How to solve math problems without talent

Noriko H. Arai
National Institute of Informatics, Japan

What is your favorite theorem?

LK

$$P_2 \vdash P_2 \qquad P_1 \vdash P_1 \qquad\qquad P_2 \vdash P_2$$

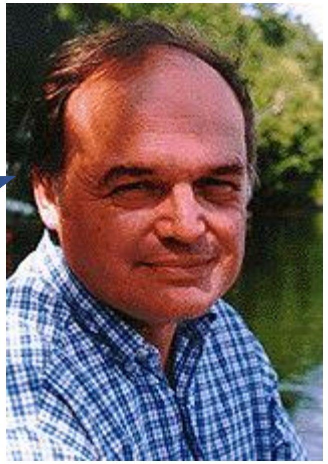
$$P_1 \vdash P_1 \qquad\qquad\qquad P_j \vdash P_j \qquad\qquad P_w \vdash P_w$$

$$\frac{A, \cdots, A_n \vdash B_1, \cdots, B_m \qquad C, \cdots, A_n \vdash B_1, \cdots, B_m}{A \vee C, \cdots, A_n \vdash B_1, \cdots, B_m}$$

Cut-Elimination Theorem (by Gentzen, 1934)

Cuts are removable!

$$\frac{A_1, \cdots, A_n \vdash B_1, \ \cdots, B_m, \mathrm{X} \qquad \mathrm{X}, A_1, \cdots, A_n \vdash B_1, \ \cdots, B_m}{A_1, \cdots, A_n \vdash B_1, \ \cdots, B_m}$$



Don’t eliminate cuts !

But the resulted proof will blow up exponentially (in most cases), or even super-exponentially…

quasi-polynomial simulation of tree resolution by analytic tableaux (Arai, Pitassi & Urquhart, STOC2001)

There is a trade-off between
TALENT and TIME.

Preliminaries 1

- The first-order predicate logic is complete. (1929)
  - $T \vDash \varphi \ \leftrightarrow \ T \vdash \varphi$
  - Unfortunately, it is undecidable.

- Peano Arithmetic is incomplete.(1931)
  - Incompleteness theorem applies for any consistent formal theories that are of sufficient complexity to express the basic arithmetic of the natural numbers.
    - Including theories of bounded arithmetic such as S_2^1.

Kurt Gödel (from Wikipedia)

Preliminaries 2

- Some interesting decidable theories
  - Presburger Arithmetic: {=, S, +} (1929)
  - The first order theories of
    - Boolean Algebra (1949)
    - Algebraically Closed Fields (1949)
    - Real Closed Fields (1949)
    - Euclidean Geometry (1949)

Alfred Tarski (from Wikipedia)

Prove $\boldsymbol{\varphi}$.

Prove the following.

Either $x^3y + y < 12$ if $y < 1$,or $y < 1$ if $x^3y + y < 12$.

$$\underbrace{\underset{q}{\underline{x^3y + y < 12}}\ \text{if}\ \underset{p}{\underline{y < 1}}}_{p \supset q}\ ,\text{or}\ \underbrace{\underset{p}{\underline{y < 1}}\ \text{if}\ \underset{q}{\underline{x^3y + y < 12}}}_{q \supset p}$$

$$p \supset q \quad \vee \quad q \supset p$$

p	q	$p \supset q$	$q \supset p$	$(p \supset q) \vee (q \supset p)$
T	T	T	T	T
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

$$
\begin{array}{c}
p \vdash p \\
\hline
p \vdash p, q \\
\hline
q, p \vdash p, q \\
\hline
p, q \vdash p, q \\
\hline
q \vdash p, p \supset q \\
\hline
q \vdash p \supset q, p \\
\hline
\vdash p \supset q, q \supset p \\
\hline
\vdash p \supset q, p \supset q \vee q \supset p \\
\hline
\vdash p \supset q \vee q \supset p, p \supset q \\
\hline
\vdash p \supset q \vee q \supset p, p \supset q \vee q \supset p \\
\hline
\vdash p \supset q \vee q \supset p
\end{array}
$$

It depends on which THEORY⋯

- If φ is a propositional formula.
  - Write a truth table.
  - Write a cut-free LK tree.
- If φ is a first-order sentence, and if it happens to be logically vaid, you can prove it without talent!
  - Write a cut-free LK tree.
- If φ is a sentence in PA, you need talent+luck to prove it.
- If φ is a sentence in RCF, you can prove or disprove it without talent!

