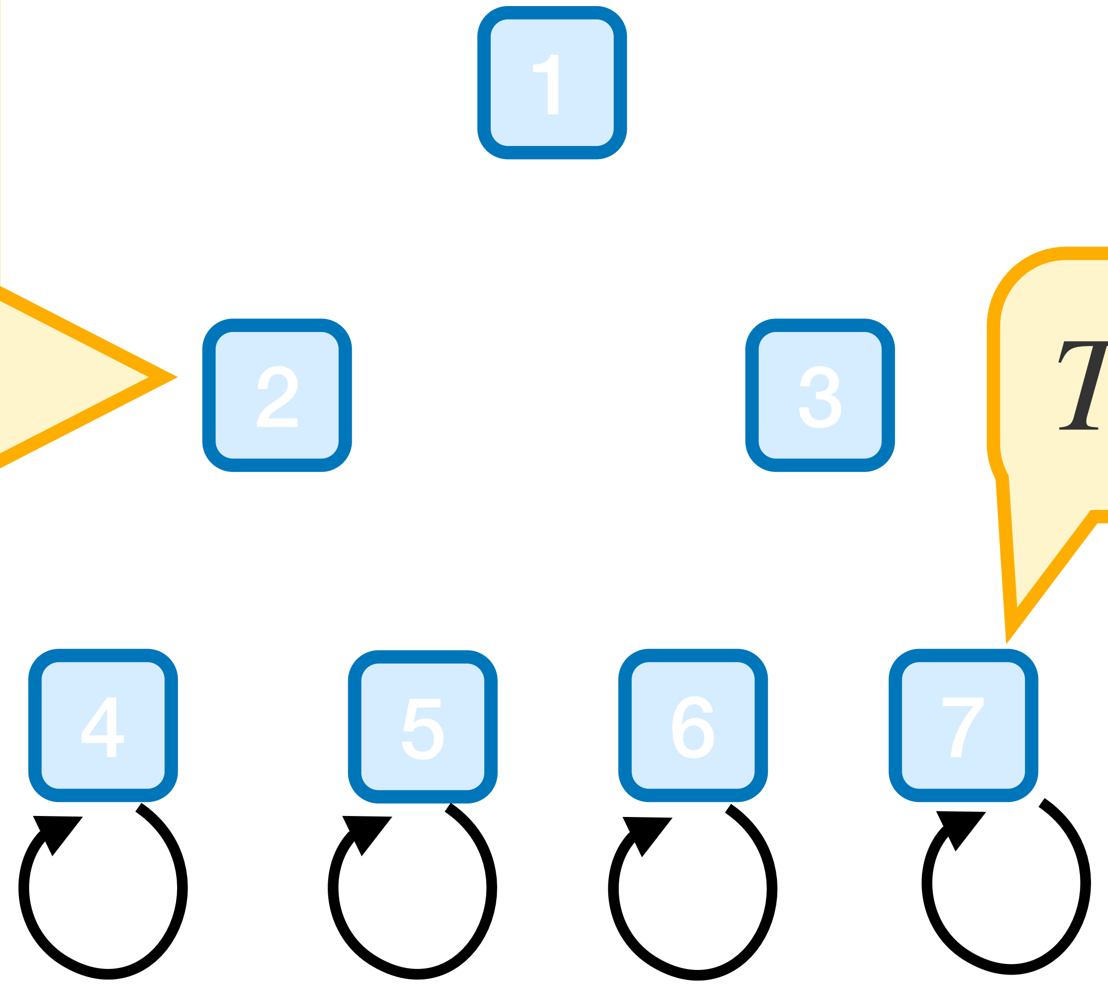


T_2 queries x, y:



$T_7 = 7$

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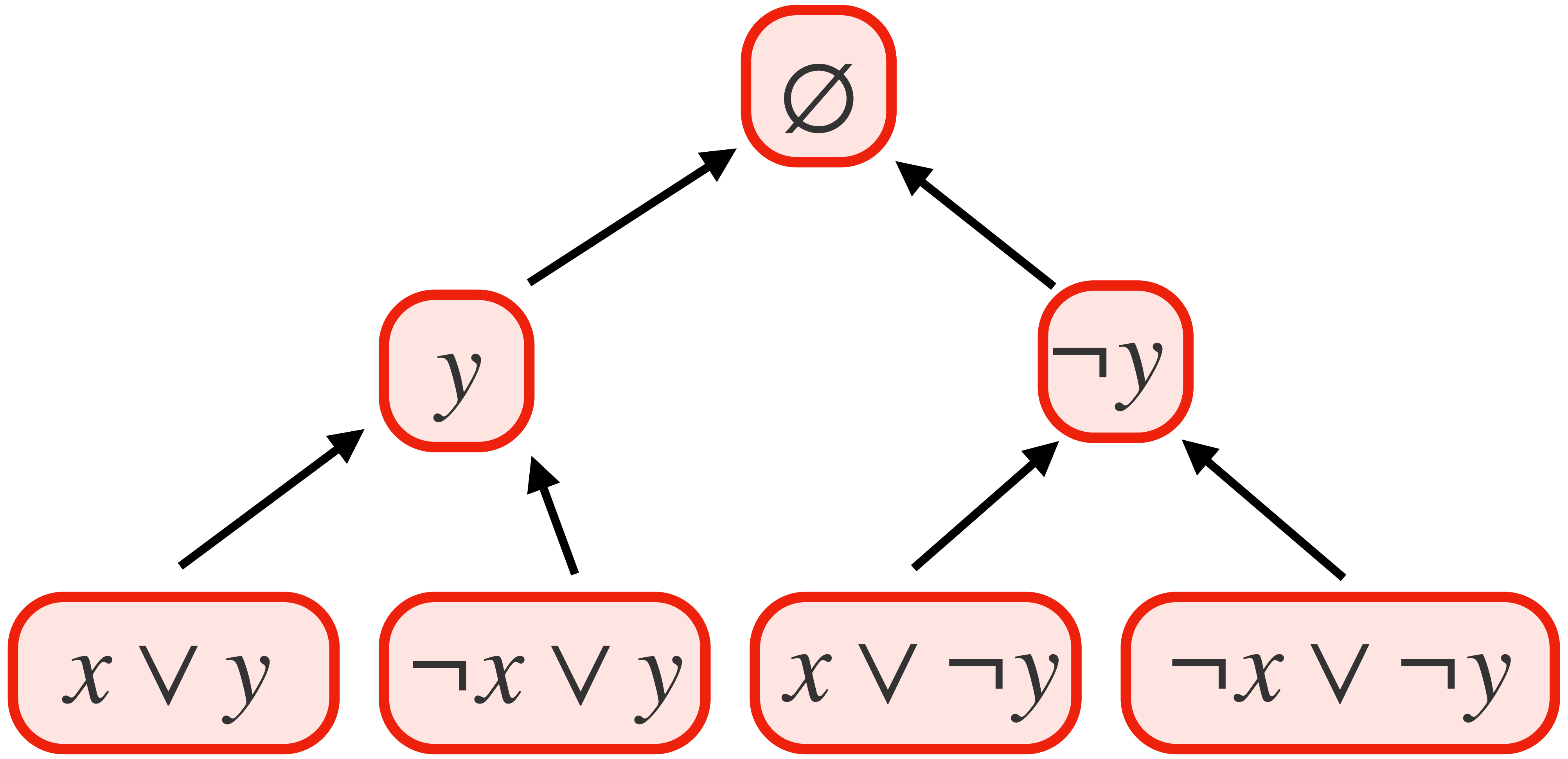
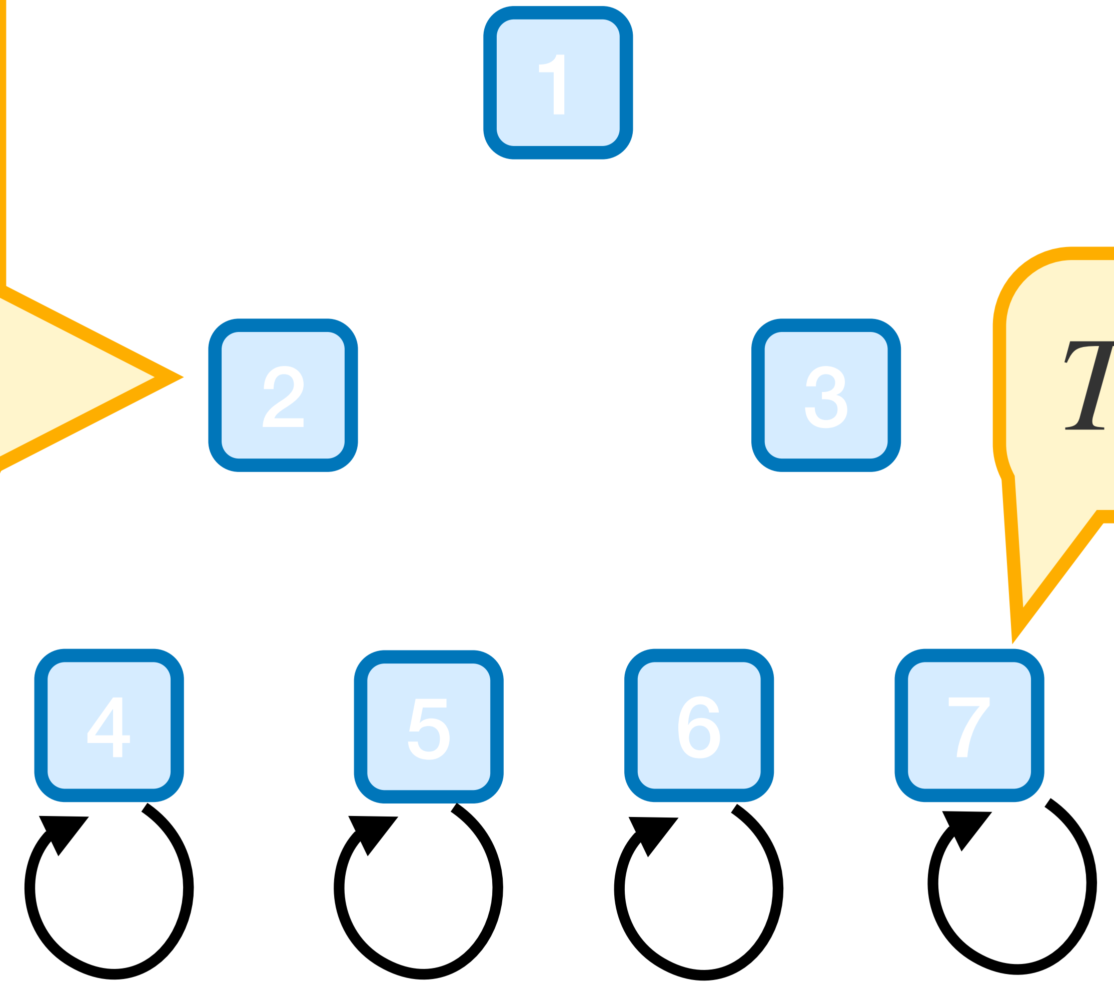


T_2 queries x, y:

$$T_2 = \begin{cases} 2 & \text{if } y = 1 \\ 4 & \text{if } x \vee y = 0 \\ 5 & \text{otherwise} \end{cases}$$



