Communication Complexity, Streaming and Computational Assumptions

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What Effect Do Crypto Assumptions have on Algorithms

Choose a setting where **randomness** helps

- Show a good algorithm against an **inactive/static** adversary
- Show what an **active/adaptive** adversary can do
- Discuss whether **crypto** can help
  - **And if it can help, show that the tools are essential**

Repeat

Minimal Assumptions

Can we **automate** the process?

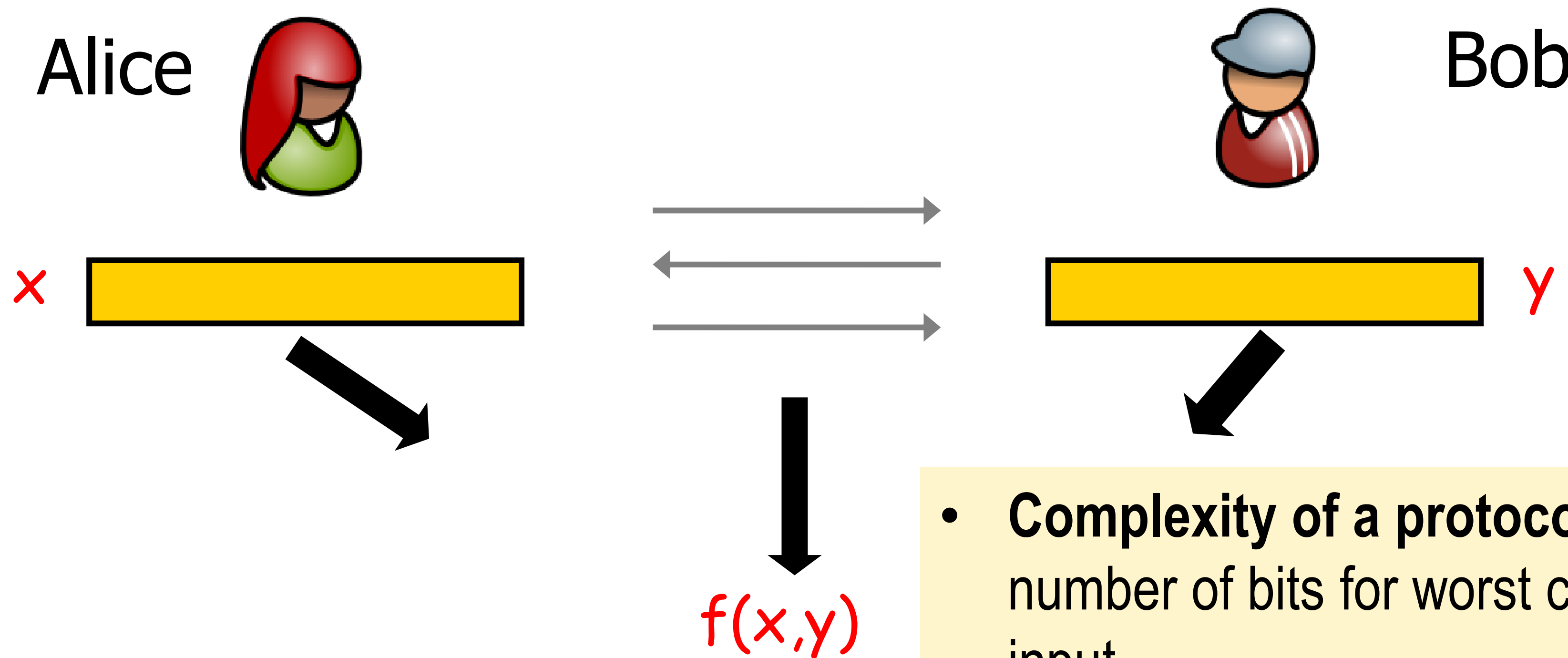
Other Examples

- **Sketching**, Mironov, Naor and Segev 2008
- **Error correction**, Lipton, Micali-Peikert-Sudan-Wilson, Grossman-Holmgren-Yogev
- **Communication vs. Computation,** Harsha, Ishai, Kilian, Nissim and Venkatesh
- **Lower Bound for Checking Correctness of Memories,** Naor and Rothblum 2005
- **Adversarially Robust Bloom Filters,** Naor-Yogev 2015
  - Bet-or-Pass TCC 2022 - Noa Oved
  - Defining the success of an Adversary with adaptive choices
- **Adversarially Robust Property Preserving Hash Functions,** Boyle, LaVigne and Vaikuntanathan

What Will We See (Time Permits…)

- **Communication Complexity**, Crypto 2022 –Shahar Cohen
  - Low Communication Complexity Protocols, Collision Resistant Hash Functions and Secret Key-Agreement Protocols
- Streaming (**card guessing**), ITCS 2022 - Boaz Menuhin
  - **Mirror Games**, FUN 2022 - Roey Magen
  - **WIP: Low Memory Permutation Generation**

Communication Complexity



- **Complexity of a protocol**: number of bits for worst case input
- **Complexity of a function**: complexity of best protocol

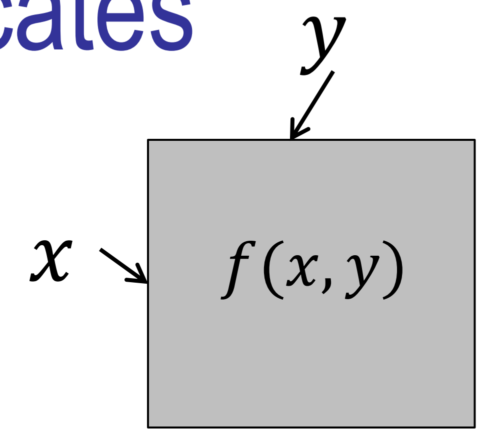
Let $f: X \times Y \mapsto Z$

Input is split between two participants

Want to compute: $z=f(x,y)$

while exchanging as few bits as possible

Equality and Other Predicates



- Our canonical example – **equality**.
  - $f(x,y) = 1$ iff $x = y$
- A non-trivial predicate: with no redundant rows and columns
  - No two rows or two columns are **identical**