For any RCF formula, there is an algorithm to compute an equivalent quantifier-free formula, but the size of the resulted formula may be doubly exponential to the size of the formula.

There is a trade-off between
TALENT and TIME.

Why do we need talent to prove math theorems?

Show that $f(x) = x^2$ is continuous everywhere.

$$\forall x \forall \varepsilon > 0 \exists \delta > 0 \forall y (|x - y| < \delta \rightarrow |x^2 - y^2| < \varepsilon)$$

$$\forall x \forall \varepsilon > 0 \exists \delta > 0 \forall y (|x - y| < \delta \to |x^2 - y^2| < \varepsilon)$$

Let x be a real number.

Let ε be a positive real number.

$$\frac{A_1, \cdots, A_n \vdash B_1, \ \cdots, B_m, B(a)}{A_1, \cdots, A_n \vdash B_1, \ \cdots, B_m, \forall x B(x)} \quad \forall : right$$

No talent is required to remove the universal quantifiers in most of the theories except for Theories of Natural Numbers.

Why not in Theories of Natural Numbers?

$$\frac{A(a), A_1, \cdots, A_n \vdash B_1, \ \cdots, B_m, A(a+1)}{A(0), A_1, \cdots, A_n \vdash B_1, \ \cdots, B_m, \forall x A(x)}$$

$$\forall x \forall \varepsilon > 0 \exists \delta > 0 \forall y (|x - y| < \delta \to |x^2 - y^2| < \varepsilon)$$

Let x be a real number.

Let ε be a positive number.

No talent is needed to remove the universal quantifiers in most of the theories in RCF.

Find a witness $\delta = t(x, \varepsilon)$ such that for any real number y such that

$$|x^2 - y^2| < \varepsilon \quad \text{if } |x - y| < \delta.$$

Let's find a witness!

$$
\begin{aligned}
|x^2 - y^2| &= |(x+y)(x-y)| \\
&\leq (|x| + |y|)|x-y| \\
&< ||x| + |y||\delta \\
&\leq (2|x| + \delta)\delta \\
&< \varepsilon
\end{aligned}
$$

$$(2|x| + \delta)\delta < \varepsilon$$

$$\delta^2 + 2|\mathrm{x}|\delta - \varepsilon < 0$$

$$0 < \delta < \underline{-x + \sqrt{x^2 + \varepsilon}}$$

You cannot eliminate x from the witness. $f(x) = x^2$ is not uniformly continuous.

∃:right rule in LK

You have to find a witness for B(t) to apply ∃:right rule.
(Same thing for ∀:left rule)
Try every term in ascending order of Gödel number.