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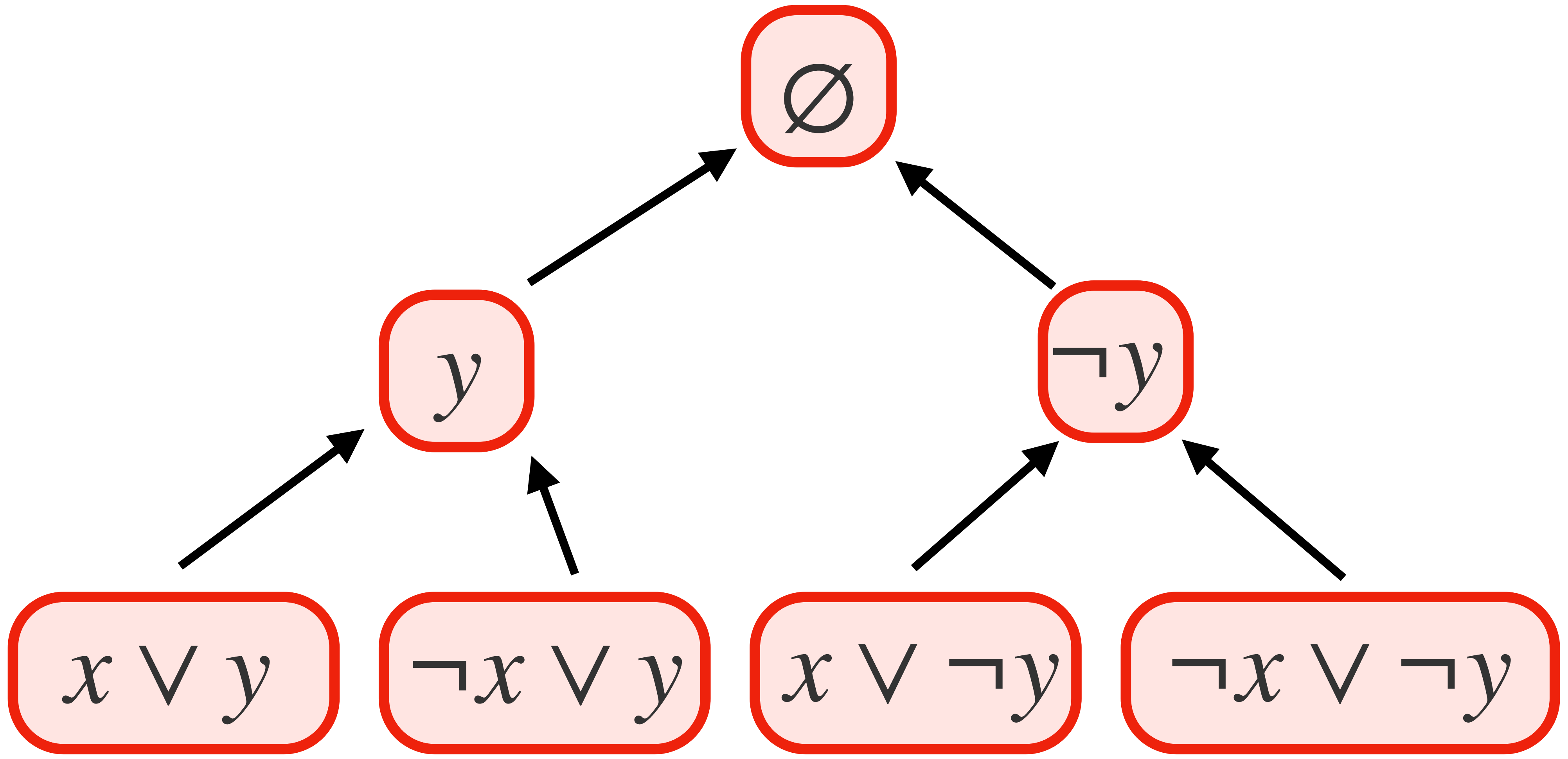
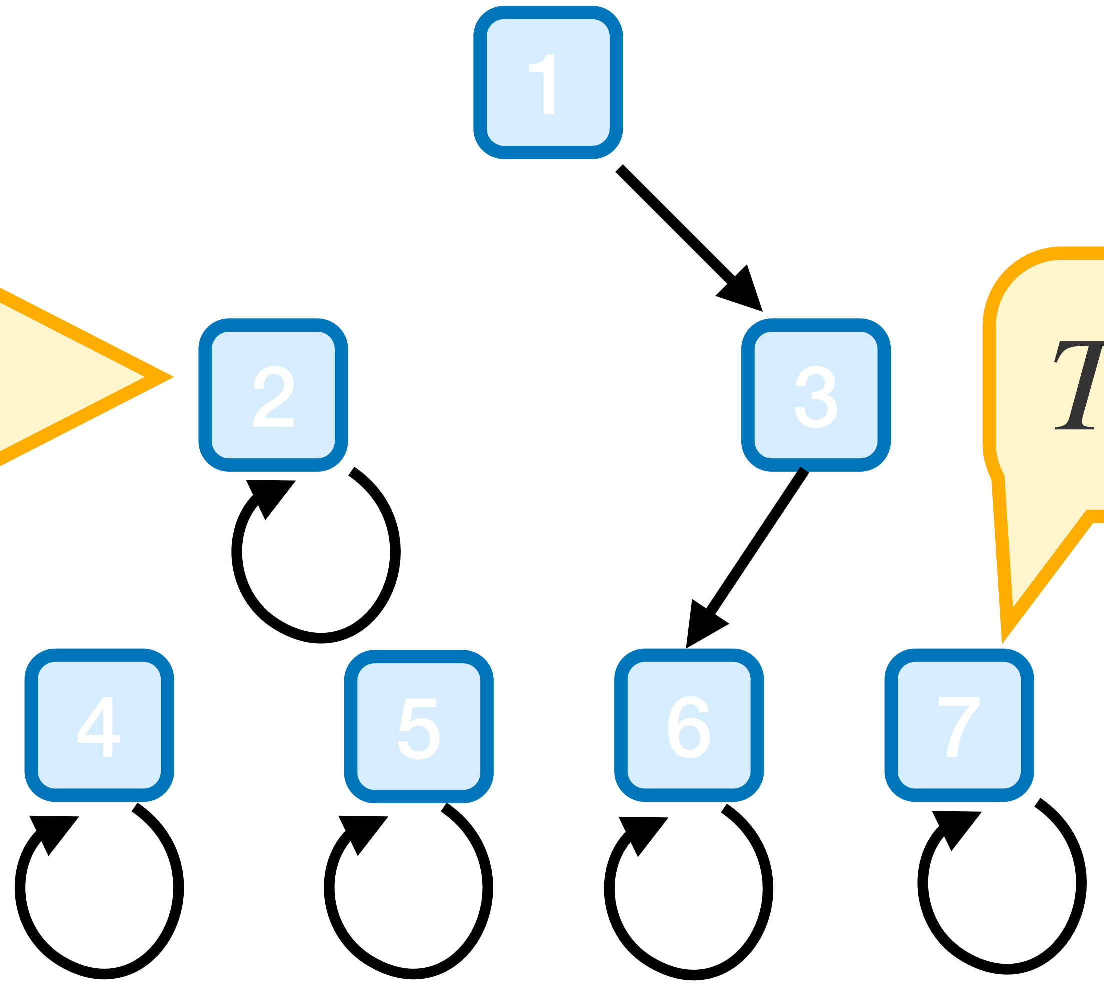


T_2 queries x, y:

$$T_2 = \begin{cases} 2 & \text{if } y = 1 \\ 4 & \text{if } x \vee y = 0 \\ 5 & \text{otherwise} \end{cases}$$

e.g. $x = 01$



$T_7 = 7$

Resolution is PLS

Resolution Complexity: of proof Π is $\log size(\Pi) + width(\Pi)$

PLS is Resolution:
$PLS^{dt} = \{F : F$ has a polylog(n)-complexity Res proof$\}$

SinkOfDag
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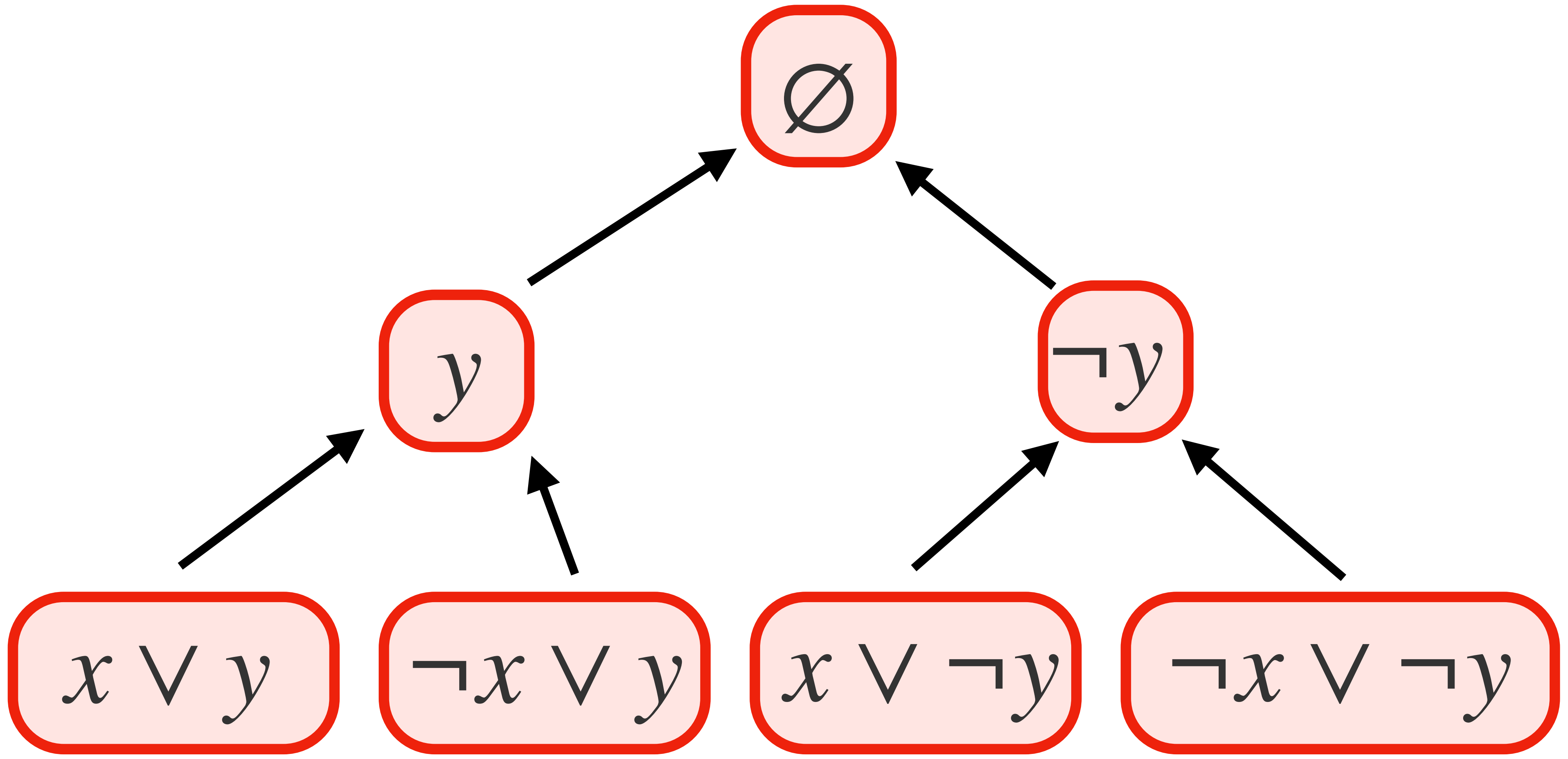
$\Longrightarrow$

Resolution is sound $\Longrightarrow$ Solutions are false clauses!



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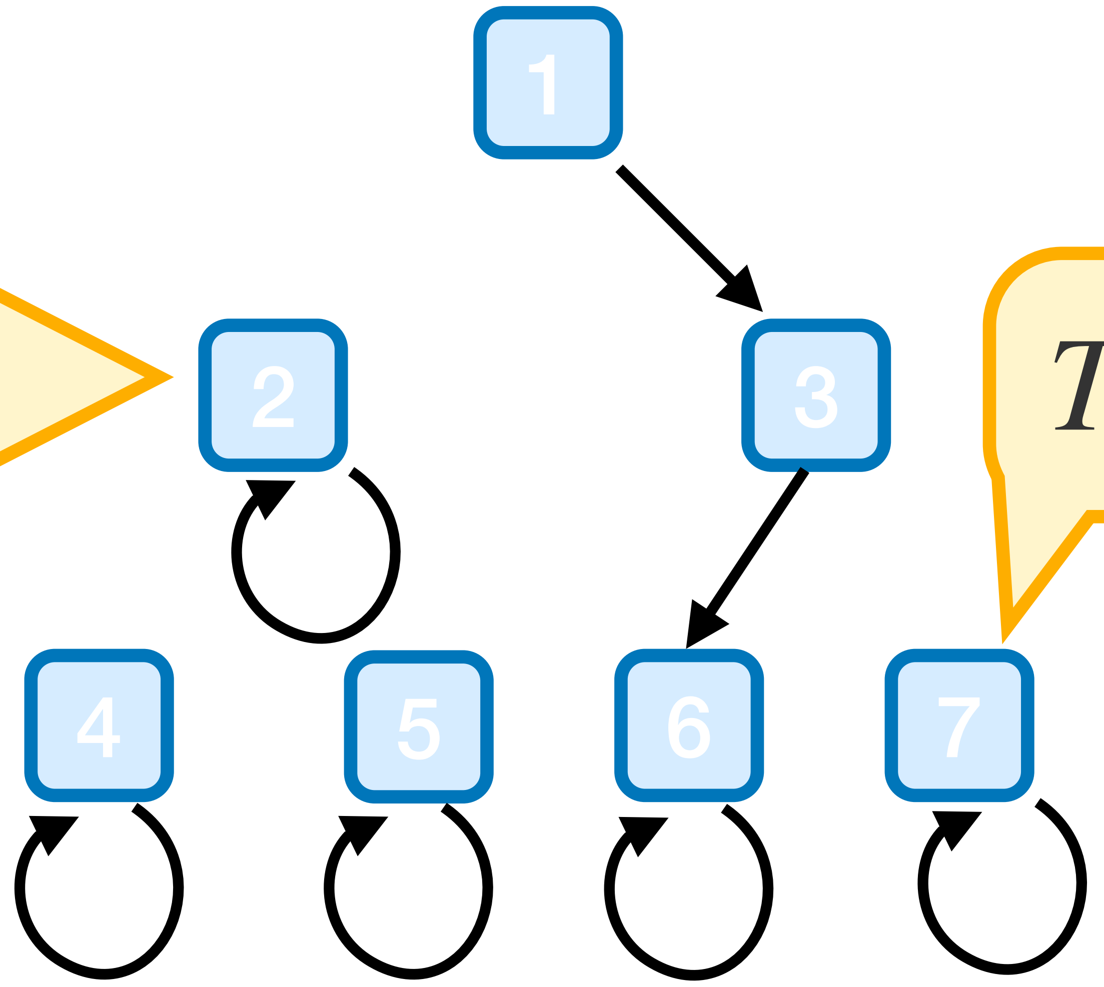

