Lower Bounds Against Non-Commutative Models of Algebraic Computation

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Algebraic Circuit Complexity

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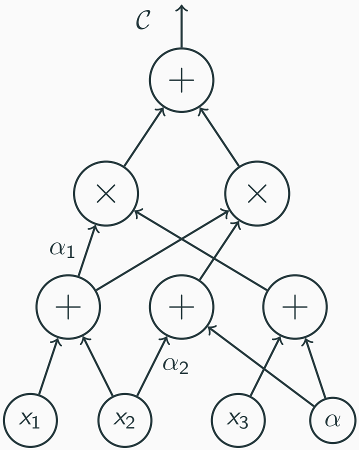
$$\mathsf{VP} = \mathsf{VNP} \overset{\mathsf{GRH}}{\Longrightarrow} \mathsf{P} = \mathsf{NP}$$

Algebraic Circuits



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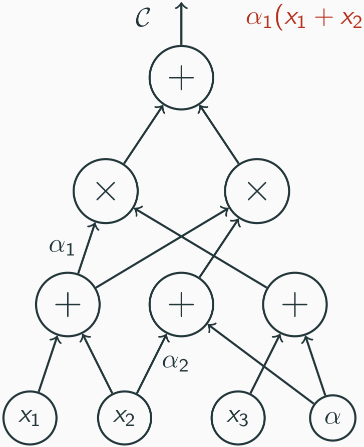

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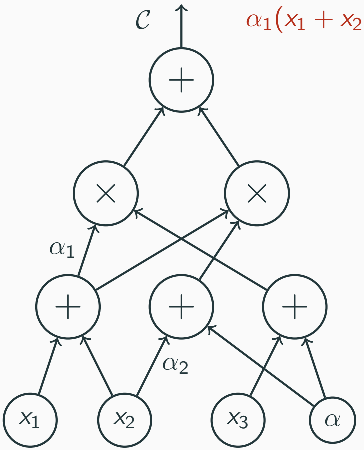

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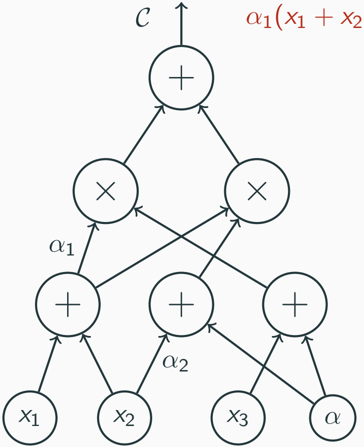

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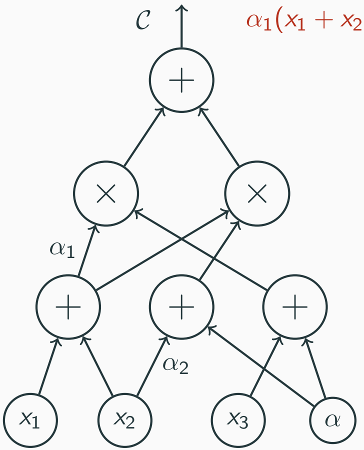

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Central Question

$\boxed{\mathrm{VP} \stackrel{?}{=} \mathrm{VNP}}$: Find explicit polynomials that cannot be computed by circuits of size poly(n,d).

What is Known?

A Superpolynomial Lower Bound against Constant Depth Circuits:

[Limaye-Srinivasan-Tavenas]: There exists an explicit n-variate degree d polynomial in VP such that any product-depth Δ circuit computing it must have size $n^{d^{\exp(-O(\Delta))}}$.

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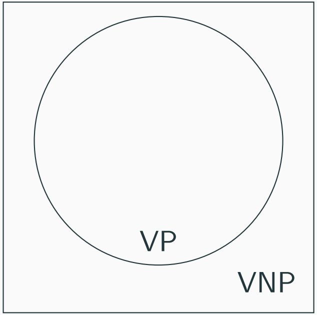
The General Setting

[Baur-Strassen]: Any algebraic circuit computing $\sum_{i=1}^{n} x_i^d$ has size at least $\Omega(n \log d)$.