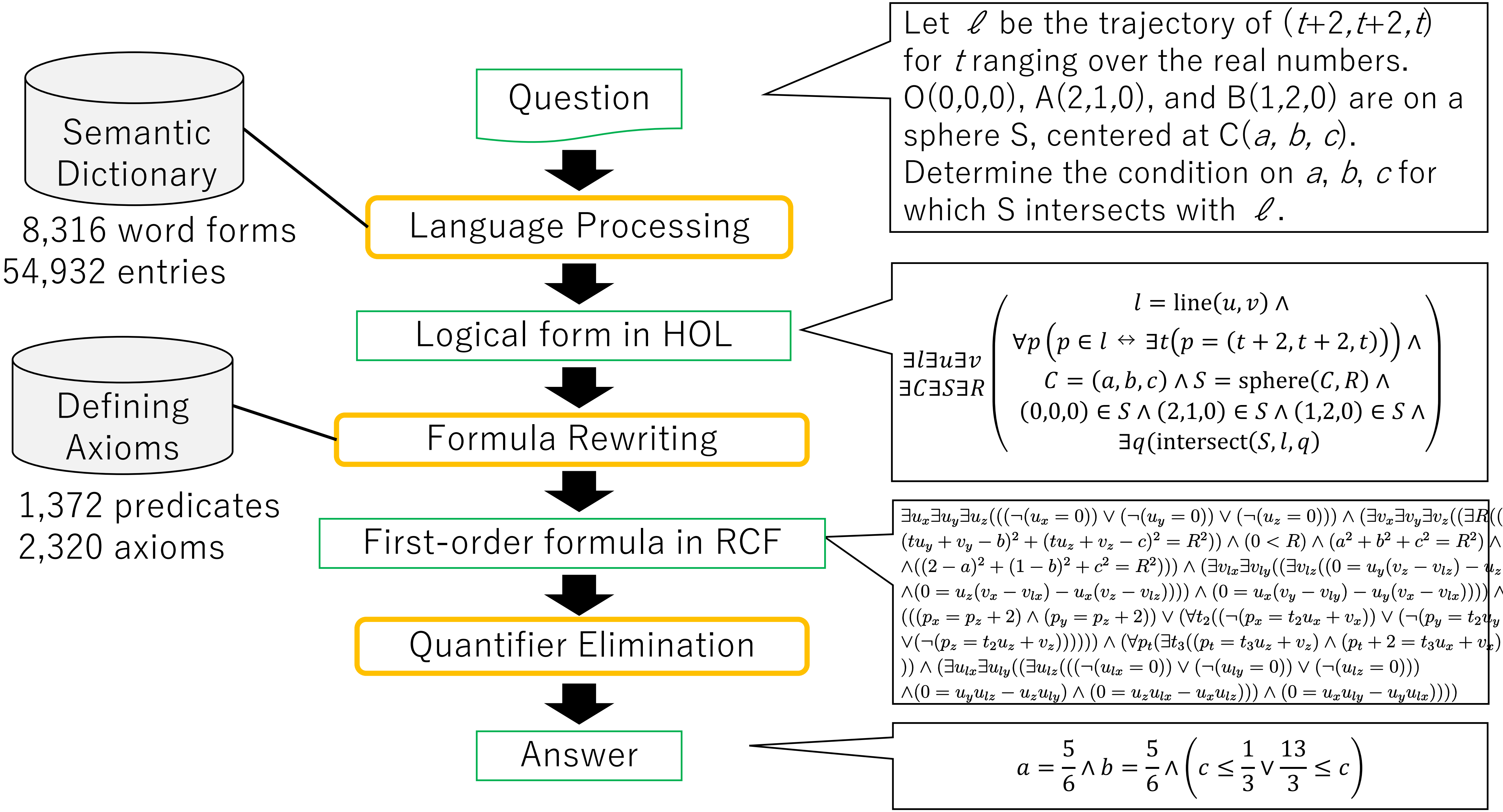
$$\frac{A_1, \cdots, A_n \vdash B_1, \ \cdots, B_m, B(t)}{A_1, \cdots, A_n \vdash B_1, \ \cdots, B_m, \exists x B(x)}$$

There is a trade-off between TALENT and TIME.

Let the least talented beings solve math problems.

(Photo by Ian Battaglia, Unsplash)

Math Problem Solver based on Quantifier Elimination



Can an AI pass the entrance exam of the U. of Tokyo?