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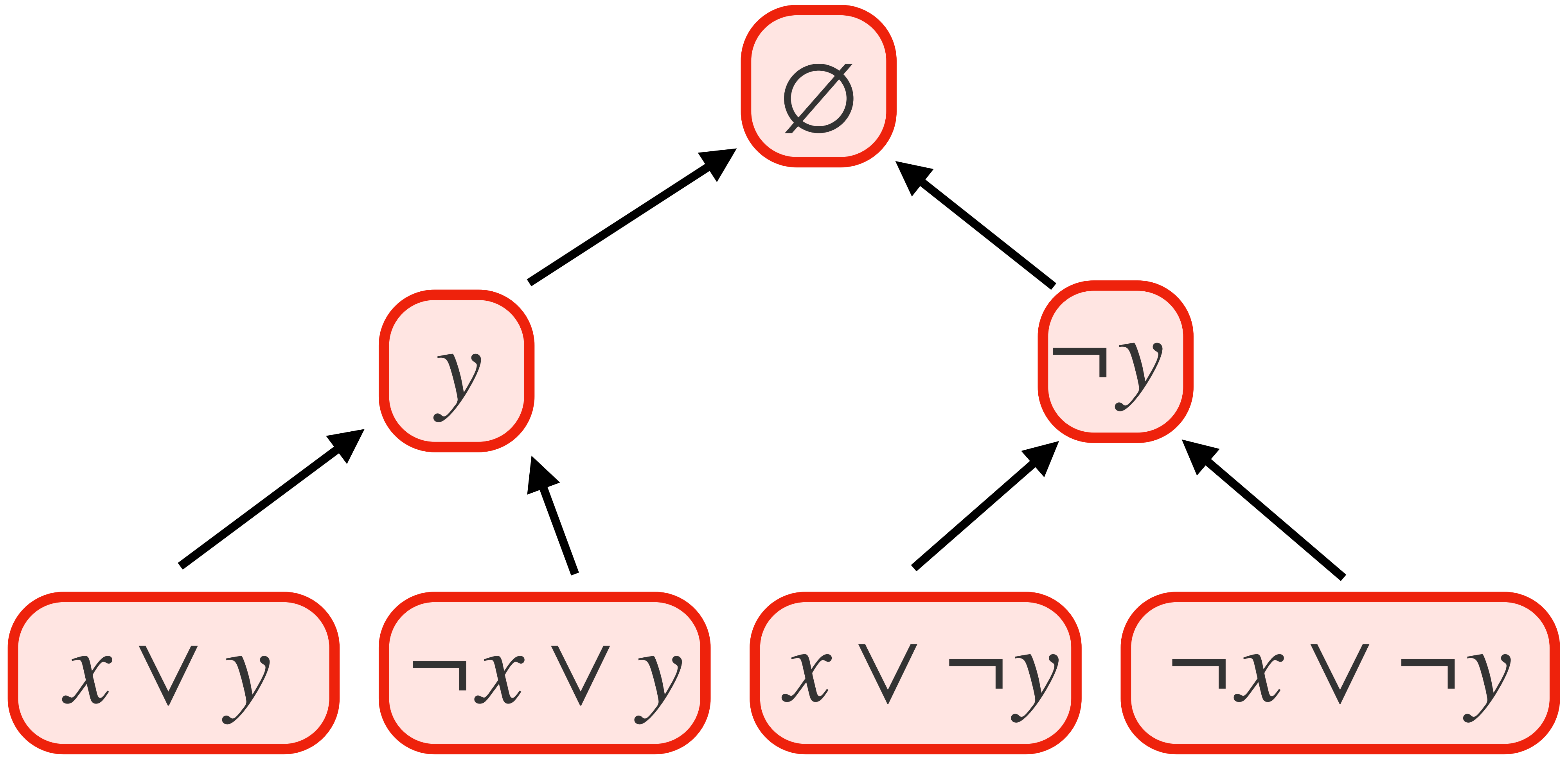

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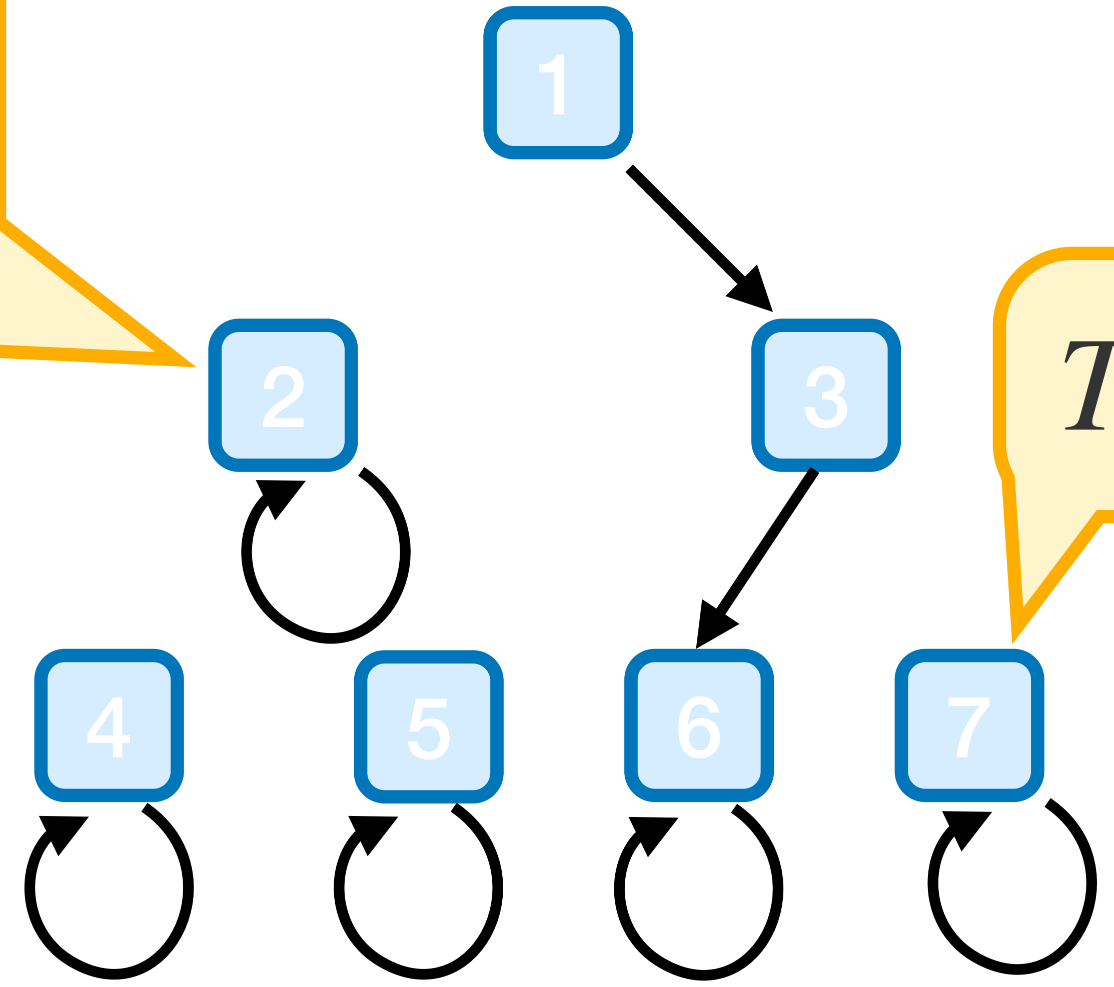

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Delayer Prover Game on F:

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w-Prover strategy $\Longrightarrow$ Complexity $w \log n$ Resolution proof

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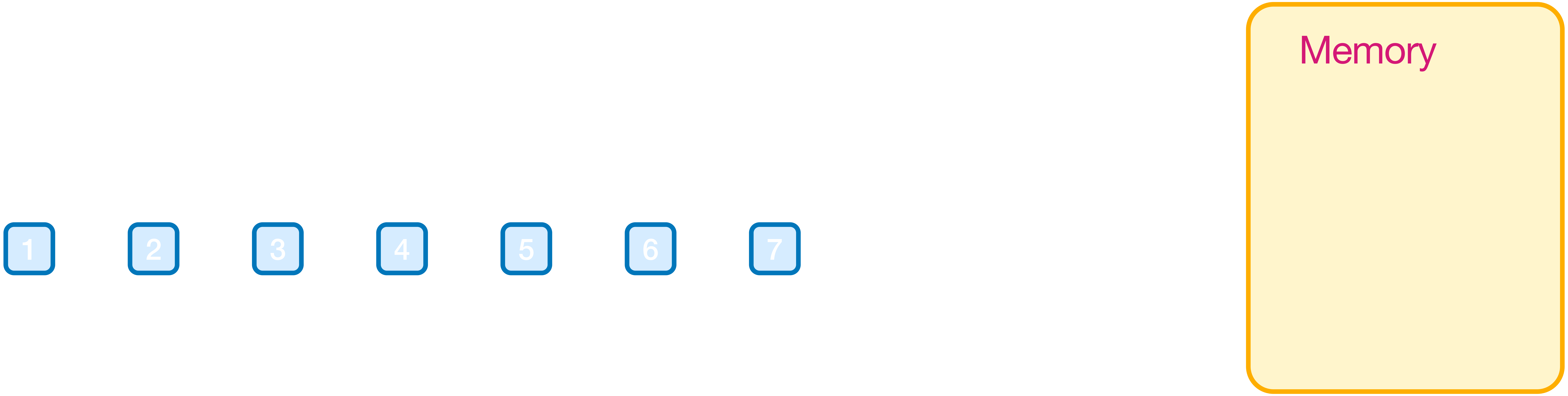

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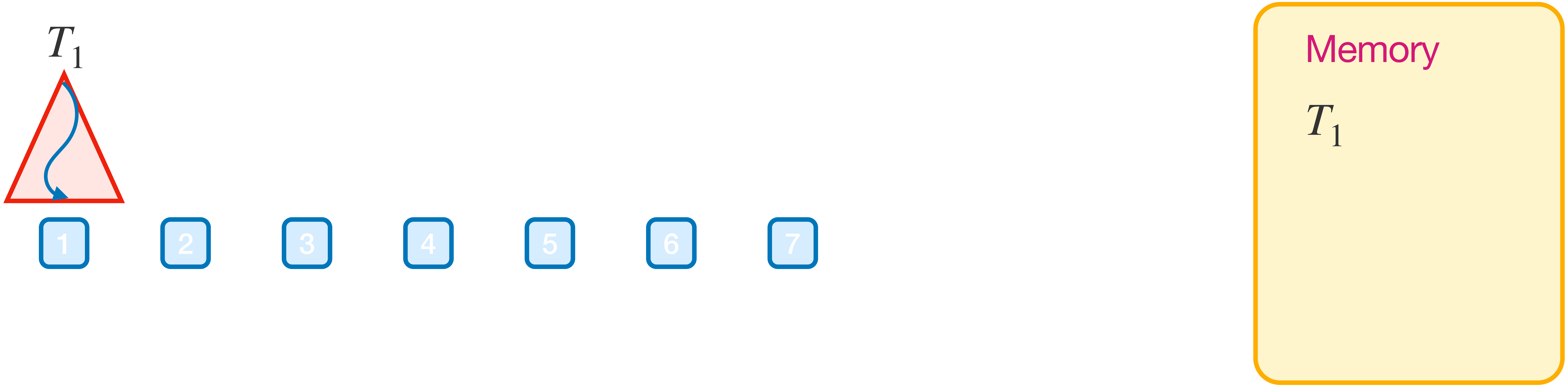

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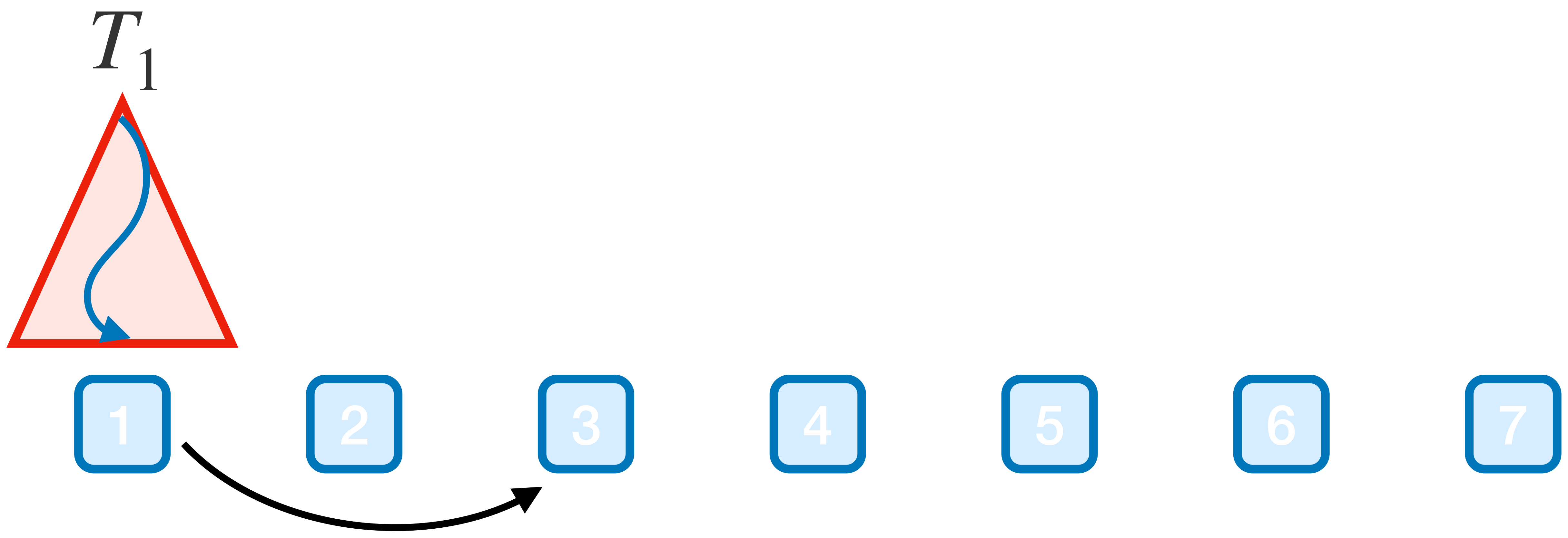