Other Important Models of Algebraic Computations



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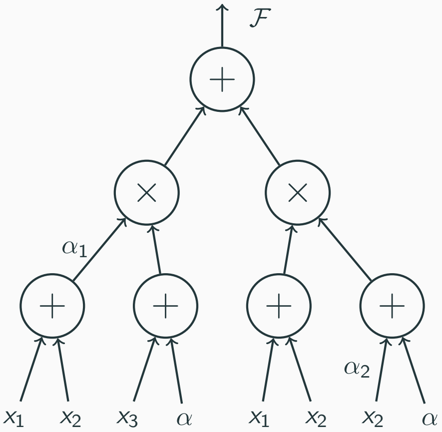

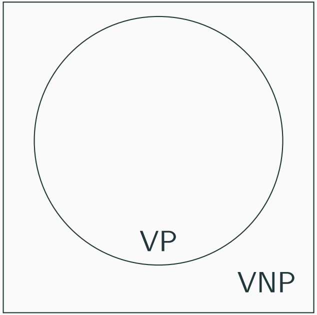

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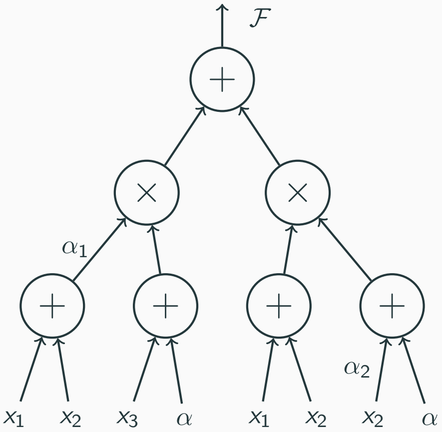

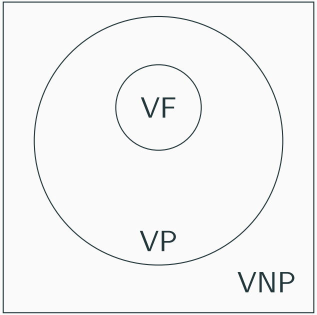

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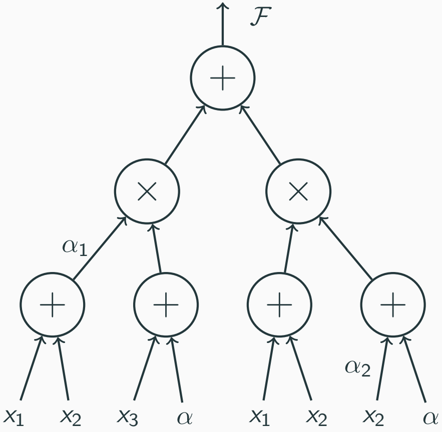

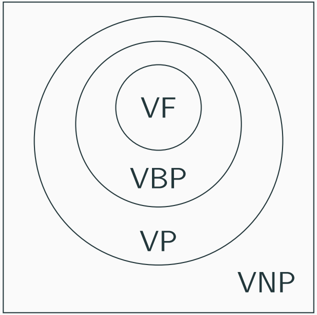

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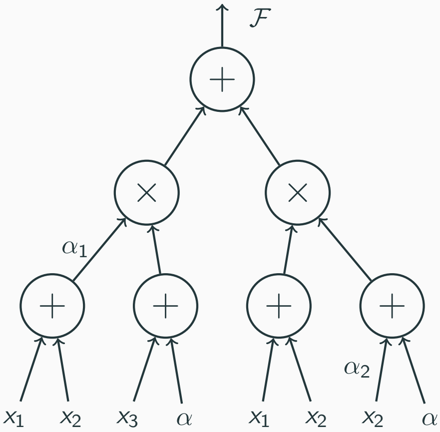

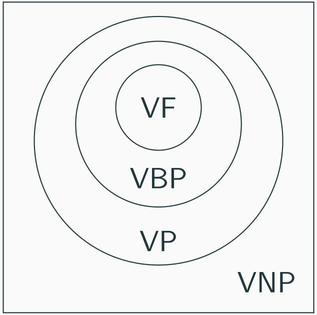

Are the inclusions tight?