Efficiently Separable Predicate:

- There is an efficient algorithm that given

$$x_1, x_2 \in X$$

finds y s.t. $f(x_1, y) \neq f(x_2, y)$

Communication Complexity Protocol Variants

Deterministic complexity is often n
- Example: equality

Protocols differ by
- Network layout
  - Who talk to who and number of rounds
    - Interactive Model
    - Simultaneous Message Model
- Use of Randomness
  - **Shared** public randomness
    - Independent of the inputs
  - Private Randomness

Newman: largest possible gap

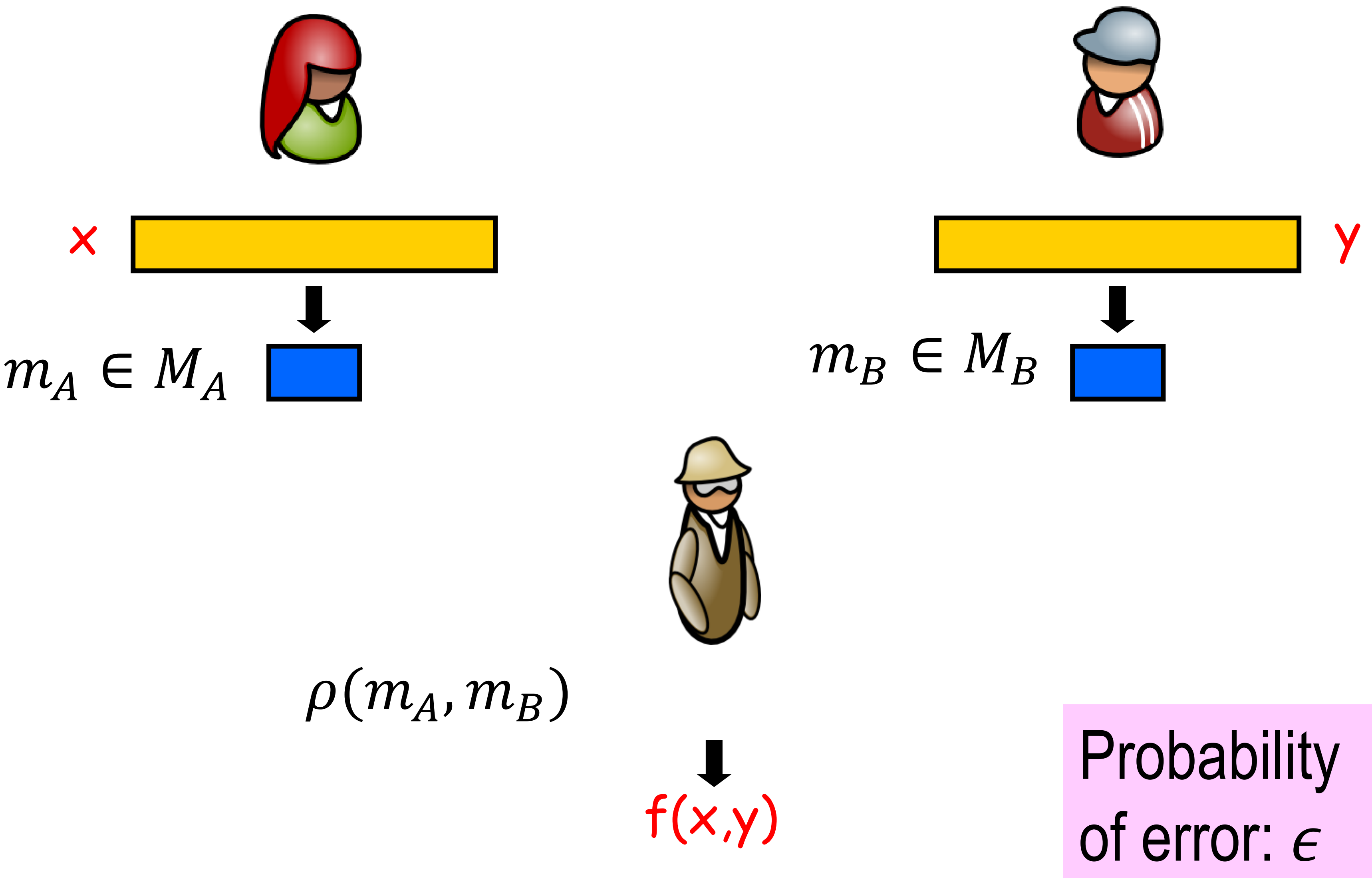
•Orthogonal!