Memory

T_1

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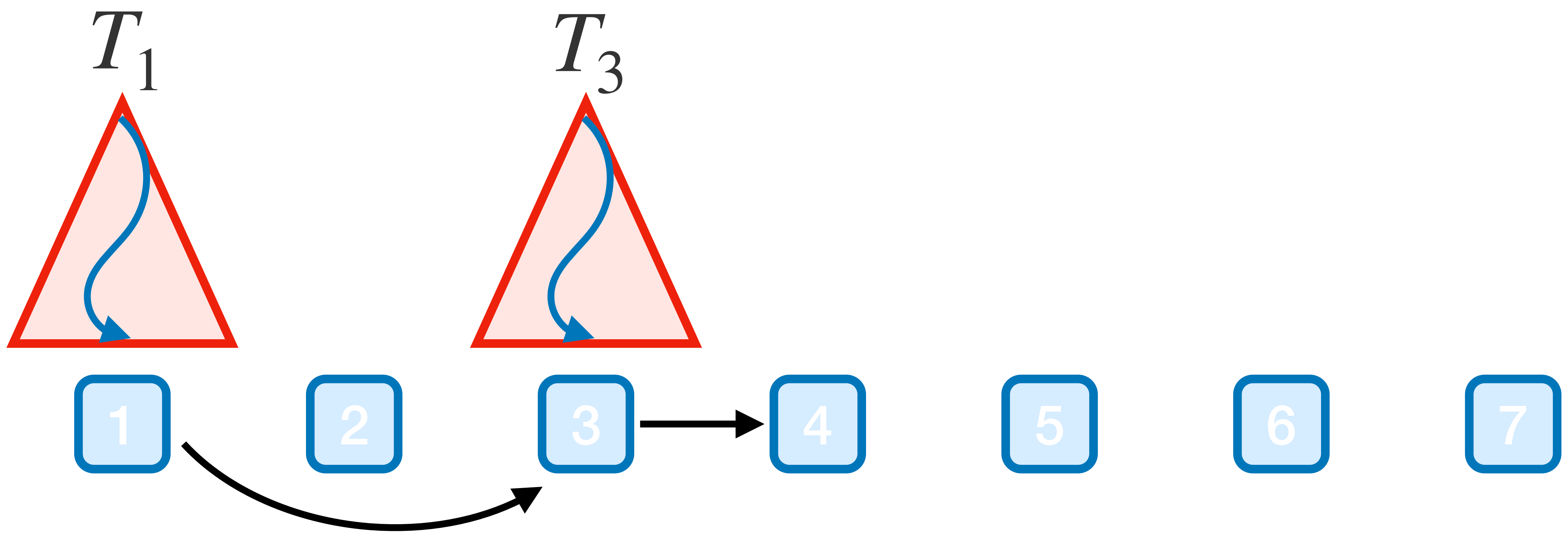

Memory
T_1
T_3

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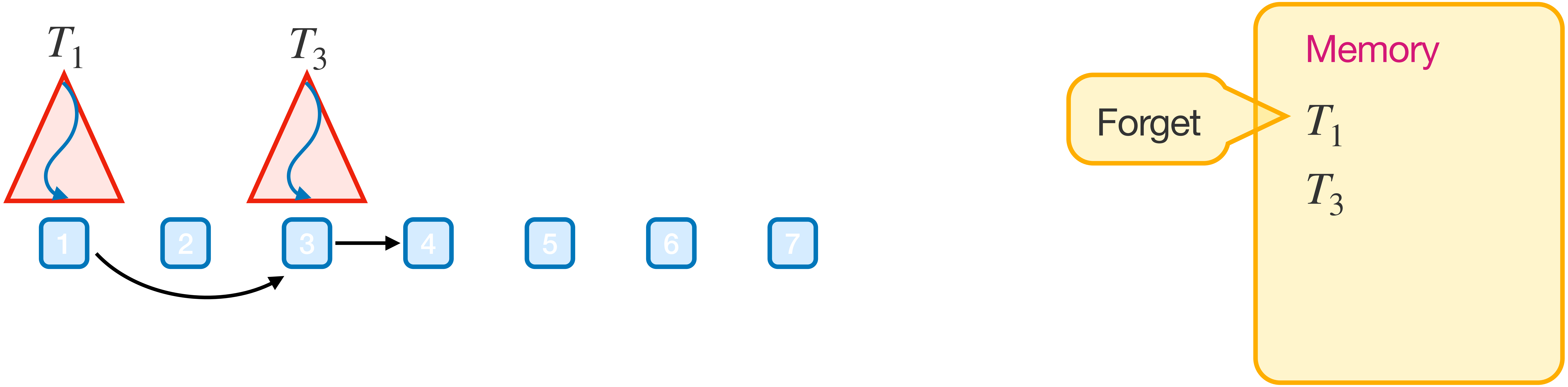

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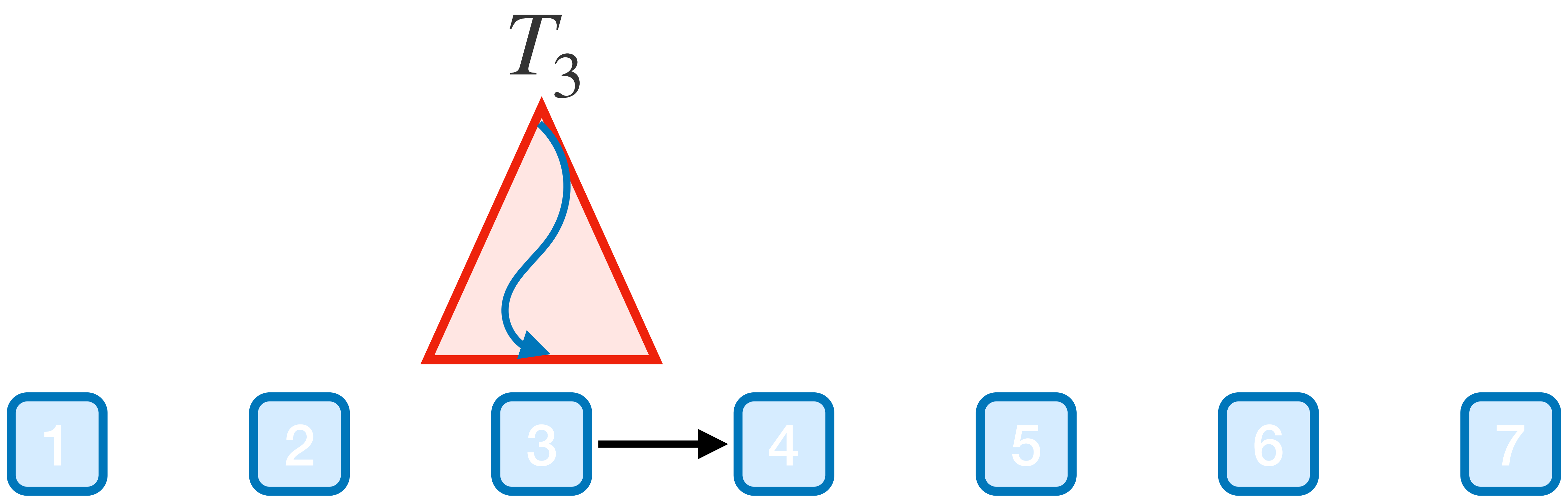

Memory

T_3

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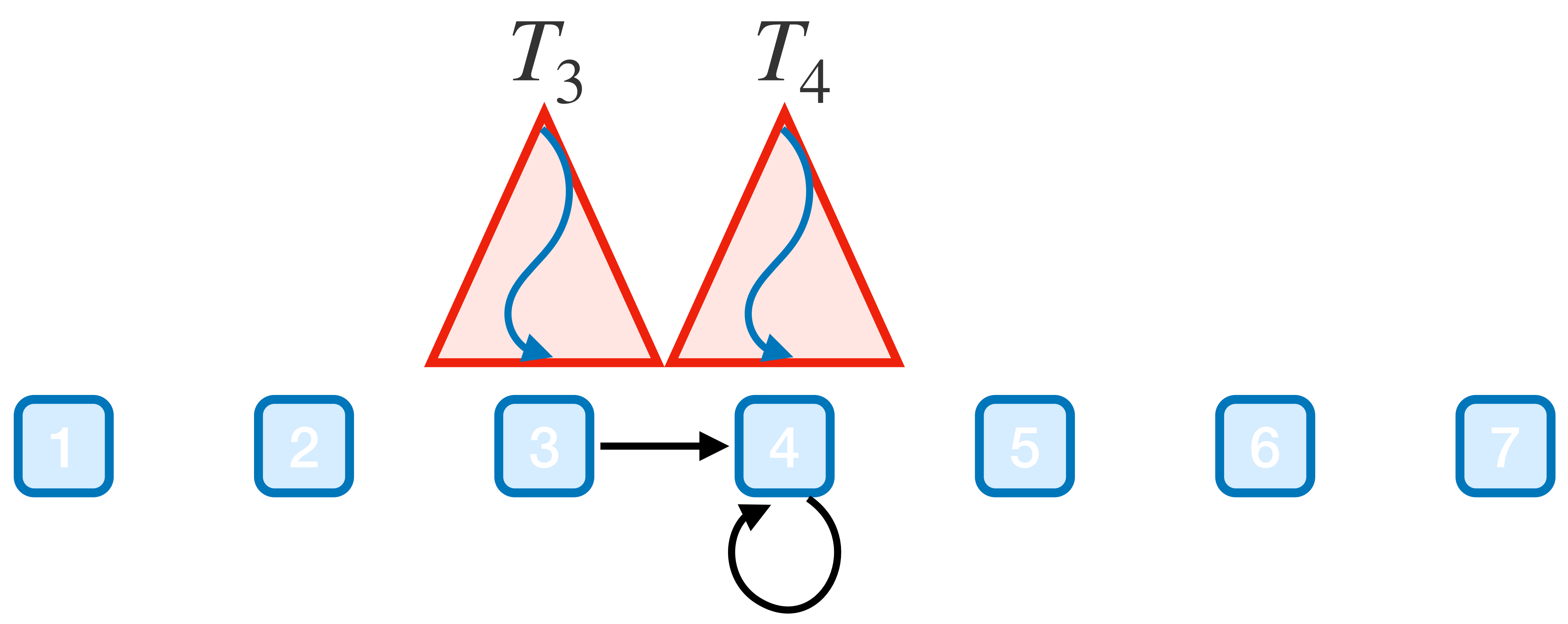

Memory
T_3
T_4

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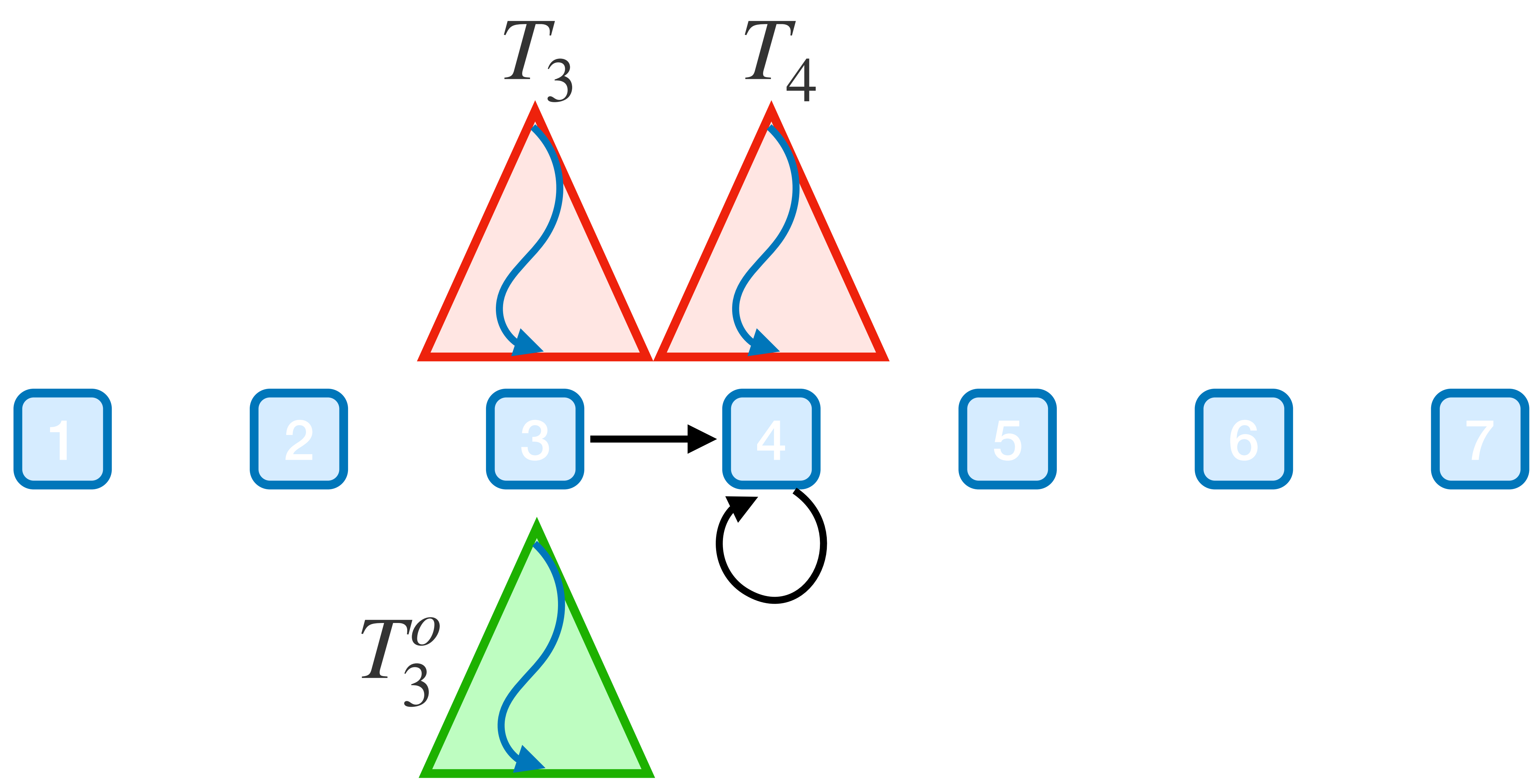