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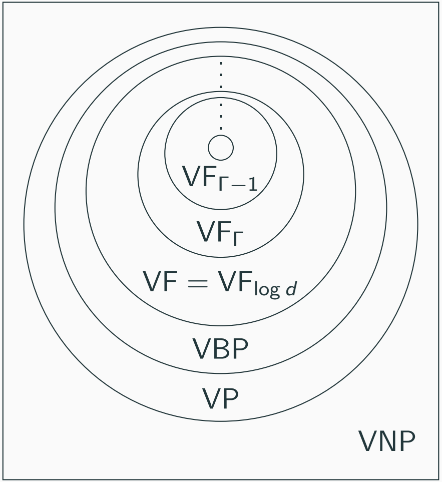

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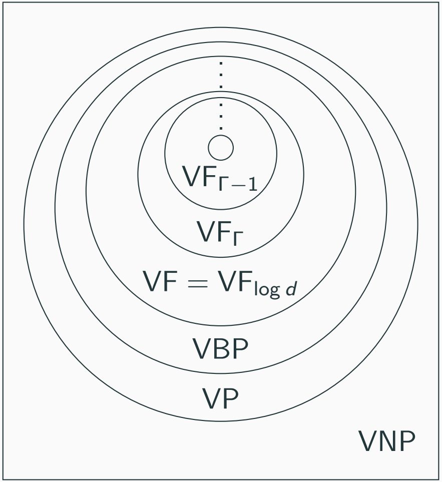

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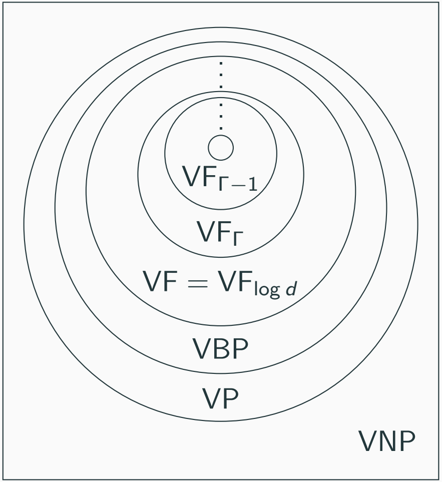

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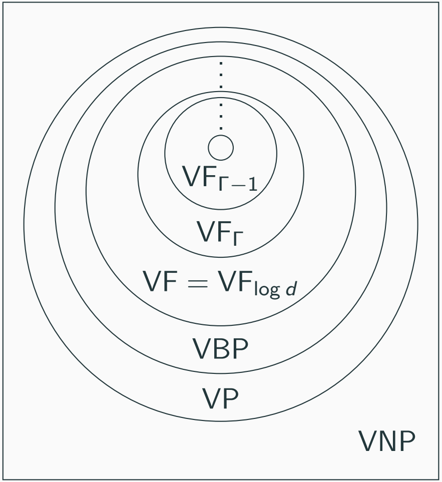

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[C-Kumar-She-Volk]

There is a polynomial over n variables of degree n s.t.

- it can be computed by a circuit of size $O(n \log^2 n)$
- any formula/layered ABP computing it must have size at least $\Omega(n^2)$

The Non-Commutative Setting

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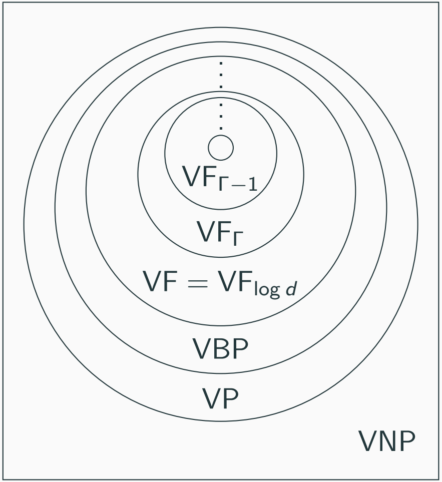
Can we do something better in this setting?