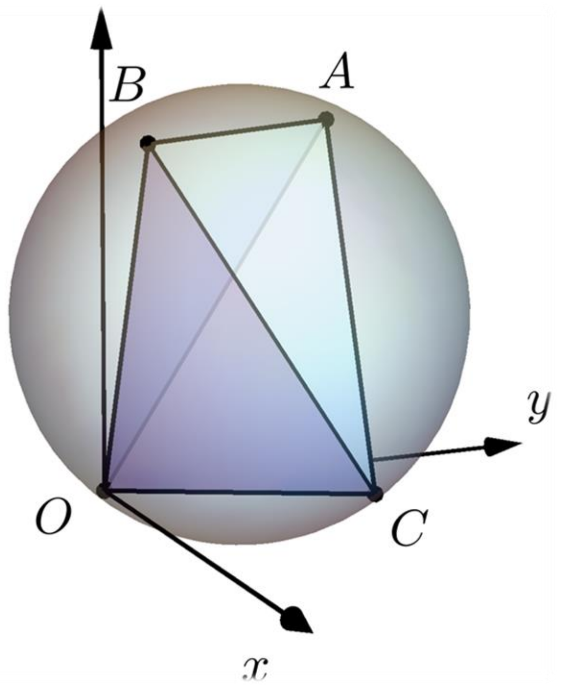
Consider the four points O(0, 0, 0), A(0, 2, 3), B(1, 0, 3), and C(1, 2, 0).

Answer the following questions:

(1) Find the coordinates of the center D of the spherical surface containing the four points O, A, B, and C.

(2) Draw a perpendicular from the point D to the plane containing the four points A, B, and C, and let F be the intersection.
Find the length of the line segment DF.

(3) Find the volume of the tetrahedron ABCD.

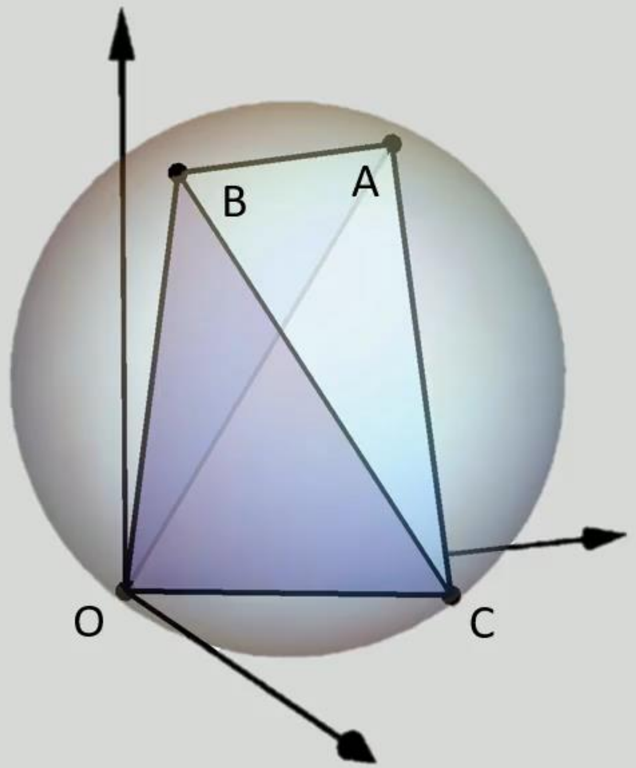


空間内の4点 O(0, 0, 0), A(0, 2, 3), B(1, 0, 3), C(1, 2, 0) を考える。
このとき，以下の問いに答えよ。

(1) 4点O, A, B, Cを通る球面の中心Dの座標を求めよ。

(2) 3点A, B, Cを通る平面に点Dから垂線を引き，
交点をFとする。線分DFの長さを求めよ。

(3) 四面体ABCDの体積を求めよ。



Is it still superior to chatGPT?

Formula Simplification via Invariance Detection by Algebraically Indexed Types,
T. Matsuzaki & T. Fujita, IJCAR2022.


Happy Birthday
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