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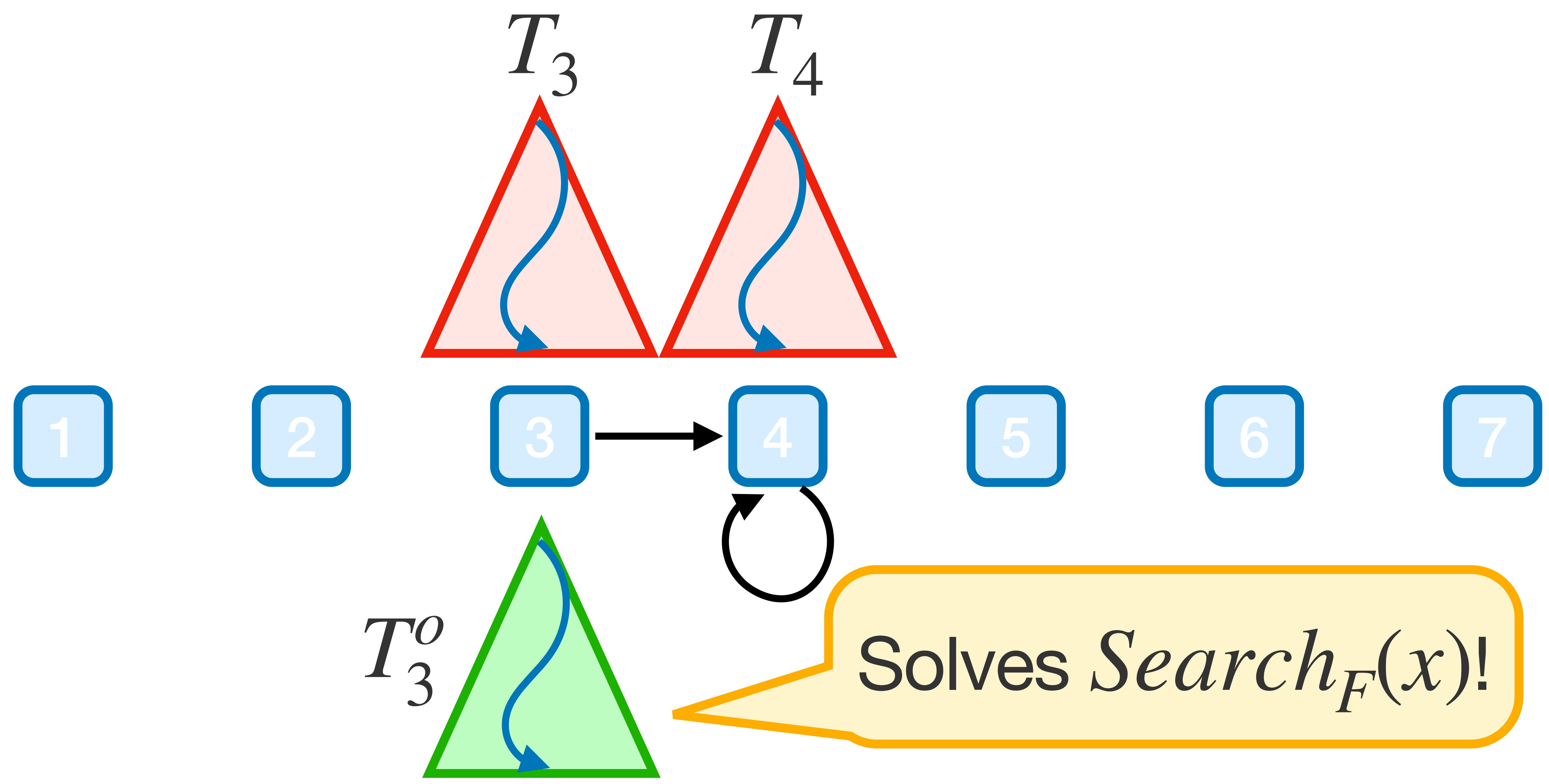
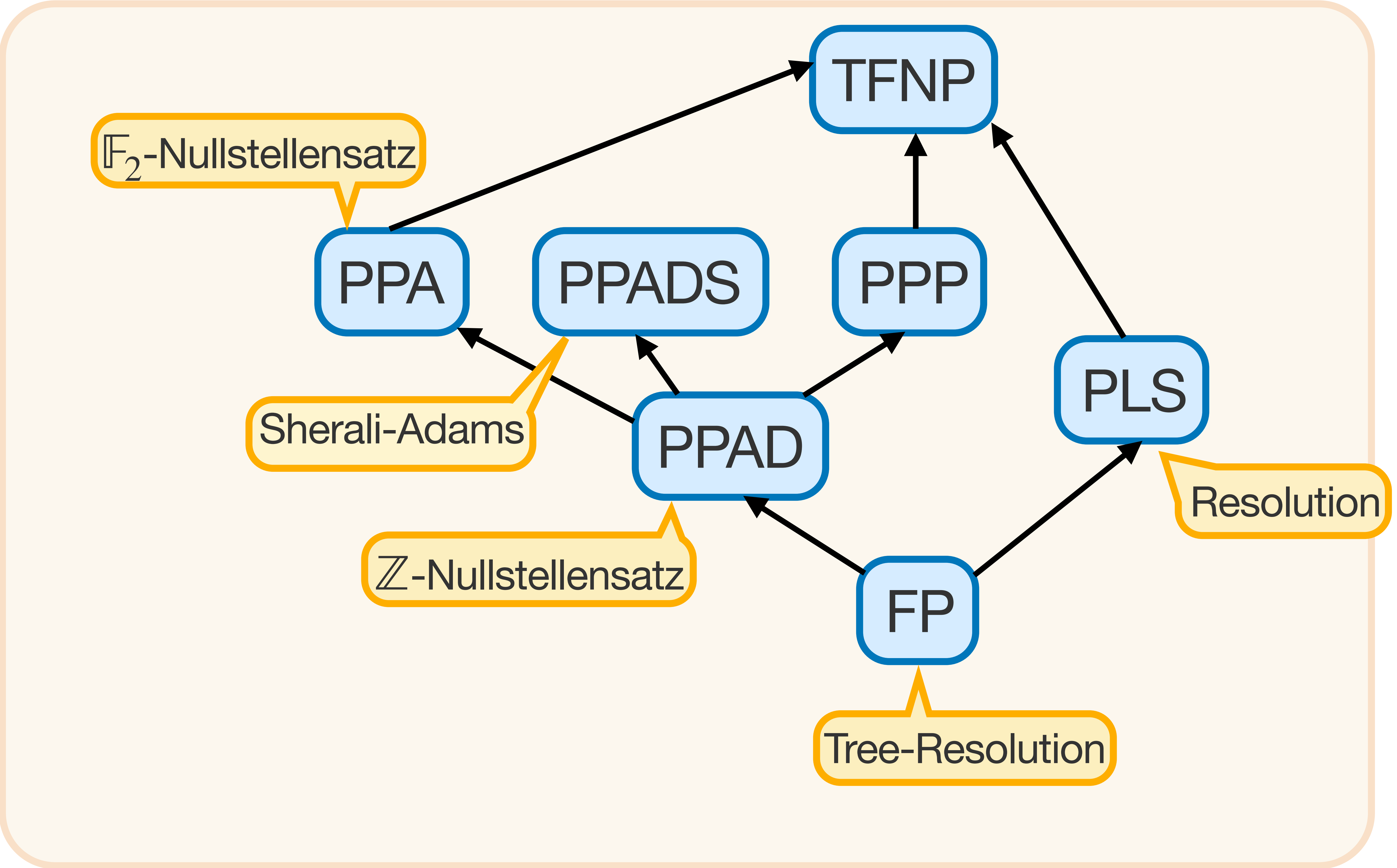