Equality function Interactive
- Shared Randomness $O(1)$
- Private Randomness $\Theta(\log n)$

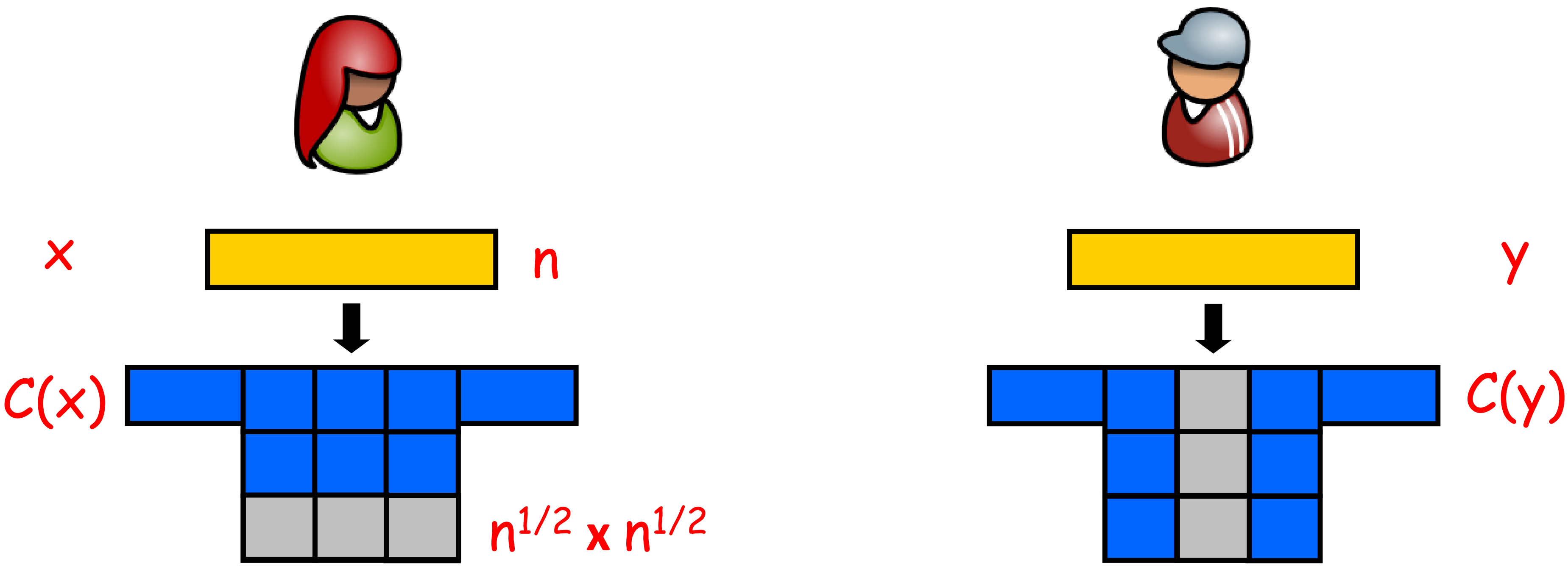
No function is $o(\log n)$ with private randomness

First proof: Ben-Sasson-Maor

Simultaneous Messages Model



Simultaneous Equality Testing



C should be a good error correcting code

Communication $O(n^{1/2})$

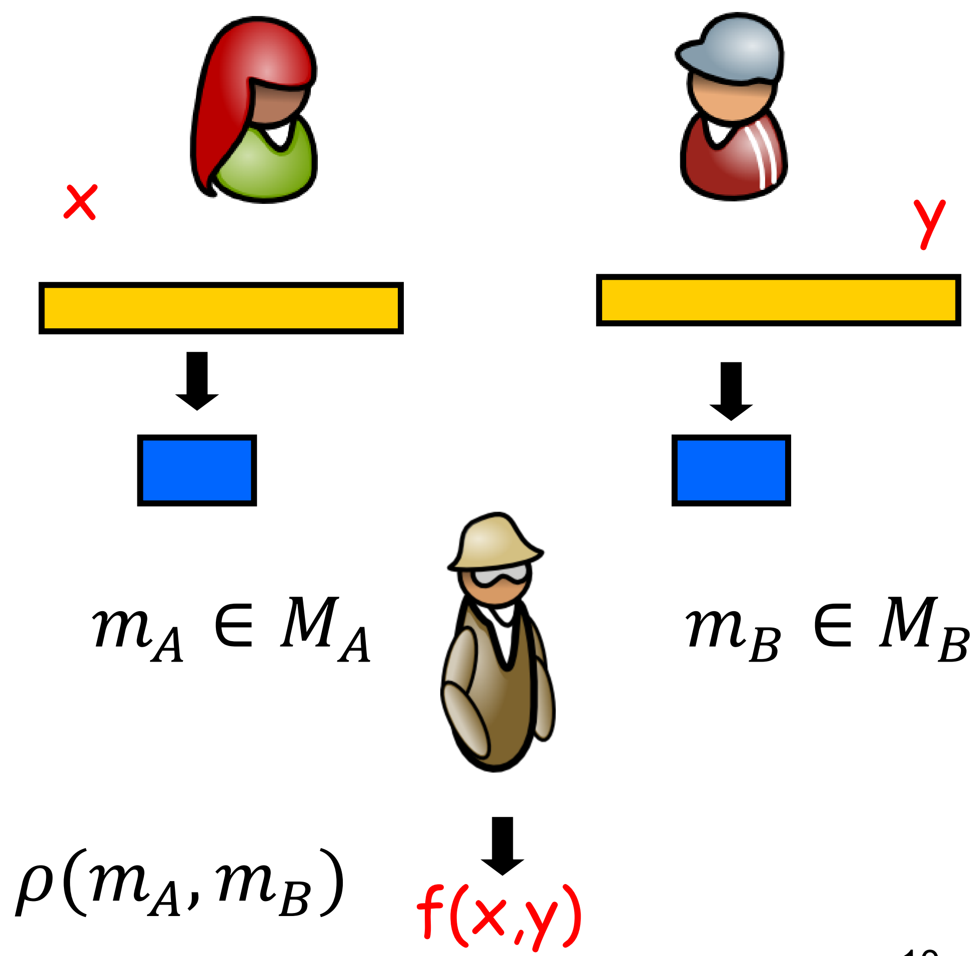
Simultaneous Messages Model Lower Bound

- Newman-Segedy 96
  $|m_A| + |m_B| = \sqrt{n}$
- Babai-Kimmel 97
  $|m_A| \cdot |m_B| = n$

In general:

Deterministic complexity

- Bottesch, Gavinsky, and Klauck 2015



Central Question

- Can we reduce communication complexity by **assuming certain hardness assumptions**
  - What assumptions do we need?
- What changes to the model do we need to make?