We should be able to...



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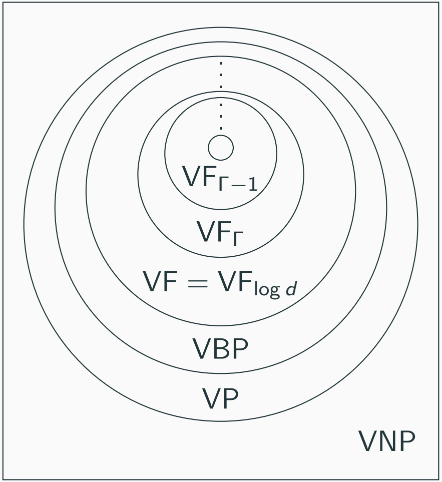

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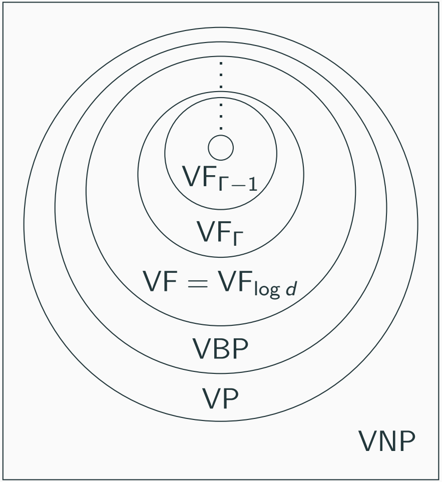

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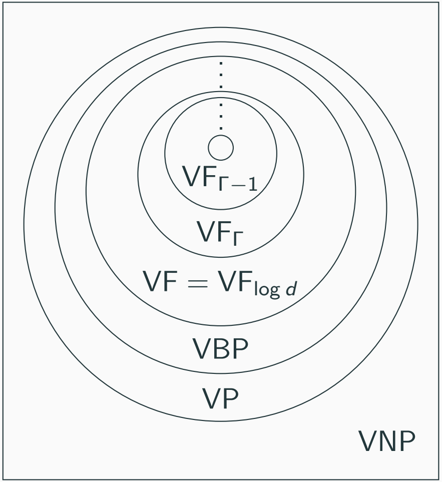

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[Nisan]: $VBP \subsetneq VP$.

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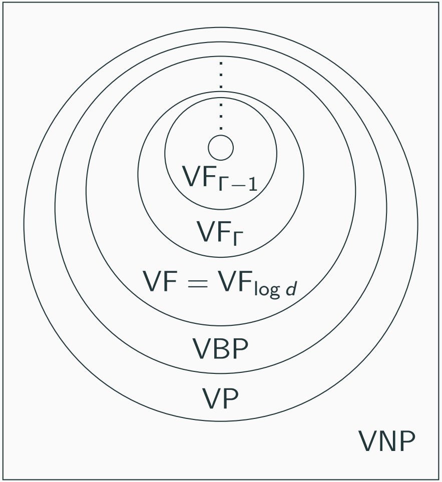

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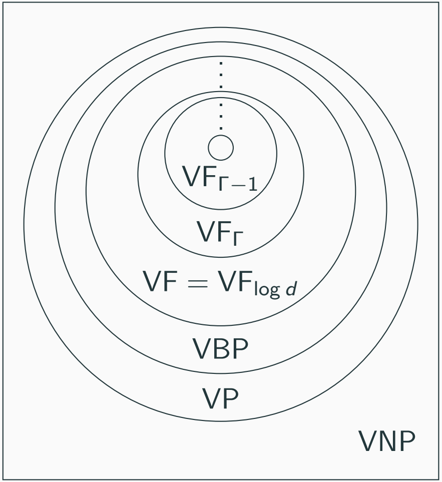

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$\mathsf{VP} \stackrel{?}{=} \mathsf{VNP}$ in the Non-Commutative Setting

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Theorem [C-Hrubeš]: Any homogeneous non-commutative circuit computing

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Further, there is a non-commutative circuit of size $O(n \log^2 n)$ that computes $\mathrm{OSym}_{n,n/2}(\mathbf{x})$.