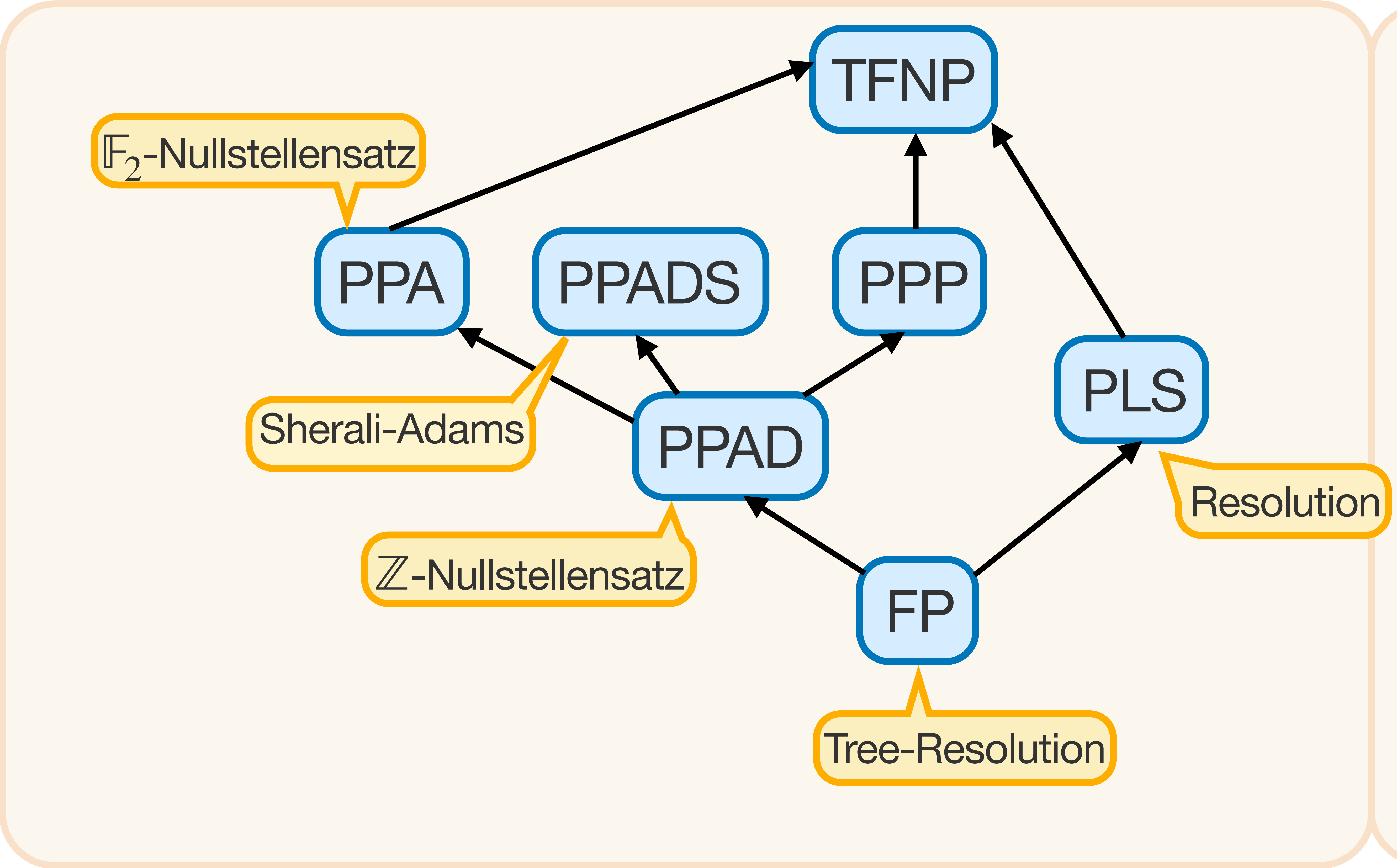
Memory
T_3
T_4
T_3^o

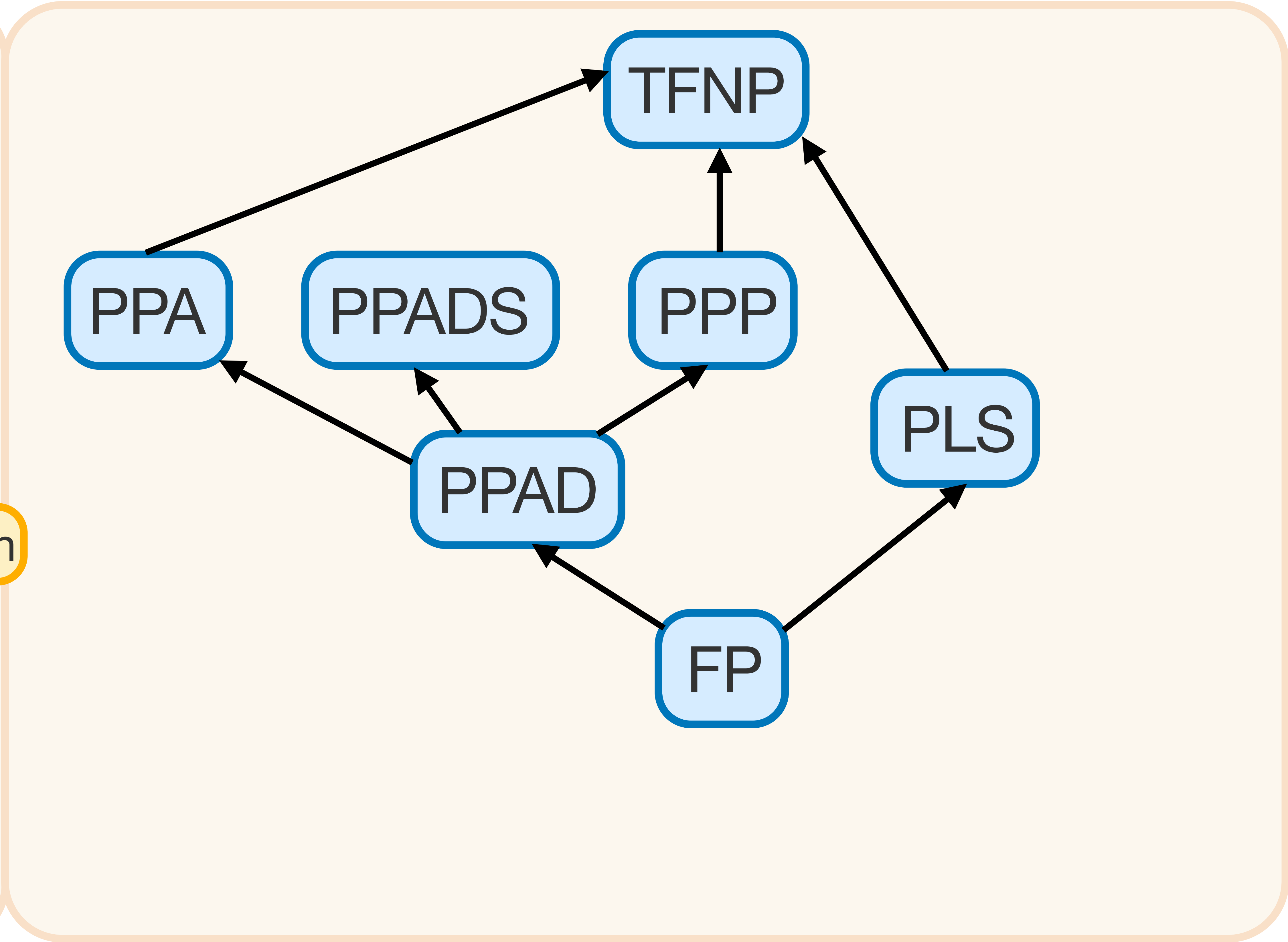
TFNP



Model of Computation: Decision Trees

TFNP

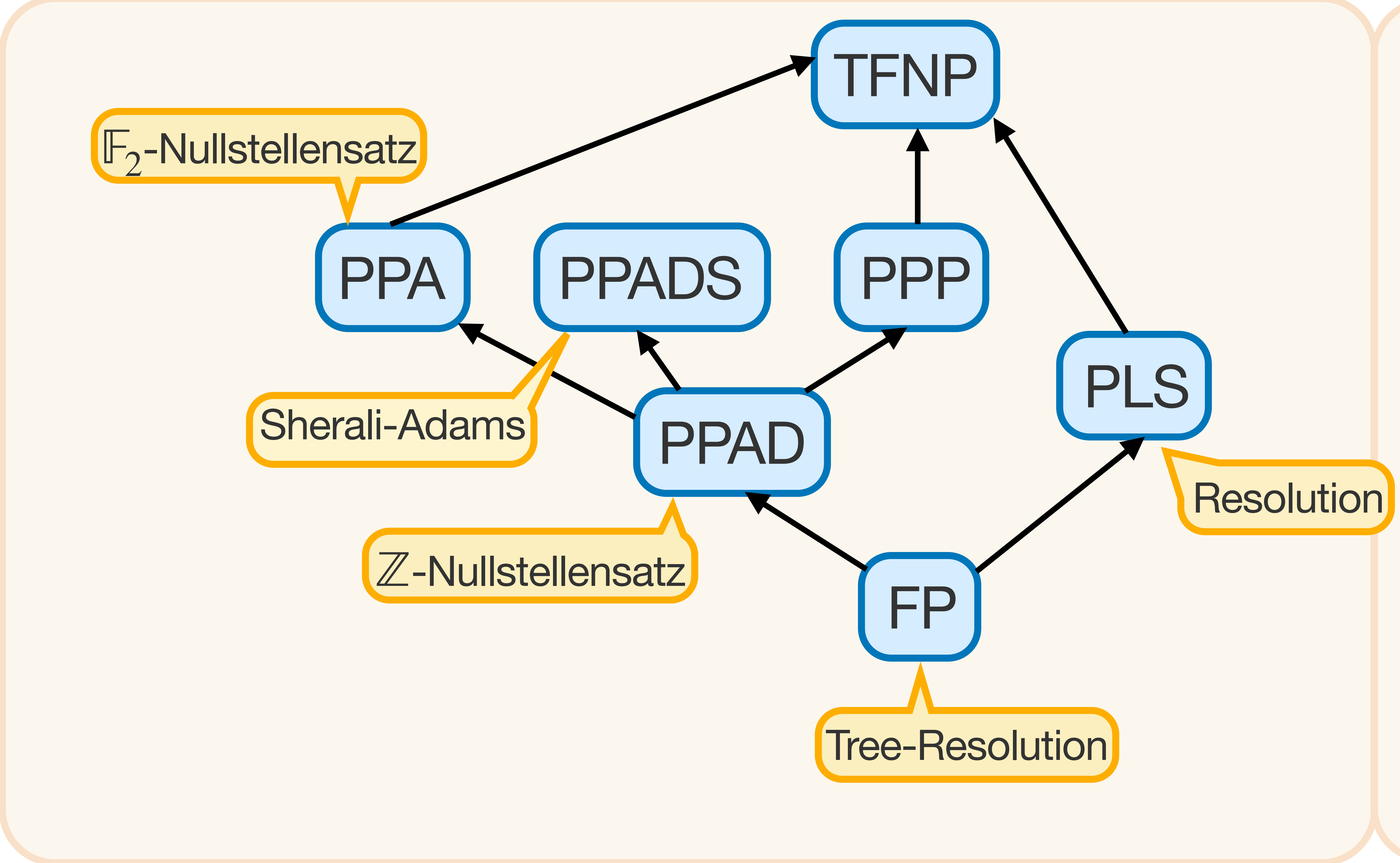


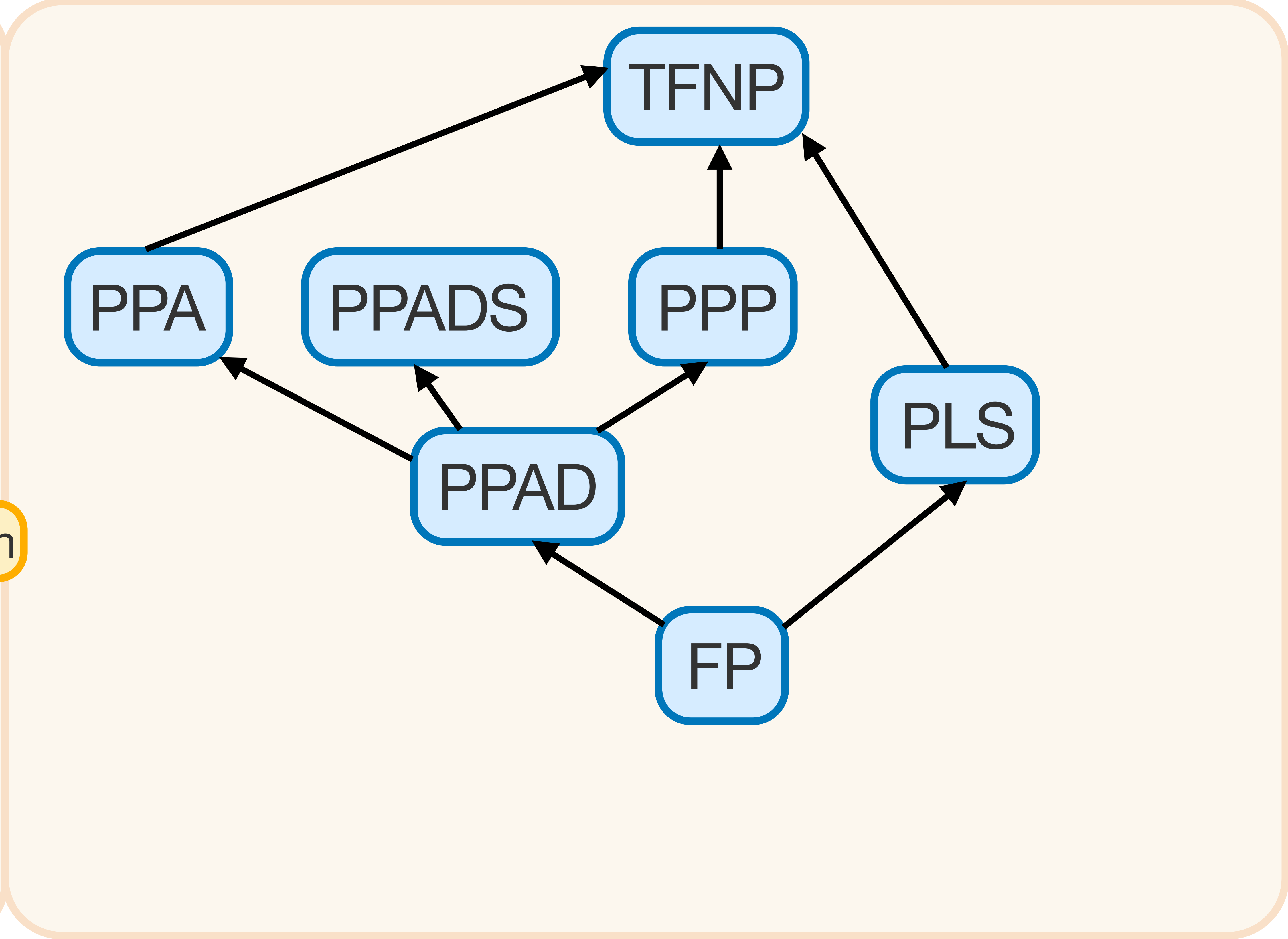


Model of Computation: Decision Trees Communication Protocols

TFNP

[GKRS18] Certain circuit models are **equivalent** to communication TFNP classes!





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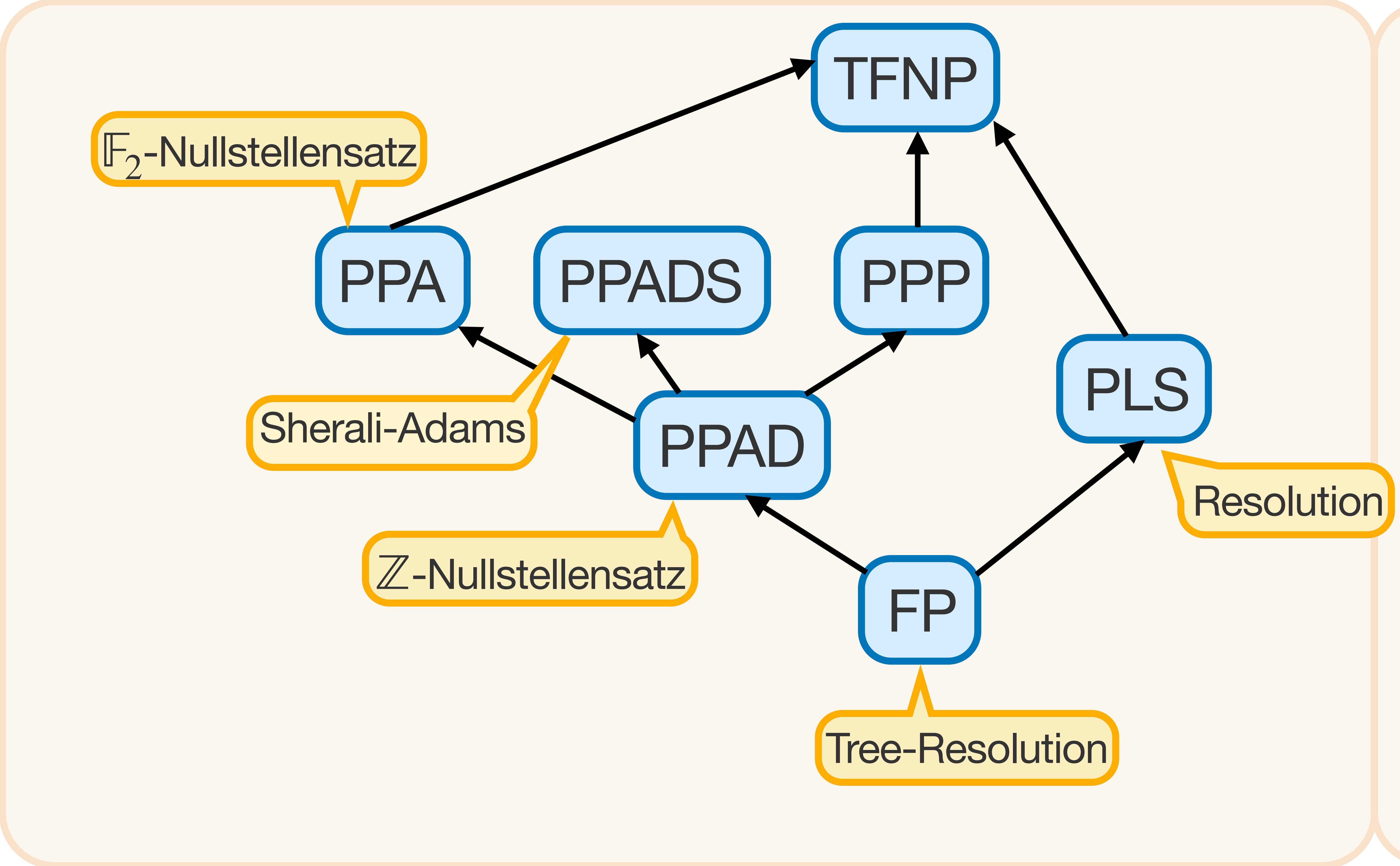