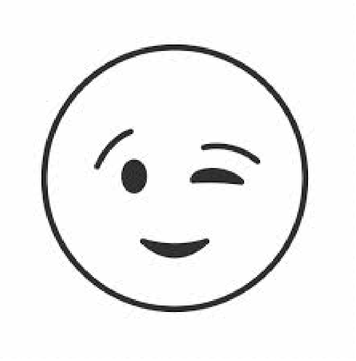
- **When** is the randomness chosen
- **Who** maintains state
- The **exact power** of the adversary

Models
•Preset Randomness
•Free talk stateful

Results

Almost Tight bounds on communication complexity, assumptions and models

When you close one eye



Results: preset randomness

- Breaking the $\sqrt{n}$ lower bound for equality in the **simultaneous message** model implies the existence of **distributional Collision Resistant Hash** (dCRH) functions in a constructive manner
- Dito for the $\log n$ bound in **interactive communication**
- There are **no protocols** of constant communication

Techniques employ the Babai-Kimmel Proof

- Assuming existence of CRH: can break the bounds

Collision Resistance Hash

Results: stateful ``free talk"

- Parties Alice and Bob talk freely **before the inputs are chosen by adversary**
  - May maintain secret states τ_A and τ_B ***respectively***
  - The communication is measured only after the preprocessing

Very efficient protocols for equality against a **rushing** adversary imply the existence of **secret-key agreement protocols**

- Assuming that for a c bit protocol the probability of error is at most $2^{-0.7c}$

Assuming SKA exist: there is a c bit protocol with error probability 2^{-c}

Assumptions in cryptography

Minicrypt

Oracle Separation

- One-way functions
  - Existentially equivalent to a whole host applications such a private key encryption
- Collision resistance Hash Function
- Secret-key Agreement.
  - Implied by Public-key encryption

- Separating OWFs from CRHs: consider a collision finder: Given a **collision finder**, OWFs do exist but CRHs do not exist
- Separating SKAs from CRHs: In the random oracle model CRHs do exist but SKAs do not exist

Collision Resistance Hash Functions

CRH

A family of hash functions H where it is **hard to find any collision**

- All functions $h \in H$ are compressing
- Efficiently computable
  - Given $h \in H$ and x

easy to evaluate $h(x)$

- Hard to find collisions: for every PPT Adv, and large enough λ, for a random $h \in_R H$

Probability $Adv(h)$ finds $x \neq x'$ s.t. $h(x) = h(x')$ is negligible in security parameter λ

Simon 98….:
- Black box separation from one-way functions

Random Collision finder

If can compress by a little –
Can compress by a lot

Distributional Collision Resistance Hash

dCRH

Dubrov and Ishai 06. Bitansky, Haitner, Komargodski and Yogev 19

Constant-round statistically hiding commitment schemes

A family of hash functions H where it is hard to find a **random** collision

Simon 98….:
- Black box separation from one-way functions

Random Collision finder

Random Collision finder **COL**

- **COL** gets $h \in H$ and outputs (x, x') s.t. x is uniformly random and x' is uniformly random from $h^{-1}(x)$
- H is a family of **distributional CRHs** if there exists poly $p(\cdot)$ s.t. for every PPT Adv, and large enough λ, for a random $h \in_R H$
$$\Delta(COL(h), Adv(h)) \geq 1/p(\lambda).$$