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leading to a lower bound on s.

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Main Lemma: For any F that is computable by a homogeneous non-commutative circuit of size s,

$$\mu(F) \leq s.$$

Proof Overview

1. [Hom. version of [Baur-Strassen]] If $F(x_1, \ldots, x_n)$ is computable by a homogeneous (non-commutative) circuit of size s, then the polynomials in $\{\partial_{1,x_1} F, \ldots, \partial_{1,x_n} F\}$ are simultaneously computable by a homogeneous (non-commutative) circuit of size $5s$.

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3. For $F_0 = \mathrm{OSym}_{n,d}(\mathbf{x})$,

$$\mu(\partial_{1,x_1} F_0, \ldots, \partial_{1,x_n} F_0) \geq nd.$$

[CILM] and Related Questions

[Carmossino-Impagliazzo-Lovett-Mihajlin]:

$$\Omega(N^{\frac{\omega}{2}+\varepsilon}) \text{ lower bound for } P_{N,D(N)}(\mathbf{x}) \implies \text{ improved lower bound for } Q_{n,d(n)}(\mathbf{x})$$

where the improvement degrades as $D(N)$ gets larger and approaches N.

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[CILM] and Related Questions

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Related Questions:

- Can we show $\Omega(N^{\frac{\omega}{2}+\varepsilon})$ lower bounds for $D(N) = \operatorname{sub}\operatorname{poly}(N)$?
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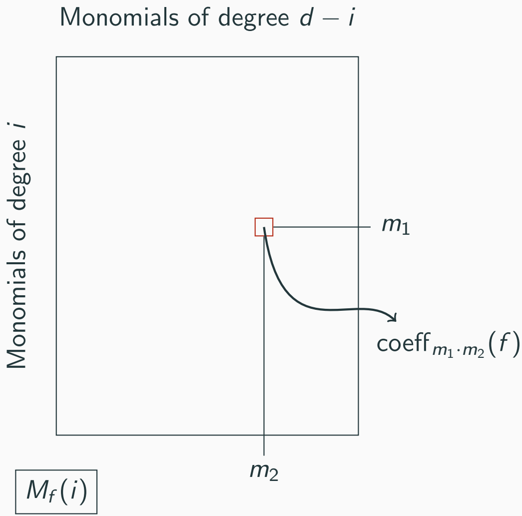
$\mathsf{VF} \stackrel{?}{=} \mathsf{VBP}$ in the Non-Commutative Setting

Nisan's Characterisation



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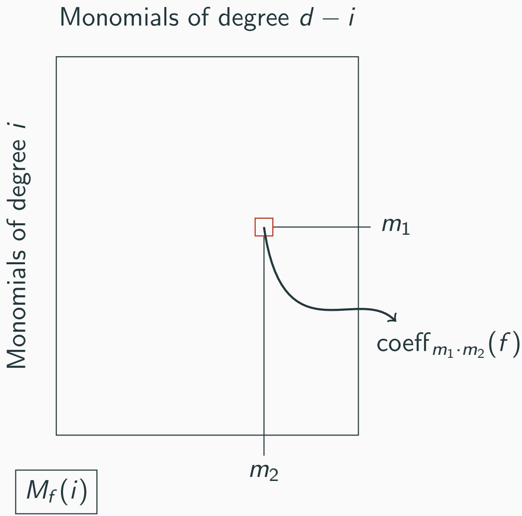