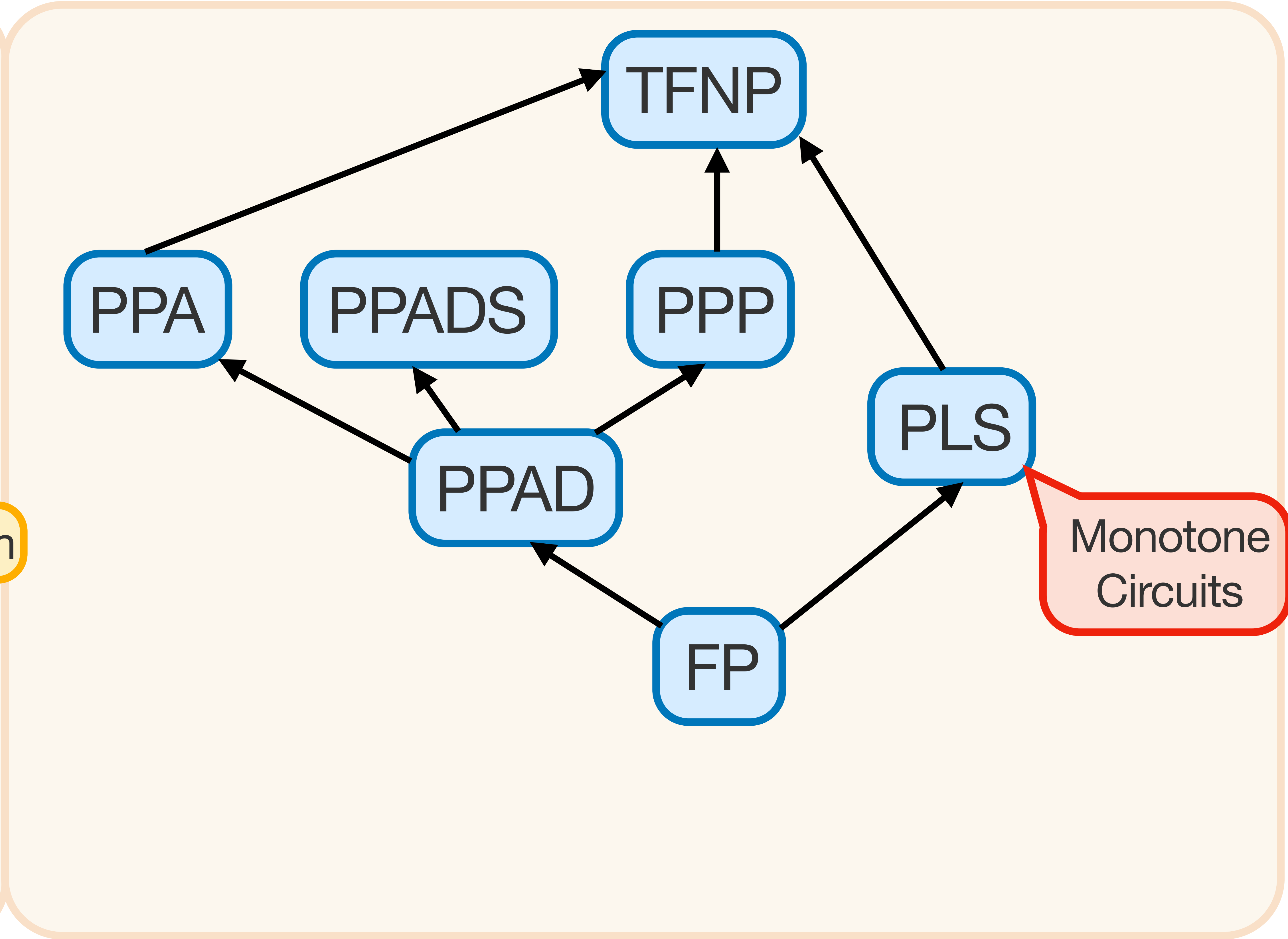

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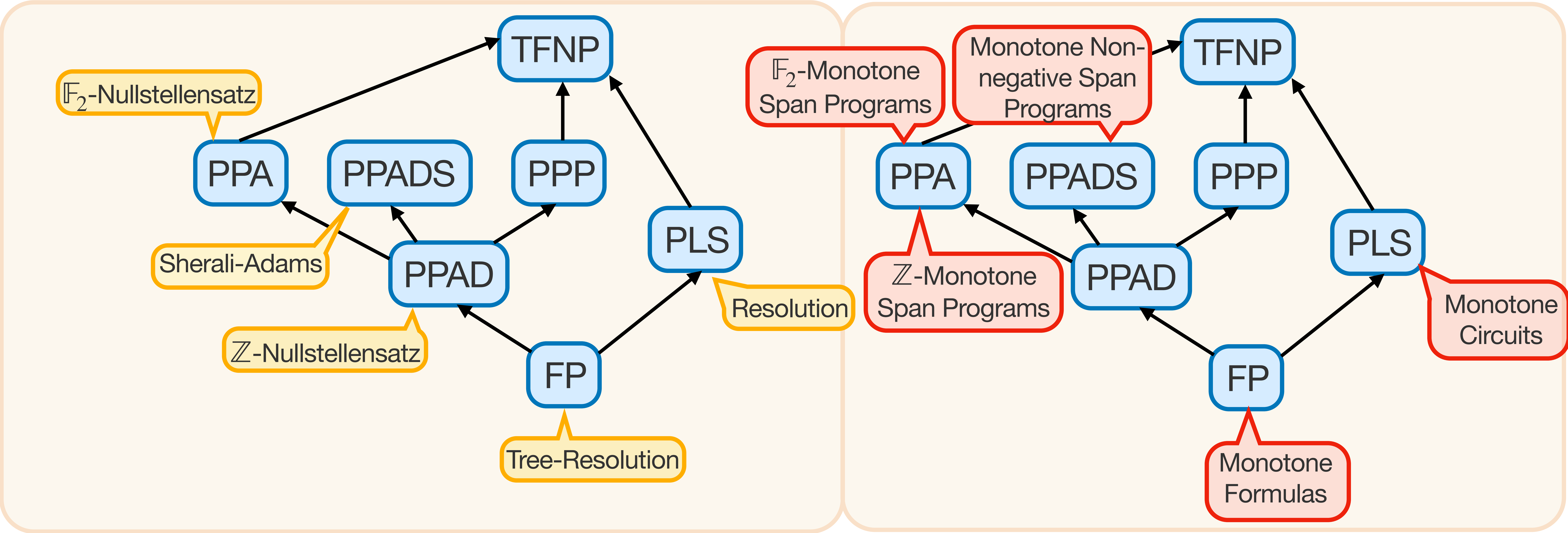

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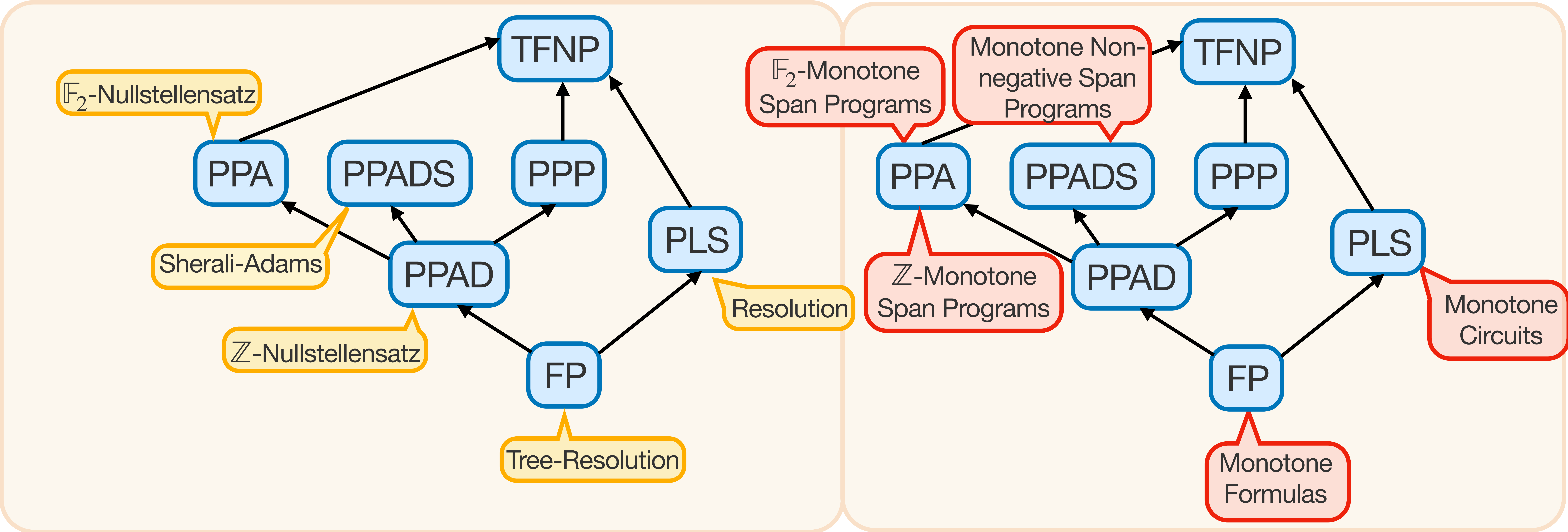
TFNP

Observation 1: When both the DT and CC versions of a TFNP class admit a characterization then we immediately get an interpolation theorem



Model of Computation: Decision Trees | Communication Protocols

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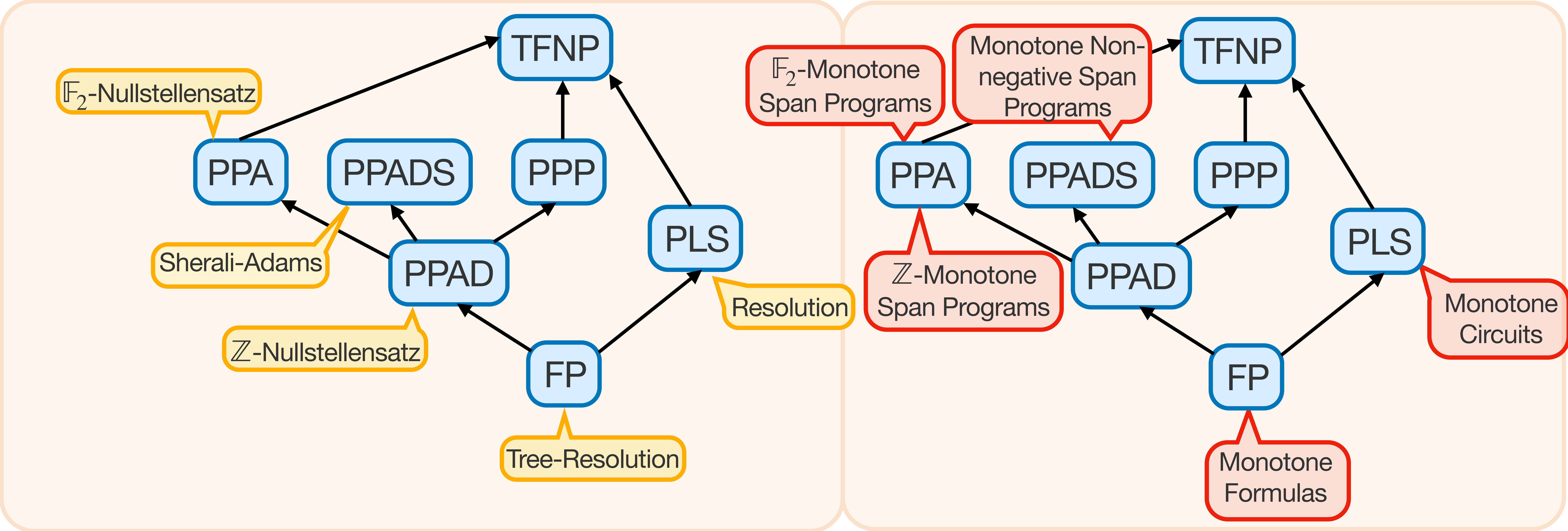