CRHs imply succinct protocols

Theorem: If CRHs exist, then given a family of CRHs

$$\{h\colon \{0,1\}^n \rightarrow \{0,1\}^\lambda\}$$

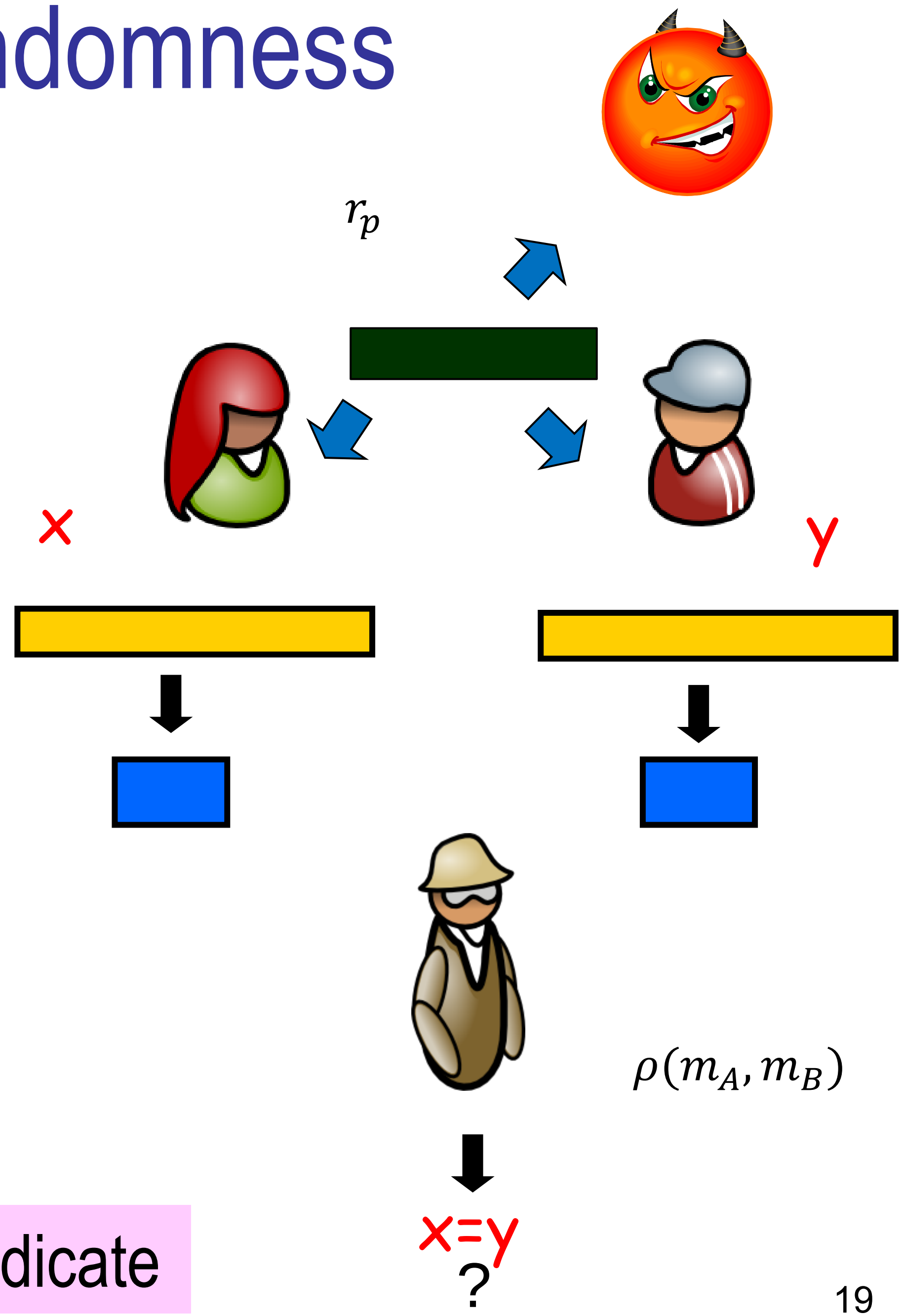
- In the preset public coins SM model: there is a protocol of complexity $O(\sqrt{\lambda})$ for the Equality predicate.
- In the preset public coins interactive model: there is a protocol of complexity $O(\log \lambda)$ for the Equality predicate.

- Public string: the hash function h
- Replace x with $h(x)$

Preset randomness

Need to show how to construct from a **succinct** protocol a **hash function**

- Inputs are chosen by the adversary depending on the public random string
- Idea: use a **characterizing multi-set** of responses as a hash function



Works for every non redundant predicate

SM Protocol Π for Equality

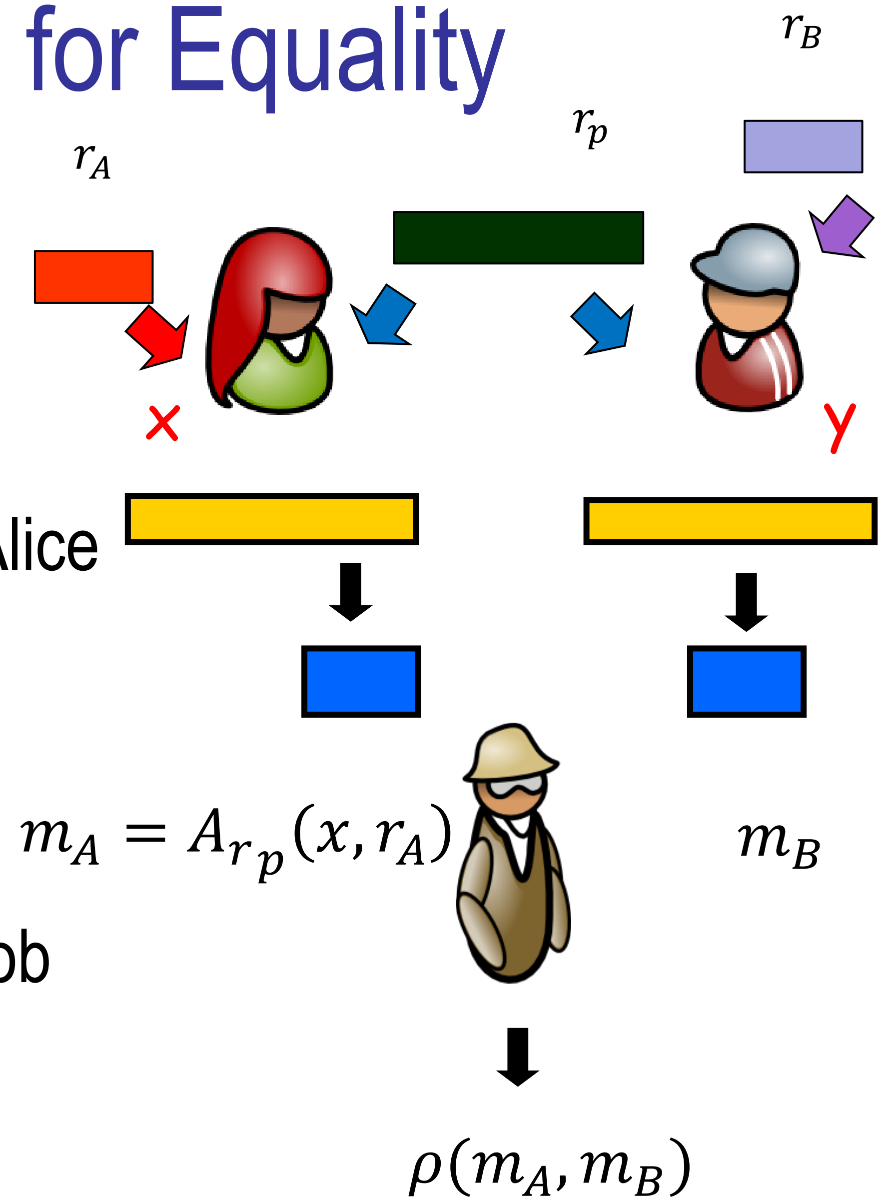
- Preset Public random string r_p
- Input space for X and Y
- Alice gets $x \in X$ and Bob $y \in Y$
- M_A and M_B message space for Alice and Bob
- Private randomness:

$$r_A \in R_A \; and \; r_B \in R_B$$

  - Random tapes for Alice and Bob
- Message Alice sends:

$$m_A = A_{r_p}(x, r_A) \in M_A$$

- Referee's Decision $\rho(m_A, m_B)$



Characterizing Multisets

input of Alice

- For every $x \in X$ there exists a multiset **characterizing** the behavior of Alice on x.
  - Instead of running Alice, can approximate the protocol's result (referee's output) by a uniform sample from the multiset.
  - Such a multiset can be found (w.h.p.) by relatively few independent samples from the distribution defined by Alice on x and r_p.

Characterizing Multisets

input of Alice

For public string r_P and input $x \in X$ a multiset of messages $T_x \subset M_A$ **characterizes** $\boldsymbol{x}$

- if $\forall\, m_B \in M_B$,

$$|Q(T_x, m_B) - Prob\left[\rho\left(A_{\mathrm{r_p}}(x, r_A), m_B\right) = 1\right]| \leq 0.1$$

over r_A

- where $Q(T_x, m_B)$ is the referee's **expected value** for the multiset T_x and Bob's message m_B.

Sampling yields characterizing multisets

Theorem:

- For any public string r_p and for and $x \in X$
- Let $r' = (r_A^1, \ldots, r_A^t)$ be t independent uniform samples from R_A where $t = \Theta(\log |M_B|)$.
- Then, for the multiset $T_x = \{A_{\mathrm{r_p}}(x, r_A^i) : i \in [t]\}$ it holds that $\boldsymbol{T_x}$ **characterizes Alice for** $\boldsymbol{x}$ with constant probability

Constructing Hash Functions From Characterizing Multisets

The function h is defined by

- The public random string r_p and
- t random tapes for Alice $r_A^1, \dots, r_A^t \in R_A$.

Output: For x ∈ X, the value of the function is the multiset

$$h(x) = \{A_{r_p}(x, r_A^i : i \in [t]\}$$

where the multiset is encoded as a sequence

$$A_{r_p}(x, r_A^1), \dots, A_{r_p}(x, r_A^t)$$

- Every message of Alice encoded using $\log|M_A| = c$ bits

The constructed function is good

- The function h is compressing

Should be characterizing to both

- Any x and x' which **share a characterizing multiset**, induce **bad inputs** for the protocol:
  Let $x, x' \in X$ and $y \in Y$ that separates them.
  If there is a multiset T that is characterizing for both x and x', then
  - the sum of the failure probability of $\pi(x, y)$ and $\pi(x', y)$ is at least 0.8.
  - At least one of them fails.

From $Adv_{collision}$ breaking h as a dCRH to Adv_{π} breaking Π

- Given an efficient adversary $Adv_{collision}$ that breaks the security of h as a **distributional CRH** for some $p \in poly(\lambda)$:

$$\Delta(Adv_{collision}(h), COL(h)) \leq \ 1/p(\lambda)$$

- Then, we can construct an adversary Adv_{π}
  - with running time of the same order as $Adv_{collision}$

that succeeds in making Π fail with probability 0.4(1-$1/p(\lambda)$)

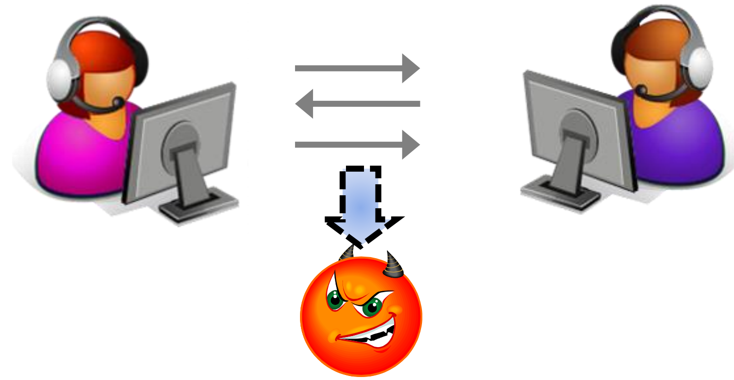
Using Collision Finder for h to Find Bad Inputs for Protocol Π

- Construct $h(x)$ using the public random string of π
- $x, x' \leftarrow Adv_{collision}(h)$.
- Find $y \in Y$ which separates x and x'
- Set Bob's input to be y and Alice input to be
  - x w.p. ½ or
  - x' w.p. ½.