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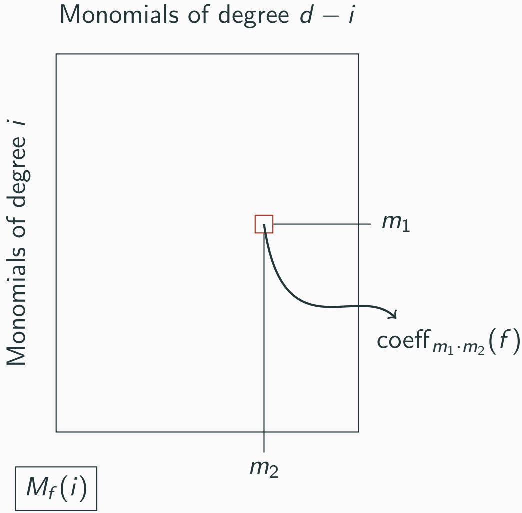

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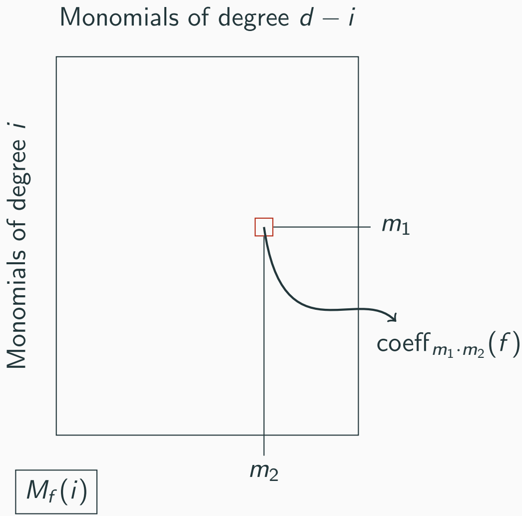

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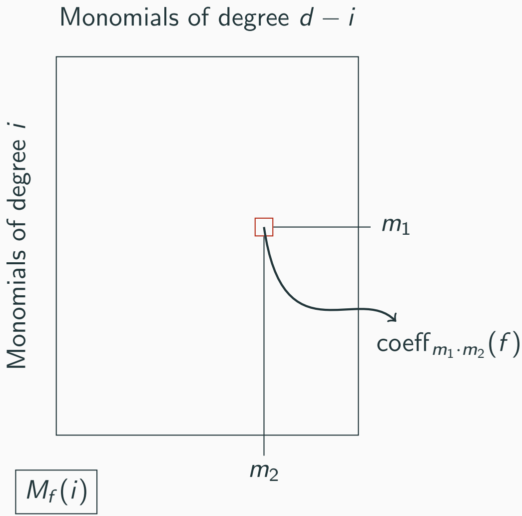

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The Lower Bound: There is a bivariate polynomial of degree $2d$ such that any formula/ABP computing it has size $\Omega(2^d)$. That is, $\mathrm{VBP}_{\mathrm{nc}} \subsetneq \mathrm{VP}_{\mathrm{nc}}$.

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Nisan's Question: $\mathsf{VF}_{\mathsf{nc}} \stackrel{?}{=} \mathsf{VBP}_{\mathsf{nc}}$

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- The lower bound is $n^{\Omega(\log \log n)}$ for a degree n polynomial.
- Proof works in a slightly more general "unordered" setting.

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[Tavenas-Limaye-Srinivasan]: $\mathsf{VF}_{\mathsf{nc,hom}} \subsetneq \mathsf{VBP}$

- The lower bound is $n^{\Omega(\log\log n)}$ for a degree n polynomial.
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- The lower bound is $n^{\Omega(\log\log n)}$ for a degree $\log n$ polynomial.
- Number of buckets is n.

Open Threads

1. Are there VP natural proofs for $\mathsf{VP}_{\mathsf{nc}}$?

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2. Further connections between proof complexity lower bounds and lower bounds in the algebraic non-commutative setting.

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Thank you!