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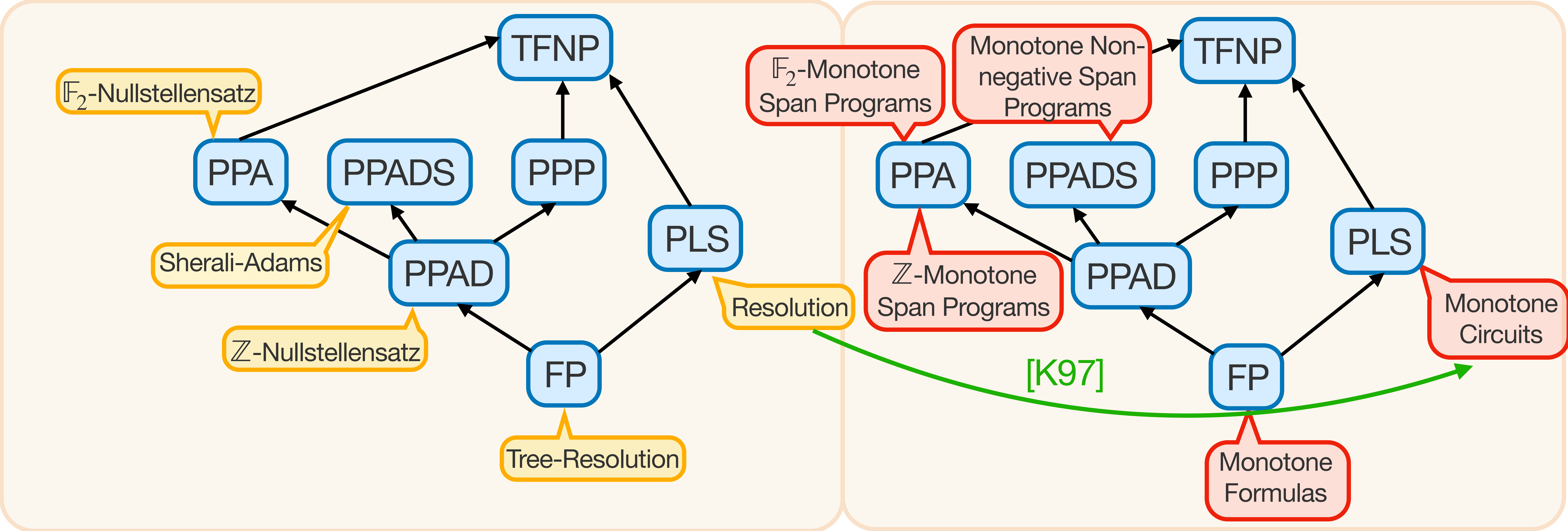

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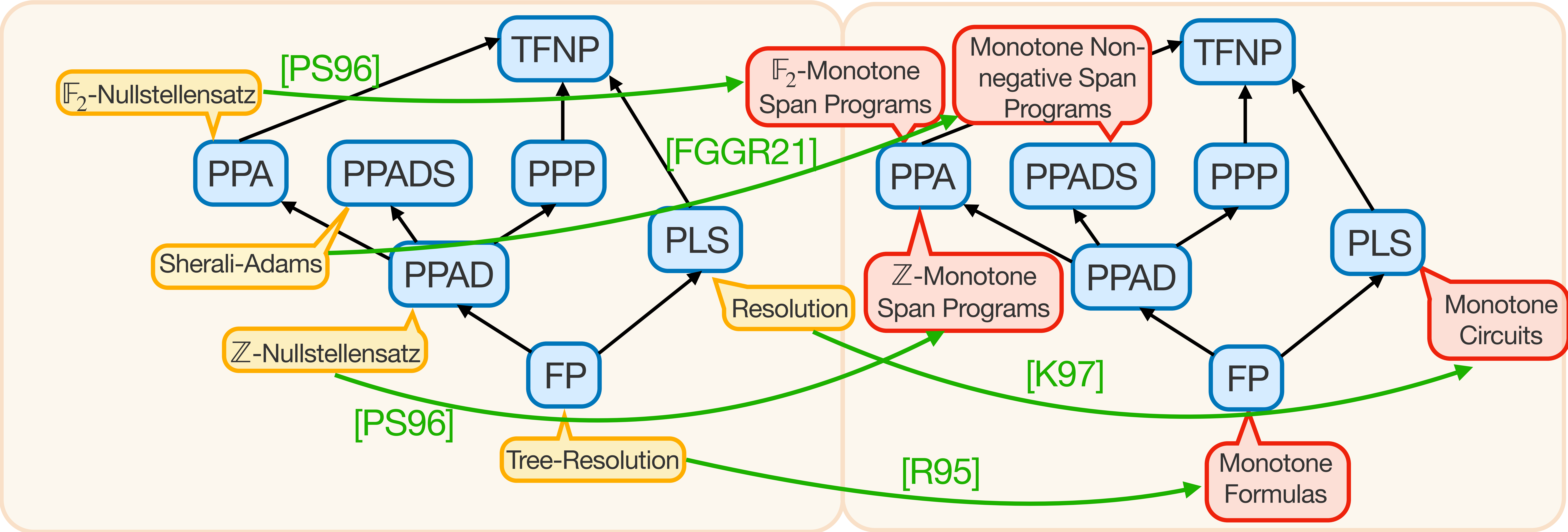

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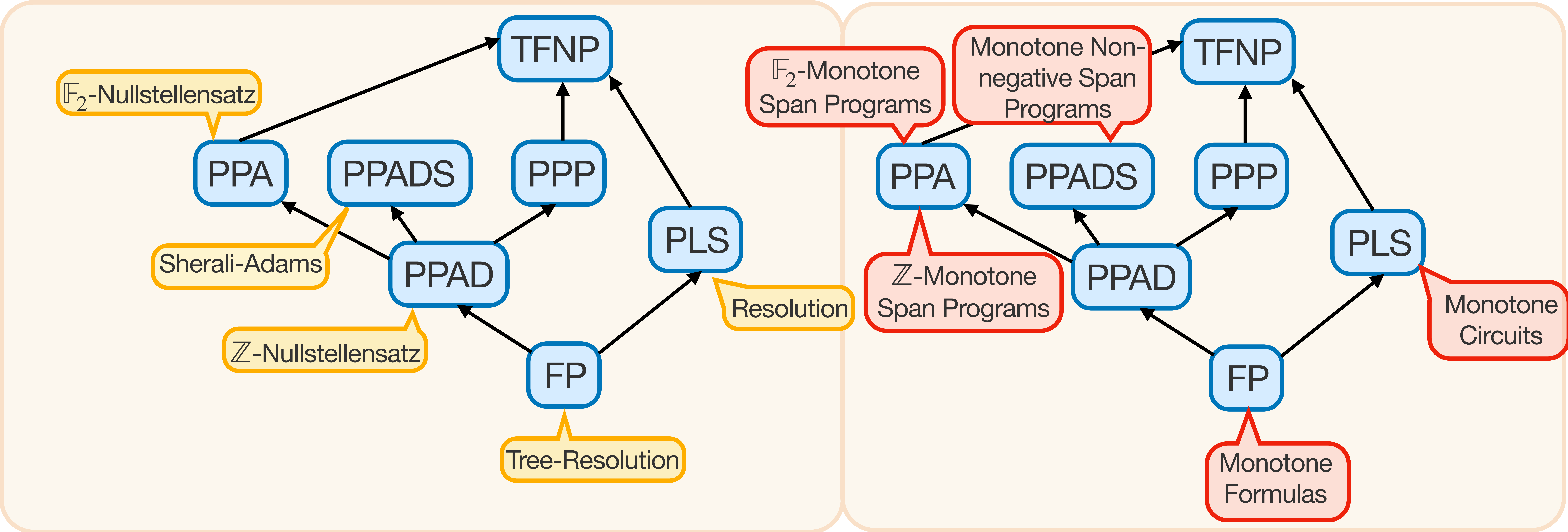

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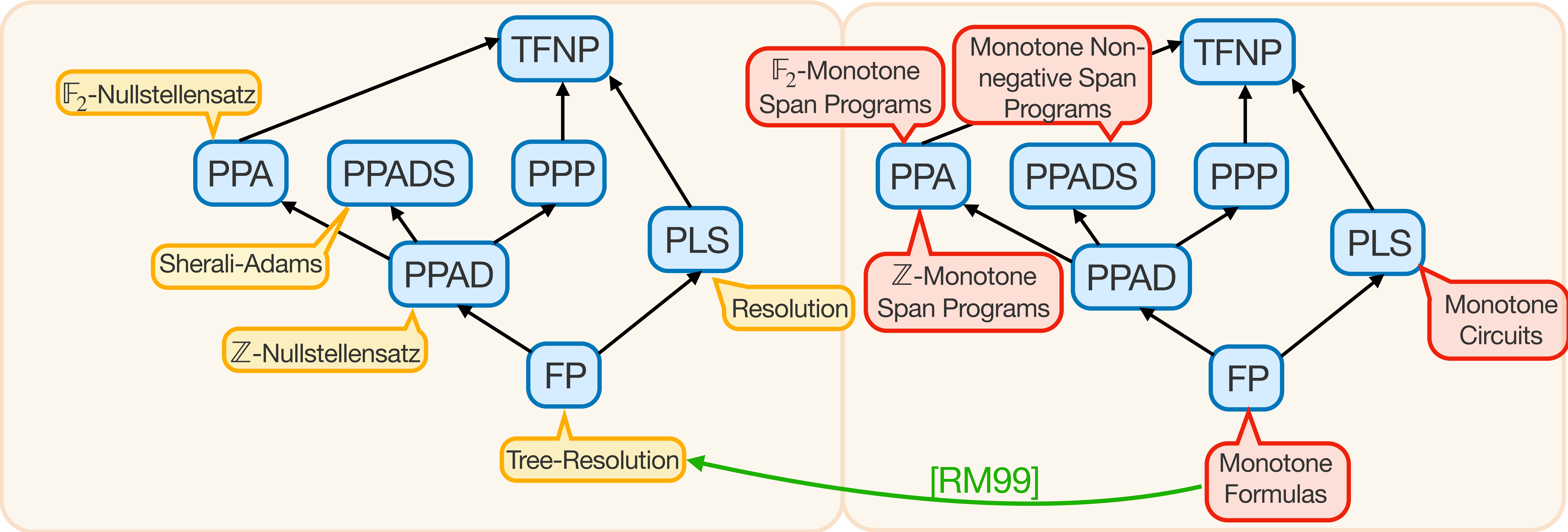

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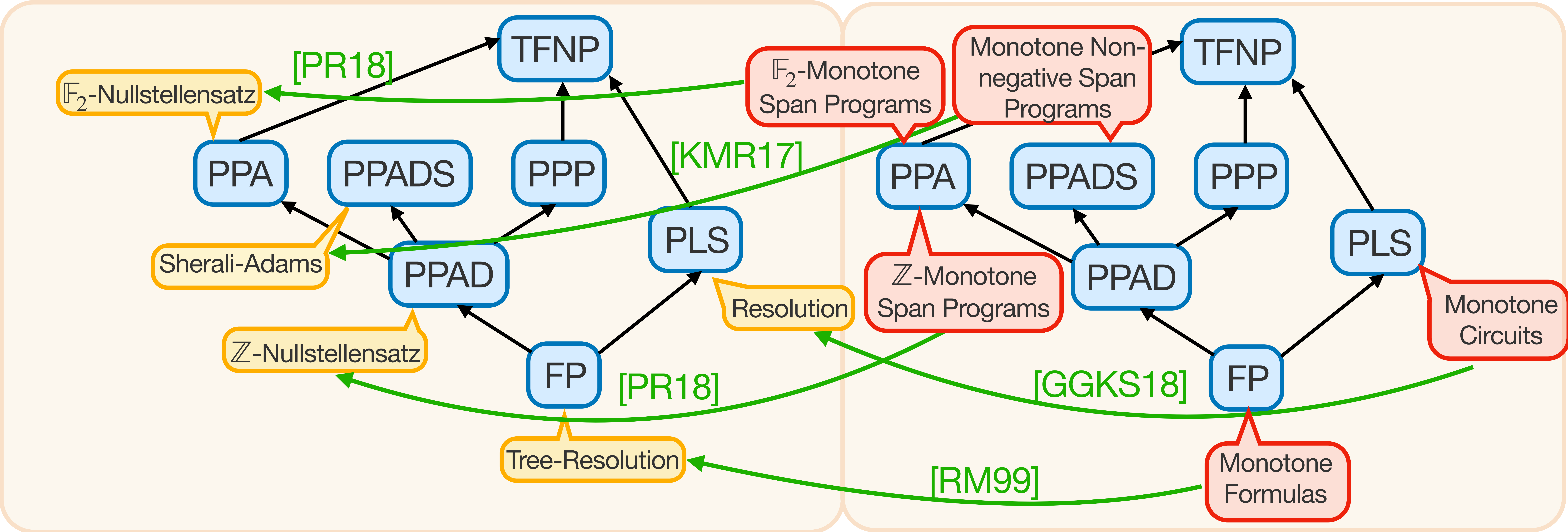

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Upshot: Understand when interpolation or query-to-communication lifting theorems occur by understanding when proof systems and monotone circuit models admit TFNP characterizations!

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Canonical proof system for C

Fix H such that $Search_H$ is equivalent to the complete problem for C

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Cook-Reckhow proof system — proofs are verifiable!

→ Just check that $(n', \{T_i\}, \{T_j^o\})$ describes a valid reduction!

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Standard proof systems satisfy this — e.g., Resolution, Sherali-Adams, Nullstellensatz…

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→ Each generates a TFNP class as everything reducible to $Search_{Ref_P}$

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Π is low complexity $\implies$ number of variables of Ref_P instance is not much more than that of F

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Corollary: A proof system admits a TFNP^{dt} characterization iff it is closed under decision tree reductions and has short proofs of a reflection principle about itself.

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A monotone function F such that

1. for any partial function g:
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Open Problem

Q. A generic lifting theorem?

A circuit and proof system characterization of a TFNP class immediately implies an interpolation theorem. Does the same hold for lifting theorems?