Why dCRH and not CRH?
- **Not all are characterizing**

Characterize the properties of h

Stateful Free Talk



- Alice and Bob talk freely

before **the inputs are chosen by adversary**

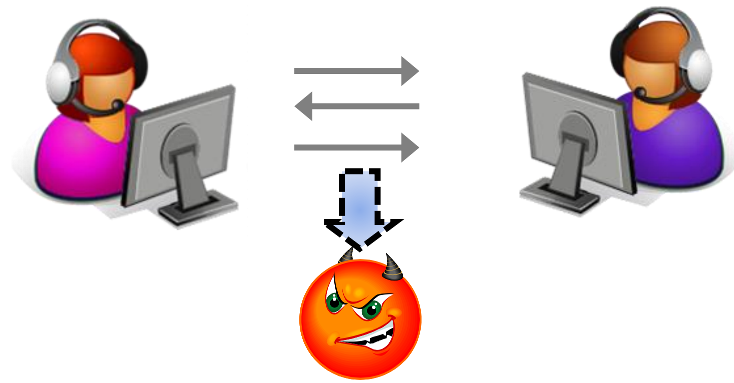
  - Maintain a secret state τ_A and τ_B
  - Adversary eavesdrops to the free talk phase and then selects inputs
- Communication is measured only **after** the free talk preprocessing phase
  - Mostly interested in SM pattern

Free Talk: Rushing Adversary

computationally bounded

- The inputs are chosen by an adversary, depending on the public discussion it witnesses in preprocessing phase.
- A **rushing adversary** can choose Bob's input at the `last moment’:
  - The adversary first chooses the input x of Alice *depending on the public random string*
  - **After** Alice sends her message m_A to the referee, the adversary chooses the input y of Bob
    - Depending on **both** the preprocessing transcript and on m_A
- **Patient adversary**: there are multiple sessions between Alice and Bob and the adversary can choose **one session to attack among them**, after seeing the message Alice sends.

Secret-Key Agreement



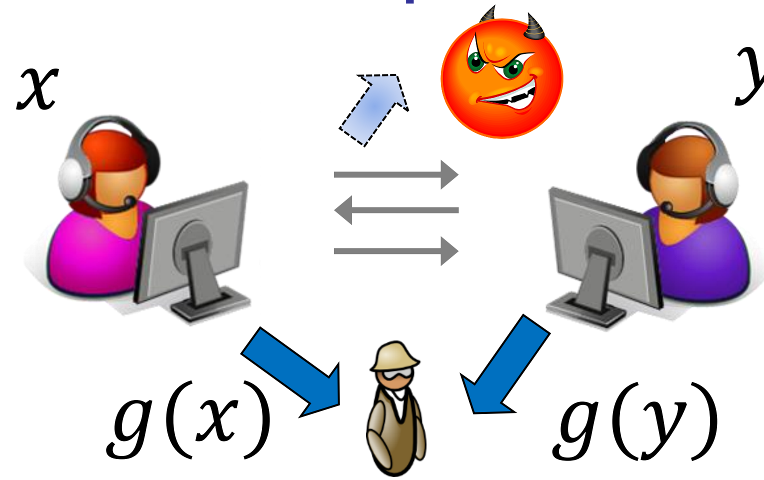
Secret key agreement (SKA)

- A protocol where two parties with **no prior common information** agree on a secret key.
- The key should be secret
  - No PPT adversary, given the transcript of the communication between Alice and Bob, can compute the key with non-negligible advantage
  - Public-key encryption implies SKA

Can “distinguish it from random”

SKA implies succinct protocol with optimal error

Execute an SKA
    Secret state is the key
Given the input use the **key** as a **pairwise ind**. hash function $g \in G$
    Send $g(x)$



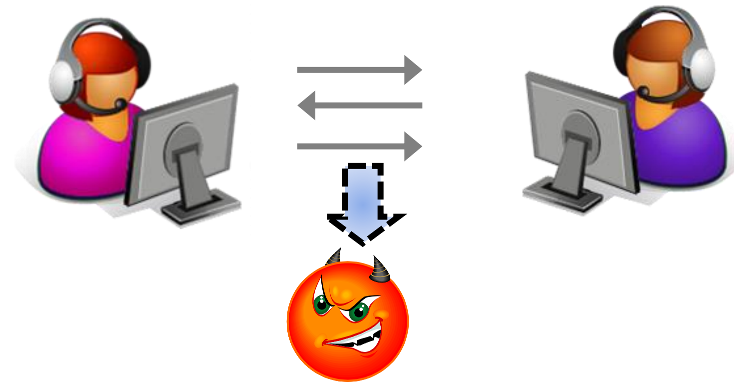
Theorem: Given a secret key agreement protocol there is in the

- Stateful preset public coins
- SM with free talk model:
- For any c(n),

a protocol for equality of complexity c(n), **where any adversary can cause an incorrect answer** with prob. at most $2^{-c} + negl(n)$

- **Even a rushing one**
- **Even a patient one**

Secret-Bit Agreement - Quantification



(α, β)-Secret bit agreement (SBA)

- The secret is one bit.
  - The two parties output b and b'.
- With probability at least (1+α)/2

$$b = b'$$

For α and β which are

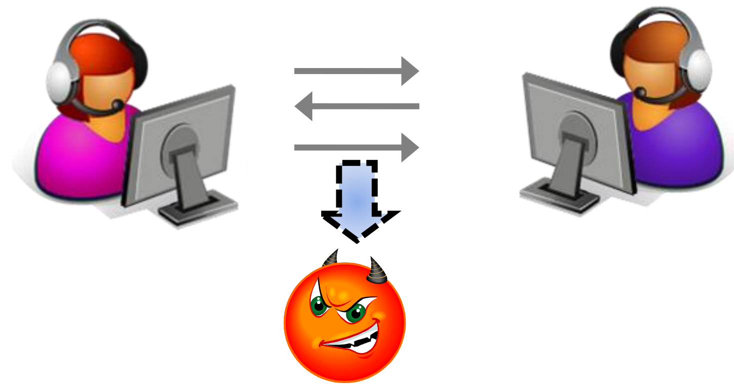
$$\mathbf{1} - \boldsymbol{negl}(\boldsymbol{\lambda})$$

we get SKA

- **Secrecy**: no PPT Adv which gets as input the transcript guesses the agreed bit given $b = b'$ with probability great than $1 - \frac{\beta}{2}$

$$Prob[Adv(\tau) = b | b = b'] \leq 1 - \frac{\beta}{2}$$

Secret-Key Agreement: Amplification



Holenstein 2006

Given an (α, β)-Secret bit agreement (SBA) where

$$\frac{1-\alpha}{1+\alpha} \leq \beta$$

- Can construct a computationally secure SKA
  - where α' **and** β' are $1 - negl(\lambda)$
- The time is $poly(\lambda)$

Succinct stateful free talk implies SKA

- An SM protocol with stateful free talk for equality of complexity $c(n) \in O(\log \log n)$ that is
  - **ε-secure** with $\varepsilon \leq 2^{-0.7c(n)}$
  - Immune to rushing and patient adversaries

  implies the existence of **secret key-agreement** protocols.

- The protocol should be *nearly* optimal in error

Protocol Π for Equality

Structure of Protocol Π :

- Alice and Bob communicate and generate secrets states
  - τ_A for Alice
  - τ_B for Bob
- On inputs x and y respectively
  - Alice sends $m_A = A(x, \tau_A)$
  - Bob sends $m_B = A(y, \tau_B)$
- Result is $\rho(m_A, m_B)$

Weak Bit Agreement from Protocol Π for Equality

- Alice and Bob communicate and toss coins according to the **free talk** phase of protocol π
  - to generate their secret states τ_A and τ_B.
- Alice selects at random a bit $b \in_R \{0,1\}$ and uniformly random inputs $x_0, x_1 \in_R \{0,1\}^n$.
- Alice evaluates $m_A = A(x_b, \tau_A)$
  - A message of the protocol Π for EQ(·, ·).
- Alice sends to Bob the pair (m_A, x_1).
- Bob evaluates $m_B = B(x_1, \tau_B)$.
- Alice outputs b and Bob outputs $b' = \rho(m_A, m_B)$

Referee's response

The SBA protocol is sufficiently good

Theorem:

The Algorithm is an ($\alpha = 1 - 2^{-\frac{c}{2}}, \beta = 2^{-\frac{c}{2}}$)-SBA protocol.

Agreement:

By the fact that the error $\epsilon \le 2^{-0.7c}$

$$\Pr[b = b'] \ \ge \ 1 - 2^{-0.7c}$$

Secrecy: construct an adversary $\boldsymbol{Adv_{eq}}$ from adversary $\boldsymbol{Adv_{sba}}$ breaking the SBA with above parameters

ADV_{Eq} from ADV_{SBA}

Algorithm for Finding Bad Inputs Using $\boldsymbol{Adv_{sba}}$

Repeat at most $6 \cdot 2^{c+1}$ times:

- Select uniformly at random $x \in \{0,1\}^n$ and set it as Alice's input.
  - Let Alice's message be $m_A \in M_A$.
- Select uniformly at random $x' \in \{0,1\}^n$.
- If $Adv_{sba}(x, m_A) = 1$ **and** $Adv_{sba}(x', m_A) = 1$:
  - Pass m_A to the referee and set Bob's input to
    - $y = x$ w.p. ½ or
    - $y = x'$ w.p. ½.
  - Otherwise, continue to the next session

Does not distinguish x and x'

Analysis of Algorithm

Guessing b when it is equal to b'

Given $\boldsymbol{Adv_{sba}}$ with success probability at least $\frac{2^{c/2}-1}{2^{c/2}}$,

we can construct an adversary $\boldsymbol{Adv_{eq}}$ with running time O(2^{c+1})

s.t.

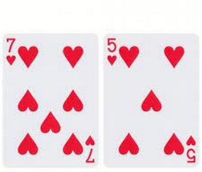
$$\text{Prob}\left[\Pi \text{ fails on inputs chosen by } \boldsymbol{Adv_{eq}}\right] > 2^{-0.7c} \geq \epsilon.$$

Further Research

- Are CRHs equivalent to preset public coins SM protocols of complexity $o(\sqrt{n})$
  - Can we break that bound using a primitive weaker than CRHs. What property do the functions we construct satisfy?
- Multi CRHs (MCRH): For $k \geq 3$, finding a k-collision of size is hard
  - Construct MCRHs from succinct protocols in a black-box manner?
- Free-talk to SKA
  - What about protocols with much worse error probability
    - Constant error probability for c which O(log log λ)
  - Do we need a rushing adversary?
- What about Rushing in the preset model? Do sublinear protocols imply (d)CRH?

Hard to Guess Permutations

- Card Guessing with Limited Memory [Menuhin Naor]
  - The Power of Adaptive Adversaries in Streams



- Mirror Games
  - Garg Schneider
  - Feige
  - Magen Naor

- WIP: Low memory generation of hard to guess permutations.